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Maximal Degree Domination in Graphs

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Abstract

A set S of vertices in a graph G is called a dominating set if every vertex in $V - S$ is adjacent to at least one vertex in S. A maximal degree dominating function $(MDDF)$ is a type of function $f : V(G)| \{0, 1, 2, 3, ..., (\triangle(G) + 1)\}$ having the property that every v in S is assigned the value $deg(v) + 1$, and all remaining vertices with zero. The weight of a maximal degree dominating function f is defined by $w(f) = \sum_{v \in S} deg(v) + 1$. The maximal degree domination number $\gamma_{mdeg}(G)$ is the minimum weight among all possible $MDDFs$. In this paper, we determine its exact value.

Keywords: maximal domination number; dominating set; maximal degree domination; maximal degree domination function.

2020 Mathematics Subject Classification: 05C69, 05C76.

1. Introduction

Let $G = (V, E)$ be a simple graph, where V is the vertex set and E is the edge set. The open neighborhood of a vertex v in a graph G is defined as $N(v) = \{u : (u, v) \in E(G)\}$. The closed neighborhood of a vertex v is defined as $N[v] = N(v) \cup \{v\}$. The order n and size m of G are the vertices and edges respectively. The total number of edges incident to a vertex v is called the degree of v in G is defined by $deg(v)$, [1]. A set $S \subseteq V$ is a dominating set if every in $V - S$ is adjacent to at least one vertex in S. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set [4]. There are many types of domination depending on the structures of dominating sets. One of these types, the weighted domination number γ_w of (G, W) is the minimum weight $W(S) = \sum_{v \in D} W(v)$ of a set $S \subseteq V$ with $N[S] = V$, i.e., a dominating set of G [2]. The Roman domination number, denoted by γ_R, is the minimum weight among all possible RDF_s, defined as a function $f : V \to \{0, 1, 2\}$ satisfying the condition that every vertex v with $f(v) = 0$ is adjacent to at least one vertex u such that $f(u) = 2$ [5]. A degree dominating function (DDF) is a function $f : V \to \{0, 1, 2, ..., (\triangle(G) + 1)\}$ with the property that every vertex $V \in S \subseteq V$ is assigned the value $deg(v) + 1$, and all remaining vertices are assigned

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zero [3]. Motivated by the concepts of maximal dominating sets [6] and degree dominating function [3], we introduce in this paper the concept of the maximal degree domination function.

A dominating set S of a graph G is a maximal dominating set if $V - S$ is not a dominating set of G. A maximal degree dominating function (MDDF) is a function $f : V(G) \to \{0, 1, 2, ..., \triangle(G) + 1\}$ having the property that every vertex V of S is assigned with $deg(v) + 1$ and all remaining vertices with zero. The weight of a degree dominating function f is defined by $W(f) = \sum_{v \in S}(deg(v) + 1)$. The maximal degree domination number $\gamma_{mdeg}(G)$, is the minimum weight of all possible $MDDF_s$. The maximal domination number $\gamma_{mdeg}(G)$ of G is the minimum cardinality of a maximal dominating set [6].

2. Maximal Degree Domination Number

Example 2.1. *Consider the following graph G:*

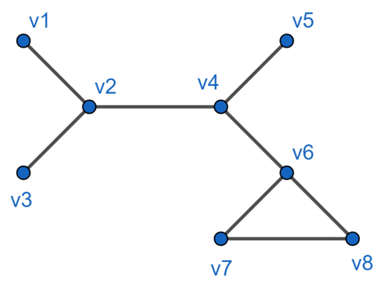

Figure 1: eight-vertices

In Figure 1, there are many maximal dominating sets, but the set that gives the minimum weight should be chosen. Here $S = \{v_1, v_2, v_5, v_6\}$ *is the minimum maximal dominating set. The maximum degree of the graph* G *is* $\triangle(G) = 3$. *By the definition of MDDF,* $f : V(G) \to \{0, 1, 2, 3, 4\}$. *Hence*

$$\gamma_{mdeg}(G) = \sum_{v \in S} f(v) = 2 + 4 + 2 + 4 = 12$$

Theorem 2.2. *For* $n \geq 3$,

$$\gamma_{mdeg}(\mu(P_n)) = \begin{cases} n+2, & if \quad n \equiv 0, 2(mod 3) \\ n+3, & if \quad n \equiv 1(mod 3) \end{cases}$$

Proof. Let $(P_n) = \{v_1, v_2, ..., v_n\}$, be a path of order n. It is known that the degree of all vertices except the pendent vertices is 2. Then we define $f : V(G) \to \{0, 1, 2, 3\}$.

