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Multifactor Balanced Asymmetrical Factorial Designs

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Abstract

This manuscript gives methods of constructing Multifactor Balanced Asymmetrical Factorial Designs (MBAFD'S). Multi-factor BAFDS are constructed from two factor BAFDS. Two methods of construction are given. The first method is the product of balanced arrays which is similar to the product of orthogonal arrays defined by [5]. The second method was given by [36] which generates a BAFD from two given BAFD's. These methods can provide efficient BAFD's if efficient two factor BAFD's are used. The designs constructed are balanced with orthogonal factorial structure.

Keywords: balanced arrays; orthogonal arrays; effiecient BAFD's; designs; balance; orthogonal factorial structure.

1. Introduction

In many situations there arise scenarios when an experimenter has to use factors at different levels. The problem of obtaining confounded plans for such cases has received a good deal of attention. To this extent, [45], by trial and hit methods obtained confounded plans of the type $3^m \times 2^n$, where m and n are any positive integers. Using orthogonal arrays of strength 2, Nair [31] gave methods for constructing Extended Group Divisible Designs $\{EGD\}$ for $s_1 \times s_2$ experiments in blocks of size $s_1 < s_2$. Thomson [42] starting from a basic $s_1 \times s_2$ design in blocks of size s_2 ($s_2 < s_1$, s_1 being a prime number or power of prime) obtained three factor designs. Rao [34] constructed some series of designs from orthogonal latin squares for $s_1 \times s_2$ experiments in block of size s_1 and $s_2 - 1$ replications. Tharthare [41] gave a class of balanced designs with OFS. Muller [28] considered the use of balanced incomplete block designs for the construction of $s_1 \times s_2$ balanced factorial designs with OFS when $s_1 > s_2$. Informative accounts and subsequent developments have been done by [23, 17, 39]. Sreenath [37] proposed a

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general method of obtaining block designs for asymmetrical confounded factorial experiments using block designs for symmetrical factorial experiments.

Rajarathinam [33] applied the methods that are used in construction of variance balanced designs in order to construct variance balanced block designs that are highly efficient. Ghosh [16] also extended the work of [33] in the construction of variance balanced block designs.

Gupta [18] describes a general method of construction of supersaturated designs for asymmetric factorials by exploiting the concept of resolvable orthogonal arrays and Hadamard matrices. El-Helbawy [14] considered three forms of a general null hypothesis on the factorial parameters of a general asymmetrical factorial paired comparison experiment in order to determine optimal or efficient designs. Prakash Kumar [22] constructed designs by using confounding through equation methods. In this case construction of confounded asymmetrical factorial experiments in row-column settings and efficiency factor of confounded effects was worked out. Agarwal [1] attempted to construct asymmetrical factorial type switch over designs having strip type arrangement of combination of the levels. To start with, two factors at different levels were considered. Jyoti Divecha [13] described a method of constructing cost-efficient response surface designs (RSDs) as compared to the replicated central composite designs (RCCDs), that are useful for modelling and optimization of the asymmetric experiments. Voss [43] identified a Kronecker product structure for a particular class of asymmetric factorial designs in blocks, including the classes of designs generated by several of the generalizations of the classical methods in literature. Dipa Rani Das [9] focuses on the construction and analysis of an extra ordinary type of asymmetrical factorial experiment which corresponds to a fraction of asymmetrical factorial experiment as indicated by [11]. Chatterjee [7] establishes a lower bound to measure optimality with respect to a main effects model in a general asymmetric factorial experiment. Mainardi [26] conceptualized the fundamental aspects of the Complete, Fractional, Central Composite Rotational and Asymmetrical factorial designs. Recent applications of these powerful tools were described. Bahl [3] developed a method for the construction of $p \times 3 \times 2$ asymmetrical factorial experiments with $(p-1)$ replications. Sreenath [38] proposed A general method of obtaining block designs for asymmetrical confounded factorial experiments using the block designs for symmetrical factorial experiments. Zi [46] Constructed asymmetrical factorial designs containing clear effects. Metrika [27] explained how to choose an optimal $(s^2)s^n$ design for the practical need, where s is any prime or prime power and accordingly considered the clear effects criterion for selecting good designs. Murthy [29] delt with situations where there was a need for designing an asymmetrical factorial experiment involving interactions. Failing to get a satisfactory answer to this problem from literature, the authors have developed an adhoc method of constructing the design. It is transparent that the design provides efficient estimates for all the required main effects and interactions. The later part of this paper deals with the issues of how this method is extended to more general situations and how this adhoc method is translated into a systematic approach. Ulrike Gromping [32] developed The R package DoE.base which can be used for creating full factorial designs and general factorial experiments based on orthogonal arrays. Besides design creation, some analysis functionality is also available, particularly (augmented) half-normal effects plots.

Jalil [20] Published a monograph that is an outcome of the research works on the construction of factorial experiments (symmetrical and asymmetrical). In this booklet, construction frameworks have been described for factorial experiments. The construction frameworks include general construction method of p^n factorial experiments, construction methods with confounded effects and detection methods of confounded effects in a confounded plan. The concepts of combinatorial matrix operations and linear equation technique have been deployed to develop the methods. [10] discussed an Alternative Method of Construction and Analysis of Asymmetrical Factorial Experiment of the type 6×22 in Blocks of Size 12. Dipa Rani Das [9] focuses on the construction and analysis of an extra ordinary type of asymmetrical factorial experiment which corresponds to a fraction of a symmetrical factorial experiment as indicated by [11]. For constructing this design, they used 3 choices and for each choice they used 5 different cases. Finding the block contents for each case shows that there are mainly two different cases for each choice. In case of analysis of variance, it is seen that, for the case where the highest order interaction effect is confounded in 4 replications, the loss of information is same for all the choices.

Klaus Hinkelmann [19] in his book chapter discusses different methods of constructing systems of confounding for asymmetrical factorial designs, including: Combining symmetrical systems of confounding via the Kronecker product method, use of pseudo-factors, the method of generalized cyclic designs, method of finite rings (this method is also used to extend the Kempthorne parameterization from symmetrical to asymmetrical factorials), and the method of balanced factorial designs. He showed the equivalence of balanced factorial designs and extended group divisible designs, establishing again a close link between incomplete block designs and confounding in factorial designs.

