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Conformal Ricci Solitons with Respect to Generalazed Symmetric Metric Connection on Riemannian and LP-Sasakian Manifolds

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Abstract

In this paper, I have studied conformal Ricci solitons with respect to generalized symmetric metric connection and obtain a necessary condition so that conformal Ricci soliton equation remains invariant with respect to generalized symmetric metric connections on Riemannian and LP-Sasakian manifolds. If the vector field ξ of a LP-Sasakian manifolds is torqued vector field, then some important results are obtained.

Keywords: Conformal Ricci solitons; Generalized symmetric connection; Torqued vector fields.
2020 Mathematics Subject Classification: 53C15, 53C25.

1. Introduction

In 1982, Hamilton [4] introduced the notion of Ricci flow to find a canonical metric on a smooth manifold. Then Ricci flow has become a powerful tool for the study of Riemannian manifolds, especially for those manifolds with positive curvature. Perelman ([9], [10]) used Ricci flow and its surgery to prove Poincare conjecture. The Ricci flow is an evolution equation for metrics on a Riemannian manifold defined as follows [4]:

$$\frac{\partial}{\partial t}g_{ij}(t) = -2R_{ij}.$$

The conformal Ricci flow equations are analogous to the Navier-Stokes equations of fluid mechanics. In 2015, Basu and Bhattacharyya [1] introduced the notion of conformal Ricci soliton equation as

$$\pounds_V g + 2S + \left[2\lambda - (p + \frac{2}{n})\right] g = 0, \tag{1}$$

where λ is constant. The equation is the generalization of the Ricci soliton equation and it also satisfies the conformal Ricci flow equation.

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The concept of Ricci almost soliton was first introduced by Pigola et al. [11]. Later studied by [5]. Sharma has also done excellent work in Ricci almost soliton [12]. A Riemannian manifold (M^n, g) admits an Ricci almost soliton, if there exist a complete vector field X and a smooth soliton function $\lambda : M^n \to \mathbb{R}$ satisfying,

$$R_{ij} + \frac{1}{2}\left(X_{ij} + X_{ji}\right) = \lambda g_{ij}, \tag{2}$$

where R_{ij} and $X_{ij} + X_{ji}$ stand for the Ricci tensor and the Lie derivative $(\pounds_X g)$ in local coordinates respectively. A gradient Ricci soliton on a Riemannian manifold (M^n, g) is defined by [1]

$$S + \nabla\nabla h = \nu\, g,$$

for some constant ν and for a smooth function h on M, h is called a potential function of the Ricci soliton and ∇ is the Levi-Civita connection on M. In particular a gradient shrinking Ricci soliton satisfies the equation,

$$S + \nabla\nabla h - \frac{1}{2\tau}\, g = 0,$$

where $\tau = T - t$ and T is the maximal time of the soliton. Again if the vector field of conformal Ricci soliton is the gradient of a function h then we call it as a conformal gradient shrinking Ricci soliton [1] and such equation is

$$S + \nabla\nabla h = \left(\frac{1}{2\tau} - \frac{2}{n} - p\right) g,$$

where h is the Ricci potential function.

Many authors have studied Ricci- solitons on LP-Sasakian manifolds. Here in this paper, I have studied conformal Ricci-soliton on LP-Sasakian manifolds. More over I have obtained some results when the connection is generalized symmetric metric connection instead of Riemannian connection. After introduction, section 2 is concerned with some preliminaries. In section 3, conformal Ricci solitons with respect to generalized symmetric metric connection on Riemannian manifolds and on LP-Sasakian manifolds are studied. Section 4 concerned with the study of application of torqued vector field to conformal Ricci soliton with respect to generalized symmetric metric connection on LP-Sasakian manifolds.

2. Preliminaries

Let R, S and r be the curvature tensor, Ricci tensor and scalar curvature of M respectively. In 1924, the semi-symmetric linear connection have been introduced by Friedmann and Schouten [6].

Definition 2.1. *A linear connection ∇ is said to be a semi-symmetric connection if the torsion tensor T is of the form $T(X, Y) = \eta(Y)X - \eta(X)Y$, where η is a 1-form and $X, Y \in \chi(M)$.*

Again in 1975, Golab [7] introduced quarter-symmetric connection.