If $n \equiv 0 \ (mod\ 3)$, then $f(V_{3i-1}) = 3$ for $1 \leq i \leq \frac{n}{3}$ and $f(v_n) = 2$.

If $n \equiv 2 \ (mod\ 3)$, then $f(V_{3i-1}) = 3$ for $1 \leq i \leq \frac{n-2}{3}$, $f(v_1) = 2$ and $f(v_n) = 2$.

If $n \equiv 1 \ (mod\ 3)$, then $f(V_{3i-1}) = 3$ for all $1 \leq i \leq \frac{n-1}{3}$, $f(v_1) = 2$ and $f(v_n) = 2$.

For all remaining vertices $f(v) = 0$. It is easy to generalize that f is MDDF of (P_n) weight

$$3.\frac{n}{3} + 2 = n + 2, \ \ if \ \ n \equiv 0 \ (mod \ 3),$$
$$3.\frac{n-2}{3} + 4 = \frac{3n+6}{3} = n + 2, \ \ if \ \ n \equiv 2 \ (mod \ 3),$$
$$3.\frac{n-1}{3} + 4 = n + 3, \ \ if \ \ n \equiv 1 \ (mod \ 3).$$

Thus,

$$\gamma_{mdeg}(P_n) = \begin{cases} n+2, & if \quad n \equiv 0,2 \ (mod \ 3) \\ n+3, & if \quad n \equiv 1 \ (mod \ 3) \end{cases}$$

□

Theorem 2.3. *For $n \geqslant 3$,*

$$\gamma_{mdeg}(C_n) = \begin{cases} \frac{n+6}{,} & if \quad n \equiv 0(mod3) \\ \frac{n+5}{,} & if \quad n \equiv 1(mod3) \\ \frac{n+4}{,} & if \quad n \equiv 2(mod3) \end{cases}$$

Proof. Let $C_n = \{v_1, v_2, ..., v_n\}$ be a cycle of order n. It is a regular graph of degree 2. Then, we define $f : V(G) \rightarrow \{0, 1, 2, 3, \}$.
If $n \equiv 0 \ (mod \ 3)$, then $f(v_{3i-1}) = 3$ for $1 \leq i \leq \frac{n}{3}, f(v_{n-1}) = 3$ and $f(v_n) = 3$.
If $n \equiv 1 \ (mod \ 3)$, then $f(v_{3i-1}) = 3$ for $1 \leq i \leq \frac{n-1}{3}, f(v_{n-1}) = 3$ and $f(v_n) = 3$.
and If $n \equiv 2 \ (mod \ 3)$, then $f(v_{3i-1}) = 3$ for $1 \leq i \leq \frac{n-2}{3}, f(v_{n-1}) = 3$ and $f(v_n) = 3$.
For all remaining vertices of $f(v) = 0$. It is easy to generalize that f is MDDF of C_n of weight

$$3.\frac{n}{3} + 6 = n + 6, \ \ if \ \ n \equiv 0 \ (mod \ 3),$$
$$3.\frac{n-1}{3} + 6 = n + 5, \ \ if \ \ n \equiv 1 \ (mod \ 3)$$
$$3.\frac{n-2}{3} + 6 = n + 4, \ \ if \ \ n \equiv 2 \ (mod \ 3)$$

Thus,

$$\gamma_{mdeg}(C_n) = \begin{cases} \frac{n+6}{,} & if \quad n \equiv 0(mod3) \\ \frac{n+5}{,} & if \quad n \equiv 1(mod3) \\ \frac{n+4}{,} & if \quad n \equiv 2(mod3) \end{cases}$$

□

Theorem 2.4. *For, $K_{r,t}$, with respect to, $\gamma_{mdeg}(Kr, t) = r(t + 2) + 1$.*