Angela Dean [12] in her book chapter discusses confounding in single replicate experiments in which at least one factor has more than two levels. First, the case of three-levelled factors is considered and the techniques are then adapted to handle m-levelled factors, where m is a prime number. Next, pseudofactors are introduced to facilitate confounding for factors with non-prime numbers of levels. Asymmetrical experiments involving factors or pseudofactors at both two and three levels are also considered, as well as more complicated situations where the treatment factors have a mixture of $2, 3, 4$, and 6 levels. Analysis of an experiment with partial confounding is illustrated using the SAS and R software packages.

Gachii [15] shows that Asymmetrical single replicate factorial designs in blocks are constructed using the deletion technique. Results are given that are useful in simplifying expressions for calculating loss of information on main effects and interactions, due to confounding with blocks. Designs for estimating main effects and low order interactions are also given.

Conto Lopez [25] in his work presents the results of a systematic literature review (SLR) and a taxonomical classification of studies about run orders for factorial designs published between 1952 and 2021. The objective here is to describe the findings, main and future research directions in this field. The main components considered in each study and the methodologies they used to obtain run sequences are also highlighted, allowing professionals to select an appropriate ordering for their problem. This review shows that obtaining orderings with good properties for an experimental design with any number of factors and levels is still an unresolved issue.

Rahul Mukerjee [44] in his present book gives, for the first time, a comprehensive and up-to-date account of modern theory of factorial designs. Many major classes of designs are covered in the book. While maintaining a high level of mathematical rigor, it also provides extensive design tables for research and practical purposes. Bagchi [2] in his work discusses the construction of 'inter-class orthogonal' main effect plans (MEPs) for asymmetrical experiments. In such a plan, the factors are partitioned into classes so that any two factors from different classes are orthogonal. The researcher also defined the concept of "partial orthogonality" between a pair of factors. In many of his plans, partial orthogonality has been achieved when (total) orthogonality is not possible due to divisibility or any other restriction. He presented a method of obtaining inter-class orthogonal MEPs. Using this method and also the method of 'cut and paste' he obtained several series of inter-class orthogonal MEPs. One of them happens to be a series of orthogonal MEP (OMEPs), which includes an OMEP for a 330 experiment on 64 runs.

Ching-Shui Cheng [8] in his book, provides a rigorous, systematic, and up-to-date treatment of the theoretical aspects of factorial designs. To prepare readers for a general theory, the author first presents a unified treatment of several simple designs, including completely randomized designs, block designs, and row-column designs. As such, the book is accessible to readers with minimal exposure to experimental design. In [6], Lee discrepancy has wide applications in design of experiments, which can be used to measure the uniformity of fractional factorials. An improved lower bound of Lee discrepancy for asymmetrical factorials with mixed two-, three- and four-level is presented. The new lower bound is more accurate for a lot of designs than other existing lower bounds, which is a useful complement to the lower bounds of Lee discrepancy and can be served as a benchmark to search uniform designs with mixed levels. In this paper, we shall use already known methods and some known balanced factorial designs to construct multifactor balanced factorial designs and in particular type I designs. We are especially interested in designs in which main effects and lower order interactions can be estimated with higher efficiencies.

2. S^m Symmetrical BFD's with Block Size s

An experiment involving $2 \leq m$ factors $F_1, F_2, \ldots, F_m$ that appear at $s_1, \ldots, s_m$ (≥ 2) levels is called an $s_1 \times \cdots \times s_m$ factorial experiment (or an $s_1 \times \cdots \times s_m$ factorial for brevity). If $s_1 = \cdots = s_m = s$, we have s^m symmetrical BFD with $r =$ no of replications, λ_i =no of blocks in which any two treatments are

i^{th} associates, k =block size, b =no of blocks. The s^m symmetrical balanced factorial design (BFD) has been shown by [35] to be equivalent to a PBIB with a hypercubic association scheme.We shall consider the construction of such designs with block size s in this section, we have

$$r(s-1) = \sum_{i=1}^{m} \binom{m}{i} (s-1)^i \lambda_i \tag{1}$$

hence

$$r = \sum_{i=1}^{m} \binom{m}{i} (s-1)^{i-1} \lambda_i \tag{2}$$

r is completely determined by the values of $\lambda_i's$. When s is a prime power [40] showed that there exists an s^m symmetrical BFD with block size s for any given $\lambda_1, \lambda_2, \ldots, \lambda_m$.

Lemma 2.1. *If s is a prime power, then given j $(1 \leq j \leq m)$ there exists an s^m symmetrical balanced factorial design with block size s and parameters $\lambda_j = 1$, $\lambda_i = 0 \quad for \quad all \quad i \neq j$.*

The efficiencies of the symmetrical balanced factorial design constructed in Lemma 2.1, can be calculated by equation (28) and Theorem A.9 and hence

$$E_i = 1 - \frac{1}{s} - \frac{P_j(i; m, s)}{\binom{m}{j}(s-1)^{j-1}s} \quad i = 1, 2, \ldots, m \tag{3}$$

In particular when $j = m$, $P_m(i; m, s)$

$$= (-1)^i (s-1)^{m-i}$$

and equation (3) becomes (4)

$$E_i = 1 - \frac{1}{s} - \frac{(-1)^i}{(s-1)^{i-1}s}; \quad i = 1, 2, \ldots, m. \tag{4}$$

This balanced design has been constructed by [4], the main effects are estimated with full efficiency since $E_1 = 1$ in equation (4)

Theorem 2.2. *If s is a prime power, then for any given $\lambda_1, \lambda_2, \ldots, \lambda_m$ there exists an s^m symmetrical balanced factorial design with block size s and parameters $\lambda_1, \lambda_2, \ldots, \lambda_m$.*

