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Automorphism Group and Distinguishing Number of Some Shadow Graphs and Some Split Graphs

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Abstract

The proposed study asserts the automorphism group and distinguishing number of some shadow graphs along with some split graphs. Automorphism group of path graph P_n, cycle graph C_n and star graph $K_{1,n}$ are well-known groups and also their distinguishing number are well-known. It is full of zest to know what would be the automorphism group of shadow graphs and split graphs of that graphs whose automorphism group are known. Also, it would be interesting to determine the distinguishing number of shadow graphs and split graphs of that graphs whose distinguishing numbers are known. In this paper, shadow graph as well as split graph of path graph P_n, cycle graph C_n and star graph $K_{1,n}$ have been taken into the account to investigate their the automorphism groups and distinguishing numbers. The results are illustrated with the help of examples and the applicability of the proposed theory, related to some shadow graphs and some split graphs, is elaborated successfully.

Keywords: Automorphism of graph; Distinguishing number, Shadow graph; Split graph.
2020 Mathematics Subject Classification: 05C78, 05C92.

1. Introduction

The automorphism group of path graph P_n, cycle graph C_n and star graph $K_{1,n}$ has been characterised earlier [1]. It is widely known that for a given finite group X, there exists a finite graph G such that the automorphism group of a graph G isomorphic to X, where the automorphism group of a graph G is the set of all automorphisms of graph G form a group under the composition. Symbolically, it is written as $Aut(G)$. Further, to explore the automorphism group of larger graphs, we consider a finite, simple and undirected graph G with the vetrex set $V(G)$ and the edge set $E(G)$.
The bipartite Kneser graph $H(n,k)$, is elaborated with an algebraic properties. It is explored that the automorphism group of the above mentioned graph exist for all n, k, where $2k < n$. It is evident from

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the result that $Aut\left(H(n,k)\right) \cong S_n \times \mathbb{Z}_2$. A new approach based on Johnson graph used to elaborate the automorphism group of this graph [2]. In continuation of finding automorphism groups, a different method is introduced to explore connected bipartite irreducible graphs. The automorphism groups of some classes of connected bipartite irreducible graphs along with a class based on Grassmann graphs are evaluated. It is also elaborated that Johnson graph is a stable graph [3]. Recently, a study based on Andrasfi graph $And(k)$ is carried out and shown that $Aut(And(k))$ is isomorphic to the dihedral group D_{2n} [4]. The Andrasfi graph $And(k)$ was introduced in 1977 firstly [5]. Mirafzal proved that $Aut(And(k))$ is isomorphic to the dihedral group D_{2n}. Further some work on determining automorphism groups of some graphs available in the literature [6–11]. The distinguishing number was introduced by Albertson and Collins firstly [12]. A graph G with no nontrivial automorphisms is $1-$distinguishing. For a graph G, $D(G) = |V(G)|$ if and only if $G = K_n$, where $|V(G)|$ is the number of vertices of G. $D\left(P_n\right) = 2$ for $n \geq 3$; $D\left(C_n\right) = 3$ for $n = 3,4,5$; $D\left(C_n\right) = 2$ for $n \geq 6$ and $D\left(K_{n,n}\right) = n+1$ for $n \geq 4$ [13].

Some work on determining distinguishing numbers of some graphs available in the literature [14–17,19,20]. The present work, emphasize that the graphs have been obtained with the help of some graph operations like shadow graph and split graph on that graphs whose automorphism groups are known. The automorphism groups and distinguishing number of the graphs are investigated which are obtained with the help of graph operations. The two graph operations, shadow graph and split graph, are incorporated on path P_n, cycle C_n and star graph $K_{1,n}$. Further, we have determined the automorphism group and distinguishing number of shadow graph of path P_n; automorphism group and distinguishing number of shadow graph of cycle C_n; and automorphism group and distinguishing number of shadow graph of star graph $K_{1,n}$. Further, in view of split graph operation, the automorphism group and distinguishing number of split graph of path P_n; automorphism group and distinguishing number of split graph of cycle C_n; and automorphism group and distinguishing number of split graph of star graph $K_{1,n}$ have been determined.