Proof. Let $G \cong Kr, t$ be complete bipartite graph with bipartite sets V_1 and V_2 of order r and t, for $r \leq t$, we know that $\gamma_{md}(Kr, t) = r + 1$, and $S = \{x_1, x_2, x_3, ..., x_r, t\}$, where $x_i \in V_i, i = 1, 2, 3, ..., r$ and $t \in V_2$

is the maximal dominating set of Kr, t. Then $deg(x_i) = t$ and $deg(t) = r$. By the definition of MDDF,

$$\begin{aligned}
\gamma_{mdeg}(Kr, t) &= (deg(x_1) + 1) + (deg(x_2 + 1) + ... + (deg(x_r) + 1) + (deg(t) + 1) \\
&= (t + 1) + (t + 1) + ... + (t + 1) + (r + 1) \\
&= r(t + 1) + r + 1 \\
&= r(t + 2) + 1
\end{aligned}$$

Thus, $\gamma_{mdeg}(Kr, t) = r(t + 2) + 1$, where $r \leq t$. □

Theorem 2.5. *Let G be a complement of complete bipartite graph, where $r \leq t$. Then $\gamma_{mdeg}(\overline{K}_{r,t}) = r^2 + t$.*

Proof. Let $G \cong \overline{K}_{r,t}$. Then $\overline{K}_{r,t} = K_r \cup K_t$ of order r and t, respectively. We know that, $\gamma_{md}(\overline{K}_{r,t}) = \min\{r, t\} + 1 = r + 1$, if $r \leq t$. And $S = \{x_1, x_2, ..., x_r, v; x_i \in K_r \ \ and \ \ v \in K_t\}$ is the maximal dominating set of $\overline{K}_{r,t}$. Then $deg(x_i) = r - 1$ and $deg(v) = t - 1$. By the definition of MDDF,

$$\begin{aligned}
\gamma_{mdeg}(\overline{K}_{r,t}) &= (deg(x_1) + 1) + ... + (deg(x_r) + 1) + (deg(v) + 1) \\
&= ((r - 1) + 1) + ... + ((r - 1) + 1) + ((t - 1) + 1) \\
&= r + r + ... + r + t \\
&= r^2 + t
\end{aligned}$$

□

Theorem 2.6. *For $n \geq 1$, $\gamma_{mdeg}(K_n) = n^2$.*

Proof. K_n is a regular graph of degree $(n - 1)$, and $\gamma_{mdeg}(K_n) = n$ and $S = \{v_1, v_2, ..., v_n\}$ is the maximal dominating set of K_n. By the definition of MDDF,

$$\begin{aligned}
\gamma_{mdeg}(K_n) &= (deg(v_1) + 1) + (deg(v_2) + 1) + \cdots + (deg(v_n) + 1) \\
&= ((n - 1) + 1) + ... + ((n - 1) + 1) \\
&= n.n = n^2
\end{aligned}$$

□

Observation 2.7. *Let G be a totally disconnected graph, then $\gamma_{mdeg}(G) = 2n$.*

Theorem 2.8. *For $n \geq 4$, $\gamma_{mdeg}(W_n) = n + 12$.*

Proof. Consider any wheel graph W_n with n vertices formed by sum of the complete graph with one vertex v_1 and cycle graph with $n - 1$ vertices are $v_2, v_3, \ldots, v_{n-1}, v_n$, that is the wheel W_n can be defined as the graph $K_1 + C_{n-1}$. Hee v_1 has degree $n - 1$ so it is the internal vertex to all other vertices and $deg(v_2) = deg(v_3) = \cdots = deg(v_n) = 3$. We know that $\gamma_m(W_n) = 4$ and $S = \{v_1, v_2, v_3, v_4\}$ is the maximal dominating set of W_n. Then $deg(v_1) = n-1, deg(v_2) = deg(v_3) = deg(v_4) = 3$. By the definition of MDDF,

$$\begin{aligned}\gamma_{mdeg}(W_n) &= (deg(v_1)+1) + (deg(v_2)+1) + (deg(v_3)+1) + (deg(v_4)+1)\\ &= (n-1)+1+(3+1)+(3+1)+(3+1)\\ &= n+12\end{aligned}$$