Proof. Let D_j denote the design constructed in Lemma 2.1. The symmetrical balanced factorial design consists of λ_j $D_j's$ for $j = 1, 2, \ldots, m$ has parameters $\lambda_1, \lambda_2, \ldots, \lambda_m$. Now consider the case when s is not a prime power. In an s^2 symmetrical balanced factorial design with block size s if we can construct a design with $\lambda_1 = 0$ and $\lambda_2 \neq 0$ then the design is equivalent to a $TA[\lambda_2 s(s-1), s, s, 2]$ by Corollary A.11 and the main effects are estimated with full efficiency. In the case of s^m symmetrical balanced factorial designs with block size s, if we can construct a design with parameters $\lambda_m \neq 0$, and $\lambda_i = 0$ for $i = 1, 2, \ldots, m-1$, then the main effects are estimated with full efficiency. If a $TA[\lambda s(s-1), s, s, 2]$ exists,we can multiply (see Theorem A.10) $m-1$ such transitive arrays to get a $BA[\lambda s(s-$

$1)^{m-1}, s, s, 2]$ with parameters $\lambda[(x_1, \ldots, x_{m-1}), (y_1 \ldots, y_{m-1})] = \lambda^{m-1}$ if $x_i \neq y_i$ for all $i = 1, 2, \ldots, m-1$ and $\lambda[(x_1, \ldots, x_{m-1}), (y_1 \ldots, y_{m-1})] = 0$ otherwise. Identifying rows with the levels of F_1, symbols with the levels of $F_2, \ldots, F_m$, and columns with blocks, we obtain an s^m symmetrical balanced factorial design in $[\lambda s(s-1)]^{m-1}$ blocks of s plots each with parameters $\lambda_m = \lambda^{m-1}$ and $\lambda_i = 0$ for $i = 1, \ldots, m-1$. Thus we have the following theorem. □

Theorem 2.3. *The existence of a $TA[\lambda s(s-1), s, s, 2]$ implies the existence of an s^m symmetrical balanced factorial design with $b = [\lambda s(s-1)]^{m-1}, k = s, r = [\lambda(s-1)]^{m-1}$, $\lambda_m = \lambda^{m-1}$ and $\lambda_i = 0$, for $i = 1, \ldots, m-1$.*

3. Methods of Constructing Multifactor BAFDS

The methods are in form of theorems:

Theorem 3.1. *If there exists a $BA[N_i, s_m, s_i, 2](i = 1, \ldots, m-1)$ with parameters $\lambda_i(x, y) = \mu_0^i$ or μ_1^i according as $x = y$ or not then there exists an $s_1 \times s_2 \times \ldots \times s_m$ BAFD with $k = s_m$, $b = N_1 \ldots N_{m-1}, \lambda_{\alpha_1\alpha_2\ldots\alpha_{m-1}0=0}, \lambda_{\alpha_1\alpha_2\ldots\alpha_{m-1}} = \mu_{\alpha_1}^1 \mu_{\alpha_2}^2 \ldots \mu_{\alpha_{m-1}}^{m-1}$, where $\alpha_i = 0$ or 1.*

Proof. Multiply the $m-1$ balanced arrays to obtain a $BA[N_1 N_2 \ldots N_{m-1},\ s_m,\ s_1 s_2 \ldots s_{m-1}, 2]$ with parameters $\lambda[(x_1, x_2, \ldots, x_{m-1}),\ (y_1,\ y_2, \ldots,\ y_{m-1})] = \mu_{\alpha_1}^1 \mu^2 \alpha_2 \ldots \mu_{\alpha_{m-1}}^{m-1}$ where $\alpha_i = 0$ or 1 according as $x = y$ or not. Identifying the symbols with the levels of F_1, $F_2, \ldots, F_{m-1}$, rows with the levels of F_m and columns with blocks, we obtain an $s_1 \times s_2 \times \ldots \times s_m$ BAFD with the specified parameters. The method used in Theorem 3.1 can usually produce efficient BAFDS if we use balanced arrays corresponding to efficient two factor BAFDS. While applying this method, the block size remains the same but the number of blocks increases very rapidly. Hence this method is used when the number of assemblies in the balanced arrays are not too large. □

Example 3.2. *Consider the product of the $OA[4, 3, 2, 2]$ in Example A.17 and the $TA[6, 3, 3, 2]$ given below*

0	1	2	0	1	2
1	2	0	2	0	1
2	0	1	1	2	0

Table 1: TA[6,3,3,2]

00	01	02	01	02	00	01	02	00	00	01	02	10	11	12	10	11	12	10	11	12	10	11	12
01	02	00	00	01	11	12	10	02	12	10	11	01	02	00	02	00	01	11	12	10	12	10	11
02	00	01	02	00	12	10	11	01	11	12	10	12	10	11	11	12	10	02	00	01	01	02	00

Table 2: BA[24,3,6,2]

which is a $BA[24, 3, 6, 2]$ with parameters $\lambda[(x_1, x_2), (y_1, y_2)] = 0$ or 1 according as $x_2 = y_2$ or not. By Theorem 3.1 this corresponds to a $2 \times 3 \times 3$ BAFD with $k = 3, b = 24, r = 4, \lambda(0, 1, 1) = \lambda(1, 1, 1) = 1$.

$$\lambda(0, 0, 1) = \lambda(0, 1, 0) = \lambda(1, 0, 0) = \lambda(1, 0, 1) = \lambda(1, 1, 0) = 0$$

$$E[0,1,0] = E[0,0,1] = 1,$$
$$E[1,0,0] = E[1,1,0] = E[1,0,1] = E[1,1,1] = \frac{2}{3}$$
$$E[0,1,1] = \frac{1}{2}.$$

Example 3.3. *The product of $BA(T)(3,2,1)$ in Table 13 and a $BAT(2,3,1)$ in Table 14 generates a $2 \times 3 \times 6$ BAFD with $r = 25$, $b = 150$, $k = 6$, $\lambda(0,1,0) = \lambda(1,0,0) = \lambda(1,1,0) = 0$, $\lambda(0,0,1) = 2$, $\lambda(0,1,1) = 4$, $\lambda(1,0,1) = 3$ and $\lambda(1,1,1) = 6$. The efficiencies are*

$$E(0,0,1) = E(0,1,0) = E(1,0,0) = 1.0$$
$$E(0,1,1) = E(1,0,1) = E(1,1,0) = \frac{4}{5} \text{ and}$$
$$E(1,1,1) = \frac{21}{25}.$$

We can also obtain an efficient $2 \times 3 \times 6$ BAFD by collapsing the first factor of the 6^2 symmetrical balanced factorial design in Example A.12 into two factors one at 2 levels and the other at 3 levels. The BAFD has parameters $r = 10$, $b = 60$, $k = 6$, $\lambda(0,0,1) = \lambda(0,1,0) = \lambda(1,0,0) = \lambda(0,1,0) = 0$ and $\lambda(0,1,1) = \lambda(1,0,1) = \lambda(1,1,1) = 2$. The efficiencies are

$$E(0,0,1) = E(0,1,0) = E(1,0,0) = E(1,1,0) = 1.0$$

and

$$E(0,1,1) = E(1,0,1) = E(1,1,1) = \frac{4}{5}$$

All the main effects are also estimated with full efficiency like in Example 3.3 but we only need 10 replications in this design.