Definition 2.2 ([7]). *A linear connection ∇ is said to be a quarter-symmetric connection if the torsion tensor T satisfies $T(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y$, where η is a 1-form, ϕ is a (1, 1) tensor field and $X, Y \in \chi(M)$.*

Definition 2.3. *A semi-symmetric (or quarter-symmetric) connection ∇ satisfying $\nabla g = 0$, is called a semi-symmetric metric connection (or quarter-symmetric metric connection). Otherwise it is called a semi-symmetric non-metric connection (or quarter-symmetric non-metric connection).*

Definition 2.4 ([2]). *A linear connection ∇ is said to be a generalized symmetric connection if the torsion tensor T satisfies $T(X,Y) = \alpha\{\eta(Y)X - \eta(X)Y\} + \beta\{\eta(Y)\phi X - \eta(X)\phi Y\}$, where η is a 1-form, ϕ is a (1, 1) tensor field, $X, Y \in \chi(M)$ and α and β are smooth functions.*

$$\tilde{\nabla}_X Y = \nabla_X Y + \alpha\{\eta(Y)X - g(X,Y)\xi\} + \beta\{\eta(Y)\phi X - g(\phi Y, X)\xi\}. \tag{3}$$

We mention that if $\alpha = 1$ and $\beta = 0$, then the generalized symmetric connection reduces to semi-symmetric connection. Again if $\alpha = 0$ and $\beta = 1$, then the generalized symmetric connection turns into quarter-symmetric connection. If $\alpha = 0$ and $\beta = 0$, then the connection reduces to Levi-Civita connection. We write

$$\Phi(X,Y) = g(\phi X, Y). \tag{4}$$

In an LP-Sasakian manifold $M^n(\phi, \xi, \eta, g)$, the following results hold:

$$rank\phi = n - 1 \tag{5}$$

$$(\nabla_X \eta)(Y) = \Phi(X,Y), \quad \Phi(X,\xi) = 0 \tag{6}$$

$$g(R(X,Y)Z,\xi) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y) \tag{7}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y \tag{8}$$

$$S(X,\xi) = (n-1)\eta(X) \tag{9}$$

$$S(\phi X, \phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y) \tag{10}$$

for any vector fields X, $Y \in \chi(M)$.

Definition 2.5. *An LP-Sasakian manifold M is said to be generalized η-Einstein if the Ricci sensor S satisfies $S(X,Y) = ag(X,Y) + b\Phi(X,Y) + c\eta(X)\eta(Y)$ for all $X, Y \in \chi(M)$, where a, b and c are scalar functions on M.*

If $b = c = 0$, then it reduces to Einstein manifold and if $b = 0$, then it becomes η-Einstein manifold. A pseudo Riemannian manifold (M, g) is called a quasi-Einstein manifold if

$$S = pg + q\alpha \otimes \alpha, \tag{11}$$

where p, q are functions and α is a 1-form. A pseudo Riemannian manifold (M, g) is called almost quasi-Einstein manifold if

$$S = pg + q(\beta \otimes \mu + \mu \otimes \beta), \tag{12}$$

where p, q are functions and β, μ are 1-forms. In an LP-Sasakian manifold $M^n(\phi, \xi, \eta, g)$ admitting generalized symmetric connection, the following results hold [2]:

$$\overline{\nabla}_X \xi = (1-\beta)\phi X - \alpha X - \alpha\eta(X)\xi, \tag{13}$$

$$\begin{aligned}\overline{R}(X,Y)\xi &= (1-\beta+\beta^2)\{\eta(Y)X - \eta(X)Y\} + \alpha(1-\beta)\{\eta(X)\phi Y - \eta(Y)\phi X\} \\ &= k_1\{\eta(Y)X - \eta(X)Y\} + k_2\{\eta(X)\phi Y - \eta(Y)\phi X\},\end{aligned} \tag{14}$$

$$\overline{R}(\xi, Y)\xi = k_1\{\eta(Y)\xi + Y\} - k_2\phi Y, \tag{15}$$

where $k_1 = (1-\beta+\beta^2)$, $k_2 = \alpha(1-\beta)$,

$$\begin{aligned}\overline{S}(X,Y) &= S(X,Y) + \{-\alpha\beta + (n-2)(\alpha\beta - \alpha) + (\beta^2 - 2\beta)\text{trace } \Phi\}\Phi(X,Y) \\ &+ \{-2\alpha^2 + \beta - \beta^2 + n\alpha^2 + (\alpha\beta - \alpha)\text{trace } \Phi\}g(X,Y) + \{-2\alpha^2 + n(\alpha^2 + \beta - \beta^2)\}\eta(X)\eta(Y)\end{aligned} \tag{16}$$

Or,

$$\overline{S}(X,Y) = S(X,Y) + A\Phi(X,Y) + Bg(X,Y) + C\eta(X)\eta(Y), \tag{17}$$

where, $A = -\alpha\beta + (n-2)(\alpha\beta - \alpha) + (\beta^2 - 2\beta)\text{trace } \Phi$, $B = -2\alpha^2 + \beta - \beta^2 + n\alpha^2 + (\alpha\beta - \alpha)\text{trace } \Phi$ and $C = -2\alpha^2 + n(\alpha^2 + \beta - \beta^2)$,

$$\overline{S}(X,\xi) = S(X,\xi) + Bg(X,\xi) - C\eta(X). \tag{18}$$

Or,

$$\overline{S}(X,\xi) = \{n - 1 + B - C\}\eta(X), \tag{19}$$

$$\bar{r} = r + A\text{trace}\Phi + Bn - C, \tag{20}$$

$$\overline{R}(\xi, V)W = \{-\alpha\overline{\phi}(V,W) + (1-\beta)g(V,W) - \beta^2\eta(V)\eta(W)\}\xi - k_1\eta(W)V + k_2\eta(W)\phi V \tag{21}$$