□

Theorem 2.9. *For any graph G, $\gamma_{md}(G) \leq \gamma_{mdeg}(G)$.*

Proof. Suppose that S is a maximal dominating set and D is the maximal degree dominating set of G. Let $|S| = t$ where $t \geq 1$. By the definition of MDDF, it is clear that D consists of $deg(v)+1$, where $v \in S$. Thus $|D| = \sum_{i=1}^{k}(deg(v_i)+1)$. Therefore, $|S| = |D|$ and $\gamma_{md}(G) \leq \gamma_{mdeg}(G)$. □

Observation 2.10. *For any connected graph G, $\gamma_{mdeg}(G) \geq deg(v_i)$ for all $1 \leq i \leq n$.*

Lemma 2.11. *Let G be an r-regular graph. Then $\gamma_{mdeg}(G) = (r+1)\gamma_{md}(G)$.*

Proof. Suppose that S is a maximal dominating set and r is the degree of each vertex in G. Let $|S| = t$ where $t \geq 1$. It is clear that the degree of all vertices in S is r, by the definition of the MDDF, $\gamma_{mdeg}(G) = \sum_{i=1}^{t}(r+1)$. Therefore,

$$\begin{aligned}\gamma_{mdeg}(G) &= (r+1)t\\ &= (r+1)\,|S|\\ &= (r+1).\gamma_{md}(G)\end{aligned}$$

□

Proposition 2.12. *For any helm graph H_n, $(n \geq 4)$, the maximal degree domination number is $\gamma_{mdeg}(H_n) = 2n+5, n \geq 4$.*

Proof. Let $G \cong H_n$ be a helm graph on $2n+1$ vertices and $3n$ edges. Let $deg(v) = \triangle = n$. Let $v_1, v_2, \ldots, v_n$ be the vertices in the helm graph, each having degree 4. Also, let $u_1, u_2, \ldots, u_n$ denotes the pendent vertices of the helm graph. It can be easily verified that $S = \{v_1, u_1, \ldots, u_n\}$ is a dominating set of the helm graph G. Choose a vertex u_1 which is adjacent to v_1. Then clearly, $T = S \cup v_1$ is a maximal dominating set of the helm graph G. Therefore $\gamma_{md}(G) = n+1$, the maximum degree of the graph G is $\triangle(G) = n$. By the definition of MDDF, $f : V(G) \to \{0, 1, 2, ..., \triangle+1\}$ and the MDDF must consist of vertices, $\{(deg(v_1)+1), (deg(u_2)+1), \ldots, (deg(u_n)+1), (deg(u_1)+1)\}$. Hence the maximal degree domination number is,

$$\begin{aligned}\gamma_{mdeg}(G) &= \sum_{v \in T} f(v) \\ &= (4+1) + (1+1) + \cdots + (1+1) \\ &= 5 + 2 + \cdots + 2 \\ &= 2n + 5\end{aligned}$$

□

Proposition 2.13. *For any firecracker graph $F(m,n)$, the maximal degree domination number is $\gamma_{mdeg}(F_{m,n}) = 4m + 2$, where $n \geq 2$.*

Proof. Let $G \cong F(m,n)$ be a firecracker graph on mn vertices with $(mn-1)$ edges, and let D be a minimum dominating set of graph G. By the definition of firecracker graph, it is constructed by joining m copies of n stars in a series, linking one leaf from each. For each of the n stars, if we choose all central vertices $v_1, v_2, ...v_m$ and define the set $S = \{v_1, v_2, \ldots, v_m\}$, then S dominates all the other vertices of G. Therefore, the domination number is $\gamma(G) = m$. Let x be any pendent vertex of G. Then $T = S \cup \{x\}$ forms a maximal dominating set with the minimum possible cardinality, and hence $\gamma_{md}(G) = m + 1$. The maximum degree of the graph G is $\triangle(G) = 3$. By the definition of a MDDF, it is a function $f : V(G) \to \{0,1,2,3,4\}$, where the MDDF assigns to each vertex in a dominating set the value $\{deg(v_1) + 1, deg(v_2) + 1, \ldots, deg(v_n) + 1\}$. Hence the maximal degree domination number is,

$$\begin{aligned}\gamma_{mdeg}(G) &= \sum_{v \in T} f(v) \\ &= (3+1) + (3+1) + \cdots + (3+1) + (1+1) \\ &= 4 + 4 + \cdots + 4 + 2 \\ &= 4m + 2\end{aligned}$$