Example 3.4. *The product of the $BA(T)[4,3,2]$ in Example A.15 and the $BA(T)[3,4,2]$ in Example A.16 generates a $3 \times 4 \times 12$ BAFD with $r = 484$, $b = 5808$, $k = 12$, $\lambda(0,1,0) = \lambda(1,0,0) = \lambda(1,1,0) = 0$, $\lambda(0,0,1) = 24$, $\lambda(0,1,1) = 36$, $\lambda(1,0,1) = 32$ and $\lambda(1,1,1) = 48$. The efficiencies are*

$$E[0,0,1] = E[0,1,0] = E[1,0,0] = 1.0$$
$$E[0,1,1] = E[1,0,1] = E[1,1,0] = \frac{10}{11} \text{ and}$$
$$E[1,1,1] = \frac{111}{121}$$

The second method of constructing multifactor BAFD's we shall discuss was suggested by [45] and employed by [21, 24, 30].The general form with exact conditions for validity was proved by [36]. This method replaces different levels of a factor in one design by distinct sets of treatment combinations forming the blocks of another design. Assume that there exists a BAFD with m factors $F_1, F_2, \ldots, F_m$ at

$s_1, s_2, \ldots, s_m$ levels respectively,each of the $v * (= s_1 s_2 \ldots s_m)$ treatments replicated $r*$ times in b^* blocks of k^* plots each,with the incidence matrix.

$$N^* = [A_1^* | A_2^* | \ldots | A_b^* *] \tag{5}$$

Further assume that $b^* = pq$, and the pq blocks can be divided into p groups of q blocks each, such that the design consisting of p blocks formed by adding together all the blocks of a group is a BAFD.The incidence matrix is

$$N_{pq}^* = \left[\sum_{j=1}^{q} A^*{}_j | \sum_{j=1}^{q} A^*{}_{j+q} | \ldots | \sum_{j=1}^{q} A^*{}_{pq-q+j} \right] \tag{6}$$

for a resolvable design N^*, the corresponding $N^*{}_{pq}$ exists with $p = r^*$. The following theorem was proven by [36].

Theorem 3.5. *Let there be a BAFD with the incidence matrix N in $n+1$ factors $F_0, F_{m+1}, \ldots, F_{m+n}$ at $q, s_{m+1}, \ldots, s_{m+n}$ levels respectively in b blocks of k plots each.Also let there be two BAFDs with incidence matrices N^* and $N^*{}_{pq}$ as given by equations* (5) *and* (6) *respectively.If the level $j-1$ of the factor F_0 is replaced by the block $A_{iq+j} (j = 1, 2, \ldots, q)$ in each of the treatments of N ,then the design obtained by adjoining the p designs so formed (for $i = 0, 1, 2, \ldots, p-1$) is a BAFD in $m+n$ factors in bp blocks of kk^* plots each.*

This method generates an $m+n$ factor BAFD from an $n+1$ factor BAFD and an m factor BAFD.Thus from the two-factor BAFD's we can generate a three-factor BAFD. If the two-factor BAFD's are efficient, then three-factor BAFD is also efficient. We can therefore construct efficient multi-factor BAFD's step by step from efficient two-factor BAFD's. While applying this method, the number of blocks does not increase so quickly as in the first method, but the block size does increase. It can be seen that the Theorem A.13 is a consequence of Theorem 3.5 if we let $m = n = 1$ in Theorem 3.5.

Example 3.6. *Let N be the incidence matrix of the* 3×6 *BAFD constructed by identifying rows,columns and symbols, with the levels of the second factor, the blocks ,and the levels of the first factors respectively in the $BA(T)(2,3,1)$ given in Table 14. Let N^* be the incidence matrix of the resolvable* 3^2 *symmetrical balanced factorial design given below*

x_0	x_1	x_2	y_0	y_1	y_2
00	01	02	00	01	02
11	12	10	12	10	11
22	20	21	21	22	20

Table 3: 3^2 Symmetrical BFD

where $x_0, x_1, x_2, y_0, y_1, y_2$ represents blocks. Then by Theorem 3.5 we can construct a $3^2 \times 6$ BAFD with $r = 10$, $b = 30$, $\lambda(2,0) = 5$, $\lambda(0,1) = 2$, $\lambda(2,1) = 3$, $\lambda(1,1) = 4$, $\lambda(1,0) = 0$, $E[2,1] = \frac{9}{10}$ and all main effects and first order interactions are estimated with full efficiency. The BAFD is given below.

Blocks	1	2	3	4	5	6	7	8	9	10
Levels of F_3	Levels	of	F_1	and	F_2					
0	x_0	x_0	x_0	x_0	x_0	x_1	x_1	x_1	x_1	x_1
1	x_1	x_2	x_1	x_2	x_0	x_2	x_0	x_2	x_0	x_1
2	x_2	x_1	x_1	x_0	x_2	x_0	x_2	x_2	x_1	x_0
3	x_2	x_2	x_0	x_1	x_1	x_0	x_0	x_1	x_2	x_2
4	x_0	x_1	x_2	x_2	x_1	x_1	x_2	x_0	x_0	x_2
5	x_1	x_0	x_2	x_1	x_2	x_2	x_1	x_0	x_2	x_0