3. Conformal Ricci Soliton with Respect to Generalized Symmetric Metric Connection

Conformal Ricci soliton equation with respect to generalized symmetric metric connection is

$$(\overline{\pounds}_V g)(X,Y) + 2\overline{S}(X,Y) + \left[2\lambda - (p + \frac{2}{n})\right] g = 0 \tag{22}$$

where, $\overline{\pounds}_V$ is the Lie derivative along the vector field V on M with respect to the generalized symmetric metric connection. We have

$$\begin{aligned}(\overline{\pounds}_V g)(X,Y) &= g(\overline{\nabla}_X V, Y) + g(Y, \overline{\nabla}_X V) \\ &= g(\nabla_X V, Y) + \alpha\{\eta(V)g(X,Y) - \eta(X)g(Y,V)\} + \beta\{\eta(V)g(\phi X, Y) - g(\phi X, V)\eta(Y)\} \\ &+ g(\nabla_Y V, X) + \alpha\{\eta(V)g(X,Y) - \eta(Y)g(X,V)\} + \beta\{\eta(V)g(\phi Y, X) - g(\phi Y, V)\eta(X)\}\end{aligned} \tag{23}$$

Again we have

$$\overline{S}(X,Y) = S(X,Y) + A\Phi(X,Y) + Bg(X,Y) + C\eta(X)\eta(Y), \tag{24}$$

Therefore conformal Ricci soliton equation (22) becomes

$$\begin{aligned}\pounds_V g(X,Y) &+ 2S(X,Y) + \left[2\lambda - (p + \frac{2}{n})\right] g(X,Y) + \alpha\{2\eta(V)g(X,Y) - \eta(X)g(Y,V) - \eta(Y)g(X,V)\} \\ &+ \beta\{\eta(V)g(\phi X, Y) - g(\phi X, V)\eta(Y) + \eta(V)g(X, \phi Y) - g(\phi Y, V)\eta(X)\} \\ &+ A\Phi(X,Y) + Bg(X,Y) + C\eta(X)\eta(Y) = 0\end{aligned} \tag{25}$$

Thus we have the following

Theorem 3.1. *Conformal Ricci soliton equation remains invariant with respect to generalized symmetric metric connection is invariant provided*

$$\begin{aligned}&\alpha\{2\eta(V)g(X,Y) - \eta(X)g(Y,V) - \eta(Y)g(X,V)\} + \beta\{\eta(V)g(\phi X, Y) - g(\phi X, V)\eta(Y) \\ &+ \eta(V)g(X, \phi Y) - g(\phi Y, V)\eta(X)\} + A\Phi(X,Y) + Bg(X,Y) + C\eta(X)\eta(Y) = 0\end{aligned} \tag{26}$$

Corollary 3.2. *Let (g, ξ, λ) be Conformal Ricci soliton on a LP-Sasakian manifold $M^n(\phi, \xi, \eta, g)$. Then the Ricci soliton equation remains invariant with respect to generalized symmetric metric connection is invariant provided,*

$$-2\alpha\{g(X,Y) - 2\eta(X)\eta(Y)\} - 2\beta\Phi(X,Y) + A\Phi(X,Y) + Bg(X,Y) + C\eta(X)\eta(Y) = 0 \tag{27}$$

If the connection is quarter-symmetric then, $\alpha = 0$, $\beta = 1$, $A = -trace\ \Phi$, $B = 0$, $C = 0$ and we get from the above corollary $trace\ \Phi = -2$. Thus we have

Corollary 3.3. *Let (g, ξ, λ) be Conformal Ricci soliton on a LP-Sasakian manifold $M^n(\phi, \xi, \eta, g)$. Then the Ricci soliton equation remains invariant with respect to quarter-symmetric metric connection provided, trace $\Phi = -2$.*

If the connection is semi-symmetric then, $\alpha = 1$, $\beta = 0$, $A = -(n-2)$, $B = -(n-2) - trace\ \Phi$, $C = -(n-2)$ and we get from the above Corollary 3.2, $trace\ \Phi = 0$. Thus we have

Corollary 3.4. *Let (g, ξ, λ) be Conformal Ricci soliton on a LP-Sasakian manifold $M^n(\phi, \xi, \eta, g)$. Then the Ricci soliton equation remains invariant with respect to semi- symmetric metric connection provided trace $\Phi = 0$.*