Definition 1.1. *Let G be graph with vertex set $V(G)$ and edge set $E(G)$. A mapping $f : V(G) \to \{1,2,\ldots,d\}$ is said to be $d-$distinguishing, if there is no non-trivial automorphism of graph G preserves all of the vertex labels. That is, there is no non-trivial automorphism ϕ of graph G such that $f(v) = f(\phi(v))$, for every $v \in V(G)$. The distinguishing number of a graph G is the minimum number d such that G admits $d-$distinguishing [13].*

Definition 1.2 ([18])**.** *The shadow graph $D_2(G)$ of a connected graph G obtained by taking two copies of grapg G say G' and G'' join each vertex v' in G' to the neighbors of the corresponding vertex v'' in G''.*

Definition 1.3 ([18])**.** *The split graph of a graph G is obtained by adding a new vertex v' to each vertex v of G such that v' adjacent to every vertex that is adjacent to v in G and it is denoted by $spl(G)$.*

2. Main Results

Theorem 2.1. *$Aut\left[D_2\left(P_n\right)\right] \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, where P_n is a path.*

Proof. Let $D_2(P_n)$ be the shadow graph of a path P_n. Let $u_1, u_2, ..., u_n$ and $v_1, v_2, ..., v_n$ be the consecutive vertices of first and second copy of path P_n, respectively, in the shadow graph $D_2(P_n)$. Let $\phi \in Aut[D_2(P_n)]$. We have $\deg(v_1) = \deg(u_1) = \deg(v_n) = \deg(u_n) = 2$ and $\deg(v_i) = \deg(u_i) = 4, \forall i = 2, 3, ..., n-1$. Hence, there are four possibilities for the automorphisms of $D_2(P_n)$ which preserve degree of the vertices and adjacency of any two vertices as follows:

$$\phi_1(x) = \begin{cases} v_i & \text{if } x = v_i \\ u_i & \text{if } x = u_i \end{cases},$$

$$\phi_2(x) = \begin{cases} u_i & \text{if } x = v_i \\ v_i & \text{if } x = u_i \end{cases},$$

$$\phi_3(x) = \begin{cases} v_{n-i+1} & \text{if } x = v_i \\ u_{n-i+1} & \text{if } x = u_i \end{cases}$$

and

$$\phi_4(x) = \begin{cases} u_{n-i+1} & \text{if } x = v_i \\ v_{n-i+1} & \text{if } x = u_i \end{cases}$$

Here, $|Aut[D_2(P_n)]| = 4$ and $Aut[D_2(P_n)]$ is an abelian group. So, either $Aut[D_2(P_n)] \cong \mathbb{Z}_4$ or $Aut[D_2(P_n)] \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Here, $|\phi_2| = |\phi_3| = |\phi_4| = 2$. Hence, $Aut[D_2(P_n)] \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. □

Theorem 2.2. *$Aut[D_2(C_n)] \cong \mathbb{Z}_2 \times D_n$, where C_n is a cycle.*

Proof. Let $D_2(C_n)$ be the shadow graph of a cycle C_n. Let $u_1, u_2, ..., u_n$ and $v_1, v_2, ..., v_n$ be the consecutive vertices of first and second copy of cycle C_n, respectively in the shadow graph $D_2(C_n)$. Let $\phi \in Aut[D_2(C_n)]$. We have $\deg(u_i) = \deg(v_i)$, for all $i = 1, 2, 3, ..., n$ and all the neighbors of u_i are also the neighbors of v_i, for all $i = 1, 2, 3, ..., n$. So, either ϕ will fix u_i and v_i or ϕ will send u_i to v_i, for all $i = 1, 2, 3, ..., n$. Thus, in this case, the number of possibilities for the automorphisms of $D_2(C_n)$ is two. Therefore, in this case, the automorphism group of the portion of vertices u_i and v_i is isomorphic to $\mathbb{Z}_2$.