Blocks	11	12	13	14	15	16	17	18	19	20
Levels of F_3	Levels	of	F_1	and	F_2					
0	x_2	x_2	x_2	x_2	x_2	y_0	y_0	y_0	y_0	y_0
1	x_0	x_1	x_0	x_1	x_2	y_1	y_2	y_1	y_2	y_0
2	x_1	x_0	x_0	x_2	x_1	y_2	y_1	y_1	y_0	y_2
3	x_1	x_1	x_2	x_0	x_0	y_2	y_2	y_0	y_1	y_1
4	x_2	x_0	x_1	x_1	x_0	y_0	y_1	y_2	y_2	y_1
5	x_0	x_2	x_1	x_0	x_1	y_1	y_0	y_2	y_1	y_2

Blocks	21	22	23	24	25	26	27	28	29	30
Levels of F_3	Levels	of	F_1	and	F_2					
0	y_1	y_1	y_1	y_1	y_1	y_2	y_2	y_2	y_2	y_2
1	y_2	y_0	y_2	y_0	y_1	y_0	y_1	y_0	y_1	y_2
2	y_0	y_2	y_2	y_1	y_0	y_1	y_0	y_0	y_2	y_1
3	y_0	y_0	y_1	y_2	y_2	y_1	y_1	y_2	y_0	y_0
4	y_1	y_2	y_0	y_0	y_2	y_2	y_0	y_1	y_1	y_0
5	y_2	y_1	y_0	y_2	y_0	y_0	y_2	y_1	y_0	y_1

Table 4: $3^2 \times 6$ BAFD

Example 3.7. *Let N be the incidence matrix of the* 3×6 *BAFD constructed by identifying rows, columns, and symbols, with the levels of the second factor, the blocks and the levels of the first factor respectively in the* $BA(T)(2,3,1)$ *given in Table 14. Let* N^* *be the incidence matrix of the resolvable* 2×3 *BAFD given below.*

x_0	x_1	x_2	y_0	y_1	y_2
00	01	02	00	01	02
11	12	10	12	10	11

Table 5: Resolvable 2×3 BAFD

where $x_0, x_1, x_2, y_0, y_1, y_2$ represents blocks. Then by Theorem 3.5, we can construct a $2 \times 3 \times 6$ BAFD with $r = 10, k = 12, b = 30$

$$\lambda(0,0,1) = 2, \lambda(0,1,0) = 0$$

$$\lambda(0,1,1) = 4, \lambda(1,0,0) = 0$$

$$\lambda(1,0,1) = 4, \lambda(1,1,0) = 5$$

$$\lambda(1,1,1) = 3$$

and efficiencies

$$
\begin{aligned}
E[0,0,1] &= E[0,1,0] = E[1,0,0] = 1.00 \ \ and \\
E[1,1,0] &= E[1,0,1] = 1.00 \\
E[0,1,1] &= \frac{19}{20} \\
E[1,1,1] &= \frac{17}{20}
\end{aligned}
$$

The BAFD is given below

Blocks	1	2	3	4	5	6	7	8	9	10
Levels of F_3	Levels	of	F_1	and	F_2					
0	x_0	x_0	x_0	x_0	x_0	x_1	x_1	x_1	x_1	x_1
1	x_1	x_2	x_1	x_2	x_0	x_2	x_0	x_2	x_0	x_1
2	x_2	x_1	x_1	x_0	x_2	x_0	x_2	x_2	x_1	x_0
3	x_2	x_2	x_0	x_1	x_1	x_0	x_0	x_1	x_2	x_2
4	x_0	x_1	x_2	x_2	x_1	x_1	x_2	x_0	x_0	x_2
5	x_1	x_0	x_2	x_1	x_2	x_2	x_1	x_0	x_2	x_0

Blocks	11	12	13	14	15	16	17	18	19	20
Levels of F_3	Levels	of	F_1	and	F_2					
0	x_2	x_2	x_2	x_2	x_2	y_0	y_0	y_0	y_0	y_0
1	x_0	x_1	x_0	x_1	x_2	y_1	y_2	y_1	y_2	y_0
2	x_1	x_0	x_0	x_2	x_1	y_2	y_1	y_1	y_0	y_2
3	x_1	x_1	x_2	x_0	x_0	y_2	y_2	y_0	y_1	y_1
4	x_2	x_0	x_1	x_1	x_0	y_0	y_1	y_2	y_2	y_1
5	x_0	x_2	x_1	x_0	x_1	y_1	y_0	y_2	y_1	y_2

Blocks	21	22	23	24	25	26	27	28	29	30
Levels of F_3	Levels	of	F_1	and	F_2					
0	y_1	y_1	y_1	y_1	y_1	y_2	y_2	y_2	y_2	y_2
1	y_2	y_0	y_2	y_0	y_1	y_0	y_1	y_0	y_1	y_2
2	y_0	y_2	y_2	y_1	y_0	y_1	y_0	y_0	y_2	y_1
3	y_0	y_0	y_1	y_2	y_2	y_1	y_1	y_2	y_0	y_0
4	y_1	y_2	y_0	y_0	y_2	y_2	y_0	y_1	y_1	y_0
5	y_2	y_1	y_0	y_2	y_0	y_0	y_2	y_1	y_0	y_1

Table 6: $2 \times 3 \times 6$ BAFD

4. Examples of Multifactor BAFDS

In this section, we shall use the methods discussed in the preceding section and some known balanced factorial designs to construct examples of multifactor BAFDS. We are especially interested in BAFDS of which the main effects and lower order interactions can be estimated with high efficiencies.

Type I: If there exists $TA[s_i(s_i-1), s_m, s_i, 2]$ for $i = 1, 2, \ldots, m-1$ then by Theorem 3.1 we can construct an $s_1 \times s_2 \times \ldots s_m$ BAFD with $k = s_m$, $b = \prod_{i=1}^{m-1} s_i(s_i-1)$, $r = \prod_{i=1}^{m-1}(s_i-1)$, $\lambda(1,1,\ldots,1) = 1$ and other