□

Definition 2.14. *Let G be a caterpillar graph with a vertex set $\{v_1, v_2, \ldots, v_n\}$ of a path and number of the pendant vertices are denoted with $m_1, m_2, \ldots, m_n$ to the $v_1, v_2, \ldots, v_n$ respectively. If G is denoted by $G = C_n(m_1 + 1, m_2, \ldots, m_n + 1)$ as m figure 1.*

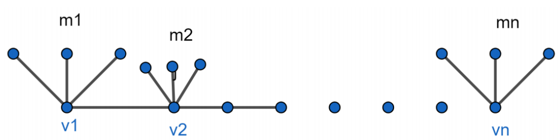

Figure 2: caterpillar-graph

Lemma 2.15. *For any caterpillar graph* $G \cong C_n(m_1+1, m_2+1, \ldots, m_n+1)$, *the maximal degree domination number is* $\gamma mdeg(G) = nt + 2n + 2$, *where* $m_1 = m_2 = \cdots = m_n = t$.

Proof. Let $G \cong C_n(m_1+1, m_2+1, \ldots, m_n+1)$ be a caterpillar graph with vertex set $\{v_1, v_2, \ldots, v_n\}$ of a path and the number of pendant vertices are denoted with $|V_1| = m_1, |V_2| = m_2, \ldots, |V_n| = m_n$, to the $v_1, v_2, \ldots, v_n$ respectively. Let $m_1 = m_2 = \cdots = m_n = t$ (say). It is known that v_1 dominates m_1 pendant vertices that are connected to it. Like this minimum dominating set is $S = \{v_1, v_2, \ldots, v_n|\}$. Therefore $\gamma(G) = n$. Choose a vertex $x \in V_1$. Then $T = S \cup \{x\}$ is a maximal dominating set of G, and so $\gamma_{md}(G)n + 1$. The maximum degree of the graph G is $\triangle(G) = t + 1$. By the definition of MDDF, $f; V(G) \to \{0, 1, 2, \ldots, \triangle + 1\}$ and the MDDF must consist of vertices $\{deg(v_1) + 1, deg(v_2) + 1, \ldots, deg(v_n) + 1, deg(x) + 1\}$. Hence, the maximal degree domination number is,

$$\begin{aligned}
\gamma mdeg(G) &= \sum_{v \in T} f(v) \\
&= [(t+1)+1] + [(t+1)+1] + \cdots + [(t+1)+1] + (1+1) \\
&= (t+2)(t+2) + \cdots + (t+2) + 2 \\
&= n(t+2) + 2 \\
&= nt + 2n + 2
\end{aligned}$$

□

Theorem 2.16. *For any Comb graph* $G = C_n(2, 2, \ldots, 2)$, *the maximal degree dominating number is,* $\gamma_{mdeg}(G) = 2n + 3$.

Proof. By above Lemma 1.15, for $m_1 = m_2 = \cdots = m_n = 1$, and $m = \sum_{i=1}^{n} m_i = n$. We know that, there are three minimum dominating sets, but the set that gives the minimum weight should be chosen. It is seen that the set of all pendant vertices $S = \{u_1, u_2, \ldots, u_n\}$ is the minimum dominating set. Let x be a vertex of degree 2. Then $T = S \cup \{x\}$ is maximal dominating set of G, and $\gamma_{md}(G) = n + 1$. The maximum degree of the graph G is $\triangle(G) = 3$. By the definition of MDDF, $f : V(G) \to \{0, 1, 2, 3, 4\}$ and the MDDF must consist of vertices $\{deg(u_1) + 1, deg(u_2) + 1, \ldots, deg(u_n) + 1, deg(x) + 1\}$

$$\begin{aligned}
\gamma_{mdeg}(G) &= \sum_{v \in T} f(v) \\
&= (1+1) + (1+1) + ... + (1+1) + (2+1) \\
&= 2 + 2 + ... + 2 + 3 \\
&= 2n + 3
\end{aligned}$$

□

3. Conclusion

In this paper, we introduced and investigated the concept of a maximal degree dominating function (MDDF) for a graph G, which assigns a value of $deg(v) + 1$ to vertices in a dominating set s, and zero to all others. We defined the maximal degree domination number, denoted $\gamma_{mdeg}(G)$, as the minimum possible weight of such a function over all dominating sets in G.

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