On the other hand, ϕ can be permute the vertices $u_1, u_2, ..., u_n$ of first copy of cycle C_n in $D_2(C_n)$ as automorphisms of cycle C_n. If ϕ sends the vertices $u_1, u_2, ..., u_n$ to each other in $D_2(C_n)$ like the automorphisms of cycle C_n, then ϕ have to send the vertices $v_1, v_2, ..., v_n$ of the second copy of cycle C_n in $D_2(C_n)$ such a way that it maintains the adjacency of u_i with v_i, u_i with v_{i+1} and u_i with v_{i-1} in $D_2(C_n)$, for all $i = 1, 2, 3, ..., n$. So, ϕ is either rotation or reflection which sends $u_1, u_2, ..., u_n$ to each other and $v_1, v_2, ..., v_n$ to each other in such a way that it maintains the adjacency of u_i with v_i, u_i with v_{i+1} and u_i with v_{i-1} in $D_2(C_n)$, for all $i = 1, 2, 3, ..., n$. Thus, in this case, the number of possibilities for the automorphisms of $D_2(C_n)$ is $2n$. Therefore, in this case, the automorphism group of the portion of vertices u_i and v_i is isomorphic to D_n. Therefore, the automorphism group of $D_2(C_n)$ must be the direct product of $\mathbb{Z}_2$ and D_n. Hence, $Aut\left[D_2(C_n)\right] \cong \mathbb{Z}_2 \times D_n$. □

Theorem 2.3. *$Aut\left[D_2(K_{1,n})\right] \cong \mathbb{Z}_2 \times S_{2n}$, where $K_{1,n}$ is a star graph.*

Proof. Let $D_2(K_{1,n})$ be the shadow graph of star graph $K_{1,n}$. Let $u_1, u_2, ..., u_n$ and $v_1, v_2, ..., v_n$ be the pendent vertices of first and second copy of star graph $K_{1,n}$, respectively in $D_2(K_{1,n})$. Let u and v be the apex vertices of first and second copy of star graph $K_{1,n}$, respectively in $D_2(K_{1,n})$. Let $\phi \in Aut\left[D_2(K_{1,n})\right]$. We have $\deg(u) = \deg(v) = 2n$, and u, v are the only vertices in $D_2(K_{1,n})$ with degree $2n$. Moreover, all the neighbors of u are also the neighbors of v in $D_2(K_{1,n})$. So, either ϕ will fix u and v or ϕ will send u to v. Thus, the number of possibilities for the automorphisms of the portion of the vertices u and v is two. Therefore, the automorphism group of the portion of vertices u and v is isomorphic to $\mathbb{Z}_2$. Since, $\deg(u_i) = \deg(v_i) = 2$, for all $i = 1, 2, 3, ..., n$ and the neighbor of all u_i and v_i are u and v in $spl(K_{1,n})$. So, ϕ can send the vertices $u_1, u_2, ..., u_n, v_1, v_2, ..., v_n$ to each other independently. Thus, the number of possibilities for the automorphisms of the portion of the vertices $u_1, u_2, ..., u_n, v_1, v_2, ..., v_n$ is $(2n)!$. Therefore, the automorphism group of the portion of all the vertices $u_1, u_2, ..., u_n, v_1, v_2, ..., v_n$ is isomorphic to S_{2n}. Therefore, the automorphism group of $D_2(K_{1,n})$ must be the direct product of $\mathbb{Z}_2$ and S_{2n}. Hence, $Aut\left[D_2(K_{1,n})\right] \cong \mathbb{Z}_2 \times S_{2n}$. □

Definition 2.4 ([18])**.** *The split graph of a graph G is obtained by adding a new vertex v' to each vertex v of G such that v' adjacent to every vertex that is adjacent to v in G and it is denoted by $spl(G)$.*

Theorem 2.5. *$Aut\left[spl(P_n)\right] \cong \mathbb{Z}_2$, where P_n is a path.*

Proof. Let $v_1, v_2, ..., v_n$ be the consecutive vertices of P_n. Let $spl(P_n)$ be the split graph of P_n obtained by taking new vertices $v_1', v_2', ..., v_n'$ corresponding to the vertices $v_1, v_2, ..., v_n$ of P_n, respectively. We have $\deg\left(v_i'\right) = 2, \deg(v_i) = 2, \forall i = 2, 3, ..., n-1$ and $\deg\left(v_1'\right) = \deg\left(v_n'\right) = 1, \deg(v_1) = \deg(v_n) = 2$. Hence, there are two possibilities for the automorphisms of $spl(P_n)$, which are preserve degree of the vertices and adjacency of any two vertices as follows:

$$\phi_1(x) = \begin{cases} v_i & \text{if } x = v_i \\ v_i' & \text{if } x = v_i' \end{cases}, \quad and \quad \phi_2(x) = \begin{cases} v_{n-1+i} & \text{if } x = v_i \\ v_{n-i+1}' & \text{if } x = v_i' \end{cases}$$