$\lambda's$ being 0. By Theorem A.18 the eigenvalues of NN^T of a BAFD are given by

$$\begin{aligned} g(y_1, y_2, \ldots, y_m) &= rk - k\rho(y_1, y_2, \ldots, y_m) && (7) \\ &= rk - \left\{ r(k-1) - \textstyle\sum_{x\in\Omega} \lambda(x) \left\{ \prod_{i=1}^{m} \left[(1-y_i)s_i - 1 \right]^{x_i} \right\} \right\} && (8) \\ &= r + \sum_{x\in\Omega} \lambda(x) \left\{ \textstyle\prod_{i=1}^{m} \left[(1-y_i)s_i - 1 \right]^{x_i} \right\} && (9) \end{aligned}$$

hence

$$\begin{aligned} E[y_1, y_2, \ldots, y_m] &= 1 - \frac{1}{rk} g[y_1, y_2, \ldots, y_m] && (10) \\ &= 1 - \frac{1}{rk} \left[r + \textstyle\sum_{x\in\Omega} \lambda(x) \left\{ \prod_{i=1}^{m} \left[(1-y_i)s_i - 1 \right]^{x_i} \right\} \right] && (11) \\ &= 1 - \frac{1}{k} - \frac{1}{rk}(1) \left\{ \textstyle\prod_{i=1}^{m} \left[(1-y_i)s_i - 1 \right]^{x_i} \right\} && (12) \\ &= \frac{-1}{s_m} + 1 - \frac{\prod_{i=1}^{m} [(1-y_i)s_i - 1]^{x_i}}{s_m \prod_{i=1}^{m-1} (s_i - 1)} && (13) \\ &= 1 - \frac{1}{s_m} - \frac{\prod_{i=1}^{m} [(1-y_i)s_i - 1]^{x_i}}{s_m \prod_{i=1}^{m-1} (s_i - 1)} && (14) \end{aligned}$$

Let $y_j = 1$ and $y_i = 0$ for $i \neq j$, equation (13) becomes

$$\begin{aligned} E[0, 0, \ldots, 1_j, \ldots, 0, \ldots, 0] &= 1 - \frac{1}{s_m} - \frac{\left\{ \begin{matrix} \left\{ (1-0)s_1 - 1 \right\}^1 \left\{ (1-0)s_2 - 1 \right\}^1 \ldots \\ \left\{ (1-1)s_j - 1 \right\}^1 \ldots \left\{ (1-0)s_m - 1 \right\}^1 \end{matrix} \right\}}{s_m \prod_{i-1}^{m-1} (s_i - 1)} && (15) \\ &= 1 - \frac{1}{s_m} - \frac{\left\{ (s_1-1)(s_2-1)\ldots(-1)\ldots(s_m-1) \right\}}{s_m \prod_{i-1}^{m-1} (s_i - 1)} && (16) \\ &= 1 - \frac{1}{s_m} - \frac{(s_1-1)(s_2-1)\ldots(-1)\ldots(s_{m-1}-1)(s_m-1)}{s_m(s_1-1)(s_2-1)\ldots(s_j-1)\ldots(s_{m-1}-1)} && (17) \\ &= 1 - \frac{1}{s_m} - \frac{-(1)(s_m-1)}{s_m(s_j-1)} && (18) \\ &= 1 - \frac{1}{s_m} + \frac{s_m - 1}{s_m(s_j-1)} && (19) \\ &= 1 + \frac{s_m - s_j}{s_m(s_j-1)} && (20) \\ &= 1 - \frac{s_j - s_m}{s_m(s_j-1)} && (21) \end{aligned}$$

This is the efficiency of the main effects of the factor F_j and it is 1 when $j = m$. Hence the main effects of F_m are estimated with full efficiency. In general, let $y_{ji} = 1$ for $i = 1, \ldots, q (q \leq m)$ and other $y's$ be 0, then equation (15) is

$$E[y_1, y_2, \ldots, y_m] = E[1, 1, \ldots, 1_q, 0, 0, \ldots, 0] \qquad (22)$$

$$= 1 - \frac{1}{s_m} - \frac{\prod_{i=1}^{m}[(1-y_i)s_i - 1]^{x_i}}{s_m \prod_{i=1}^{m-1}(s_i - 1)} \tag{23}$$

$$= 1 - \frac{1}{s_m} - \frac{\begin{bmatrix} \left\{(1-1)s_1 - 1\right\}^1 \left\{(1-1)s_2 - 1\right\}^1 \\ \ldots \left\{(1-1)s_q - 1\right\}^1 \left\{(1-0)s_{q+1} - 1\right\}^1 \ldots \left\{(1-0)s_m - 1\right\}^1 \end{bmatrix}}{s_m \left\{(s_1 - 1)(s_2 - 1)\ldots(s_q - 1)(s_{q+1} - 1)\ldots(s_{m-1} - 1)\right\}} \tag{24}$$

$$= 1 - \frac{1}{s_m} - \frac{(-1)^1(-1)^1\ldots(-1)^1(s_{q+1} - 1)\ldots(s_{m-1} - 1)(s_m - 1)}{s_m(s_1 - 1)(s_2 - 1)(s_3 - 1)\ldots(s_q - 1)(s_{q+1} - 1)\ldots(s_{m-1} - 1)} \tag{25}$$

$$= 1 - \frac{1}{s_m} - \frac{(-1)^q(s_m - 1)}{s_m \prod_{i=1}^{q}(s_{ji} - 1)} \tag{26}$$

$$= 1 - \frac{1}{s_m} - \frac{(-1)^q(s_m - 1)}{s_m \prod_{i=1}^{q}(s_{ji} - 1)} \tag{27}$$

which is efficiency of the $(q-1)^{th}$ order interaction between $F_{j1}, F_{j2}, \ldots F_{jq}$.

Example 4.1. *For any given $s_i \geq 3, (i = 1, 2, 3, \ldots, m-1)$. The $TA[s_i(s_i - 1), 3, s_i, 2]'s$ always exists. Hence we can always construct an $s_1 \times s_2 \times \ldots \times s_{m-1} \times 3$ BAFD with $k = 3, b = \prod_{i=1}^{m-1} s_i(s_i - 1), r = \prod_{i=1}^{m-1}(s_i - 1), \lambda(1, 1, \ldots 1) = 1$ and all other λ's being 0.*

Example 4.2. *Using the $TA[20, 3, 5, 2]$ in Example A.14 and the $TA[6, 3, 3, 2]$, we can construct a $5 \times 3 \times 3$ BAFD using Theorem 3.1. The parameters of this BAFD are $k = 3, b = 120, r = 8, \lambda(0, 0, 1) = \lambda(0, 1, 0) = \lambda(0, 1, 1) = \lambda(1, 0, 0) = \lambda(1, 0, 1) = \lambda(1, 1, 0) = 0, \lambda(1, 1, 1) = 1.0$*

The efficiencies of this design are

$$E[0,0,1] = E[0,1,0] = 1.00$$
$$E[1,0,0] = \frac{5}{6}$$
$$E[0,1,1] = \frac{1}{2}$$
$$E[1,0,1] = E[1,1,0] = \frac{7}{12}$$
$$E[1,1,1] = \frac{17}{24}$$

Other examples of this type include $5 \times 4 \times 4$,$5 \times 5 \times 4$, $7 \times 5 \times 4$, $7 \times 5 \times 4$,$7 \times 7 \times 5$,. . . , BAFD's and so on.