Again we recall that

$$\begin{aligned}(\bar{\pounds}_{\xi}g)(X,Y) &= g(\overline{\nabla}_X\xi,Y)+g(X,\overline{\nabla}_Y\xi) \\ &= g((1-\beta)\phi X-\alpha X-\alpha\eta(X)\xi,Y)+g(X,(1-\beta)\phi Y-\alpha Y-\alpha\eta(Y)\xi) \\ &= 2(1-\beta)\Phi(X,Y)-2\alpha g(X,Y)-2\alpha\eta(X)\eta(Y))\end{aligned} \tag{28}$$

Putting this in (18) and using (20) we get

$$S(X,Y)=-[2(1-\beta)+A]\Phi(X,Y)+\left[\beta-2\alpha+2\lambda-\left(p+\frac{2}{n}\right)\right]g(X,Y)+[C-2\alpha]\eta(X)\eta(Y) \tag{29}$$

Thus we can state that

Theorem 3.5. *Let (g,ξ,λ) be a conformal Ricci soliton on a LP-Sasakian manifold with respect to generalized symmetric metric connection. Then the manifold is generalized η-Einstein.*

If the connection is quarter symmetric, then $\alpha=0$, $\beta=1$, $A=-trace\ \Phi$. Then we have

Corollary 3.6. *If (g,ξ,λ) be a conformal Ricci soliton on a LP-Sasakian manifold with respect to quarter symmetric metric connection, then the manifold is η- Einstein provided, trace $\Phi=0$.*

4. Application of Torqued Vector Field to Conformal Ricci Soliton

A nowhere vanishing vector field τ on a Riemannian or pseudo Riemannian manifold M is called torse-forming if

$$\nabla_X\tau=fX+\mu(X)\tau, \tag{30}$$

where f is a function, μ is a 1-form. The vector field τ is called concircular ([3], [8]) if the 1-form μ vanishes identically. The vector field τ is called concurrent [13] if the 1-form μ vanishes identically and the function $f=1$. The vector field τ is called recurrent if the function $f=0$. Finally if $f=\mu=0$, then the vector field τ is called a parallel vector field. The nowhere zero vector field τ is called a torqued vector field [3] if it satisfies

$$\nabla_X\tau=fX+\mu(X)\tau,\quad \mu(\tau)=0. \tag{31}$$

where the function f is called the torqued function and 1-form μ is called the torqued form of τ.
In this section, we are going to discuss about the application of torqued vector field, i.e. the potential field ξ is a torqued vector field τ for conformal Ricci soliton. We suppose that the conformal Ricci soliton (g,ξ,λ) where the potential field ξ is a torqued vector field. From the definition of Lie-derivative we have

$$\begin{aligned}(\pounds_{\xi}g)(X,Y) &= g(\nabla_X\xi,Y)+g(X,\nabla_Y\xi) \\ &= \mu(X)g(\xi,Y)+\mu(Y)g(\xi,X)+2fg(X,Y).\end{aligned} \tag{32}$$

for any vector field X, Y tangent to M. In view of (1) and (32), we get

$$S(X,Y) = \frac{1}{2}\left[2\lambda - \left(p + \frac{2}{n}\right) - 2fg(X,Y)\right] - \frac{1}{2}[\mu(X)\eta(Y) + \mu(Y)\eta(X)] \tag{33}$$

Thus we can state that

Theorem 4.1. *Let (g, ξ, λ) be a conformal Ricci soliton on a LP-Sasakian manifold $M^n(\phi, \xi, \eta, g)$. If the potential field ξ is a torqued vector field then $M^n(\phi, \xi, \eta, g)$ is an almost quasi-Einstein manifold.*

Putting $X = Y = e_i$ in (33), where $\{e_i\}$ is orthonormal basis of the tangent space TM and summing over i we get

$$r = \frac{n}{2}\left[2\lambda - \left(p + \frac{2}{n} - 2f\right)\right] - 1.$$

It is known that for conformal Ricci soliton $r = -1$, thus we get $\lambda = [\frac{p}{2} + f]$. So we assert the following:

Theorem 4.2. *Let (g, ξ, λ) be a conformal Ricci soliton on an n-dimensional LP-Sasakian manifold $M^n(\phi, \xi, \eta, g)$. If the potential field ξ is a torqued vector field then, conformal Ricci soliton is expanding, steady or shirnking according as $p > -2f$, $p = -2f$ or $p < -2f$ respectively.*

Corollary 4.3. *Let (g, ξ, λ) be a conformal Ricci soliton on an n-dimensional LP-Sasakian manifold $M^n(\phi, \xi, \eta, g)$. If the torqued field ξ is concurrent vector field then such soliton is expanding, steady or shirnking if and only if the conformal pressure $p > 0, = 0, or < 0$ respectively.*

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