A. Appendix

Definition A.1. *A $k \times b$ array with entries from a set of v symbols is called an orthogonal array of strength t. If each $t \times b$ subarray of A contains all possible v^t column vectors with the same frequency $\lambda = \frac{b}{v^t}$. It is denoted $OA(b, k, v, t; \lambda)$; the number λ is called the index of the array. The numbers b and k are known as the number of assemblies and constraints of the orthogonal array respectively.*

Example A.2.

0	1	1	1	1	0	0	0
1	0	1	1	0	1	0	0
1	1	0	1	0	0	1	0
1	1	1	0	0	0	0	1

Table 7: OA(8,4,2,3,1)

Definition A.3. *Let A be a $k \times b$ array with entries from a set of v symbols. Consider the v^t ordered t-tuples $(x_1, \ldots, x_t)$ that can be formed from a t-rowed subarray of A, and let there be associated a non-negative integer $\lambda(x_1, \ldots, x_t)$ that is invariant under permutations of $x_1, \ldots, x_t$. If for any t-rowed subarray of A the v^t ordered t-tuples $(x_1, \ldots, x_t)$, each occur $\lambda(x_1, \ldots, x_t)$ times as a column, then A is said to ba a balanced array of strength t. It is denoted by $BA(b, k, v, t)$ and the numbers $\lambda(x_1, \ldots, x_t)$ are called the index parameters of the array.*

Example A.4.

0	1	0	1	0	1	0	1	0	1
1	1	1	0	1	1	0	0	0	0
0	0	1	1	1	0	0	0	1	1
1	1	0	0	0	0	1	0	1	1
0	0	0	0	1	1	1	1	1	0

Table 8: BA(10,5,2,2)

$\lambda(0,0) = \lambda(1,1) = 2$ *and* $\lambda(0,1) = \lambda(1,0) = 3.$

In particular we are interested in the $BA[(ks-1)s\lambda, ks, s, 2]$ with parameters $\lambda(x, y) = (k-1)\lambda$ or $(k\lambda)$ according as $x = y$ or Not. For brevity we shall call it the balanced array of type T with index λ and denote it by $BA[T][k, s, \lambda]$.

Definition A.5. *A transitive array $TA(b, k, v, t; \lambda)$ is a $k \times b$ array of v symbols such that for any choice of t rows, the $\frac{v!}{(v-t)!}$ ordered t-tuples of distinct symbols each occur λ times as a column.*

Example A.6.

0	1	2	3	0	1	2	3	0	1	2	3
1	0	3	2	2	3	0	1	3	2	1	0
2	3	0	1	3	2	1	0	1	0	3	2
3	2	1	0	1	0	3	2	2	3	0	1

Table 9: TA(12,4,4,2;1)

Example A.7.

0	1	0	0	0	0	0	1	1	1	1	1
0	1	0	0	1	1	1	0	0	0	1	1
0	1	1	0	0	1	1	1	1	0	0	0
0	1	0	1	1	0	1	0	1	1	0	0
0	1	1	1	1	0	0	1	0	0	0	1
0	1	1	1	0	1	0	0	0	1	1	0

Table 10: A Partly resolvable OA(12,6,2,2)

Definition A.8. *Suppose $(\mathcal{F},\mathcal{A})$ is a (v,k,λ)-BIBD, a parallel class in $(\mathcal{F},\mathcal{A})$ is a subset of disjoint blocks from $\mathcal{A}$ whose union is $\mathcal{F}$. A partition of $\mathcal{A}$ into r parallel classes is called a **resolution**; and $(\mathcal{F},\mathcal{A})$ is said to be a resolvable BIBD if $\mathcal{A}$ has at least one resolution. We say that $\mathcal{F}$ is a finite set of points called **treatments**, where $\mathcal{F}=\{0,1,2,\cdots,v-1\}$.*

X_0	X_1	Y_0	Y_1	Z_0	Z_1
0	2	0	1	0	1
1	3	2	3	3	2

Table 11: Table of BIBD[4,6,2]

The explicit formulae for calculating interaction efficiencies in an extended group divisible (EGD) design is given by

$$\rho(y)=k^{-1}\left[r(k-1)I-\sum_{x\in\Omega}\lambda(x)\left\{\prod_{i=1}^{m}((1-y_i)s_i-1)^{x_i}\right\}\right] \tag{28}$$

Theorem A.9. *For a connected EGD design, with parameters as stated above, the efficiency with respect to the interaction F^y is given by*

$$E(y)=k^{-1}\left[(k-1)-r^{-1}\sum_{x\in\Omega}\lambda(x)\{\prod_{i=1}^{m}((1-y_i)s_i-1)^{x_i}\}\right]$$

Theorem A.10. *The existence of $BA[N_i,k_i,s_i,t]$ for $i=1,2,...,m$ implies the existence of a $BA[N,k,s,t]$ where $N=N_1N_2...N_m$, $s=s_1s_2...s_m$ and $k=min(k_1,k_2,...,k_m)$. If the symbols of the $BA[N,k,s,t]$ are denoted by ordered k tuples then the parameters are*

$$\lambda((a_{11},a_{21},...,a_{m1})(a_{12},a_{22},...,a_{m2})...(a_{1t},a_{2t},...,a_{mt}))=\lambda(a_{11},a_{12},...,a_{1t})\lambda(a_{21},a_{22},...,a_{2t})...\lambda(a_{m1},a_{m2},...,a_{mt})$$

Corollary A.11. *In an s^2 symmetrical Factorial Design FD with block size s and if all the main effects are estimated with full efficiency then the FD has parameters $\lambda_1=0$ and $\lambda_2\neq 0$. This design is equivalent to a $TA[\lambda_2 s(s-1),s,s,2]$.*

Example A.12. *A 3×6 BAFD with $b=30,k=3,r=5$ $\lambda_{01}=\lambda_{10}=0$ and $\lambda_{11}=1$ can be constructed from a $TA[30,3,6,2]$. The efficiencies are $E[1,0]=1.0$, $E[0,1]=\frac{4}{5}$ and $E[1,1]=\frac{3}{5}$.*

Theorem A.13. *If there exists a resolvable BIBD with qs treatments and block size q, then there exists a $ps \times qs$ BAFD with block size pqs such that all main effects are estimated with full efficiency.*

Example A.14. *For $s = 5$, we can construct a $TA[20, 3, 5, 2; 1]$ by first constructing an $OA[25, 3, 5, 2]$ and then deleting five assemblies with elements that are not distinct.*

1	2	3	4	0	2	3	4	0	1	3	4	0	1	2	4	0	1	2	3
2	3	4	0	1	4	0	1	2	3	1	2	3	4	0	3	4	0	1	2
3	4	0	1	2	1	2	3	4	0	4	0	1	2	3	2	3	4	0	1

Table 12: TA[20,3,5,2;1]

Example A.15. *Let $M = [0, 1, 2]$. Among the Differences of corresponding elements of any two rows of the following array,* 0 *occurs* 6 *times whereas* 1 *and* 2 *each occur* 8 *times.*

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	2	2	0	1	0	0	2	1	2	1	2	1	1	0	2	0	0	1	2	1	2
1	1	2	2	0	1	0	0	2	1	2	2	2	1	1	0	2	0	0	1	2	1
2	1	1	2	2	0	1	0	0	2	1	1	2	2	1	1	0	2	0	0	1	2
1	2	1	1	2	2	0	1	0	0	2	2	1	2	2	1	1	0	2	0	0	1
2	1	2	1	1	2	2	0	1	0	0	1	2	1	2	2	1	1	0	2	0	0
0	2	1	2	1	1	2	2	0	1	0	0	1	2	1	2	2	1	1	0	2	0
0	0	2	1	2	1	1	2	2	0	1	0	0	1	2	1	2	2	1	1	0	2
1	0	0	2	1	2	1	1	2	2	0	2	0	0	1	2	1	2	2	1	1	0
0	1	0	0	2	1	2	1	1	2	2	0	2	0	0	1	2	1	2	2	1	1
2	0	1	0	0	2	1	2	1	1	2	1	0	2	0	0	1	2	1	2	2	1
2	2	0	1	0	0	2	1	2	1	1	1	1	0	2	0	0	1	2	1	2	2

hence we can construct a $BA[66, 12, 3, 2]$ with parameters $\lambda(x, y) = 6$ or 8 according as $x = y$ or not. i.e $BA(T)[4, 3, 2]$.

Example A.16. *Let $M = [0, 1, 2, 3]$. Among the differences of the corresponding elements of any two rows of the following array,*0 *occurs* 4 *times, wheres* $1, 2$ *and* 3 *occur* 6 *times each.*

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	1	2	0	2	1	1	3	2	3	1	0	3	2	0	2	3	3	1	2	1
3	3	0	1	2	0	2	1	1	3	2	1	1	0	3	2	0	2	3	3	1	2
2	3	3	0	1	2	0	2	1	1	3	2	1	1	0	3	2	0	2	3	3	1
3	2	3	3	0	1	2	0	2	1	1	1	2	1	1	0	3	2	0	2	3	3
1	3	2	3	3	0	1	2	0	2	1	3	1	2	1	1	0	3	2	0	2	3
1	1	3	2	3	3	0	1	2	0	2	3	3	1	2	1	1	0	3	2	0	2
2	1	1	3	2	3	3	0	1	2	0	2	3	3	1	2	1	1	0	3	2	0
0	2	1	1	3	2	3	3	0	1	2	0	2	3	3	1	2	1	1	0	3	2
2	0	2	1	1	3	2	3	3	0	1	2	0	2	3	3	1	2	1	1	0	3
1	2	0	2	1	1	3	2	3	3	0	3	2	0	2	3	3	1	2	1	1	0
0	1	2	0	2	1	1	3	2	3	3	0	3	2	0	2	3	3	1	2	1	1

hence we can construct a $BA[88, 12, 4, 2]$ with parameters $\lambda(x, y) = 4$ or 6 according as $x = y$ or not, i.e $BA(T)[3, 4, 2]$.

Example A.17. *A* 2×3 *BAFD with* $b = 4, k = 3, r = 2, \lambda_{10} = 0$ *and* $\lambda_{01} = \lambda_{11} = 1$ *can be constructed from the* $OA[4,3,2,2]$.

Blocks	1	2	3	4
Levels of F_2	levels	of	F_1	
0	0	0	1	1
1	0	1	0	1
2	0	1	1	0

In this design, the efficiencies are: $E[0,1] = 1$ *and* $E[1,0] = E[1,1] = \frac{2}{3}$.

0	0	0	0	0	1	1	1	1	1
0	0	1	1	1	0	0	0	1	1
1	0	0	1	1	1	1	0	0	0
0	1	1	0	1	0	1	1	0	0
1	1	1	0	0	1	0	0	0	1
1	1	0	1	0	0	0	1	1	0

Table 13: Table of BA(T)[3,2,1]

0	0	0	0	0	1	1	1	1	1	2	2	2	2	2
1	2	1	2	0	2	0	2	0	1	0	1	0	1	2
2	1	1	0	2	0	2	2	1	0	1	0	0	2	1
2	2	0	1	1	0	0	1	2	2	1	1	2	0	0
0	1	2	2	1	1	2	0	0	2	2	0	1	1	0
1	0	2	1	2	2	1	0	2	0	0	2	1	0	1

Table 14: Table $BA(T)[2,3,1] = BA[15,6,3,2]$

Theorem A.18. *The eigenvalues of* NN' *of a BAFD are* $g(y_1, y_2, \ldots, y_m)$*'s with corresponding eigenvectors given by the columns of* $p^{y'}$, *where* $y = (y_1, y_2, \ldots, y_m) \in \Omega$.

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