Here, $\left|Aut\left[spl(P_n)\right]\right| = 2$. Hence, $Aut\left[spl(P_n)\right] \cong \mathbb{Z}_2$. □

Theorem 2.6. *$Aut\left[spl(C_n)\right] \cong D_n$, where C_n is a cycle.*

Proof. Let $v_1, v_2, ..., v_n$ be the consecutive vertices of C_n. Let $spl(C_n)$ be the split graph of C_n obtained by taking new vertices $v_1', v_2', ..., v_n'$ corresponding to the vertices $v_1, v_2, ..., v_n$ of C_n, respectively. Let $\phi \in Aut\left[spl(C_n)\right]$. We have $\deg\left(v_i'\right) = 2$ and $\deg(v_i) = 4, \forall\ i = 1, 2, 3, ..., n$. So, ϕ can be permute the vertices $v_1, v_2, ..., v_n$ of cycle C_n in $spl(C_n)$ as automorphisms of cycle C_n. If ϕ sends the vertices

$v_1, v_2, ..., v_n$ to each other in $spl\,(C_n)$ like the automorphisms of cycle C_n, then ϕ have to send the vertices $v_1', v_2', ..., v_n'$ in $spl\,(C_n)$ such a way that it maintains the adjacency of v_i with v_{i+1} and v_i with v_{i-1} in $spl\,(C_n)$, for all $i = 1, 2, 3, ..., n$. So, ϕ is either rotation or reflection which sends $v_1, v_2, ..., v_n$ to each other and $v_1', v_2', ..., v_n'$ to each other in such a way that it maintains the adjacency of v_i with v_{i+1} and v_i with v_{i-1} in $spl\,(C_n)$, for all $i = 1, 2, 3, ..., n$. Thus, the number of possibilities for the automorphisms of $spl\,(C_n)$ is $2n$. Hence, $Aut\,[spl\,(C_n)] \cong D_n$. □

Theorem 2.7. *$Aut\,[spl\,(K_{1,n})] \cong S_n \times S_n$, where $K_{1,n}$ is a star graph.*

Proof. Let $v_1, v_2, ..., v_n$ be the pendent vertices of $K_{1,n}$ and v be the apex of $K_{1,n}$. Let $spl\,(K_{1,n})$ be the split graph of $K_{1,n}$ obtained by taking the new vertex v' corresponding to the apex vertex v of $K_{1,n}$ and taking the vertices $v_1', v_2', ..., v_n'$ corresponding to the pendent vertices $v_1, v_2, ..., v_n$ of $K_{1,n}$, respectively. Let $\phi \in Aut\,[spl\,(K_{1,n})]$. We have $\deg(v) = 2n$, and v is the only vertex in $spl\,(K_{1,n})$ with degree $2n$. Therefore, ϕ will fix v. Also, $\deg\left(v'\right) = n$, and v' is the only vertex in $spl\,(K_{1,n})$ with degree n. Therefore, ϕ will fix v'. Since, $\deg(v_i) = 2$, for all $i = 1, 2, 3, ..., n$ and the neighbors of all v_i are v and v' in $spl\,(K_{1,n})$. So, ϕ can send the vertices $v_1, v_2, ..., v_n$ to each other independently. Thus, the number of possibilities for the automorphisms of the portion of the vertices $v_1, v_2, ..., v_n$ is $n!$. Therefore, the automorphism group of the portion of all the vertices $v_1, v_2, ..., v_n$ is isomorphic to S_n. Since, $\deg\left(v_i'\right) = 1$, for all $i = 1, 2, 3, ..., n$, and the neighbor of all v_i' is v in $spl\,(K_{1,n})$. So, ϕ can send the vertices $v_1', v_2', ..., v_n'$ to each other independently. Thus, the number of possibilities for the automorphisms of the portion of the vertices $v_1', v_2', ..., v_n'$ is $n!$. Therefore, the automorphism group of the portion of all the vertices $v_1', v_2', ..., v_n'$ is isomorphic to S_n. Therefore, any automorphism ϕ of $spl\,(K_{1,n})$ must permute the set of vertices $v_1, v_2, ..., v_n$ and the set of vertices $v_1', v_2', ..., v_n'$ independently of each other. Therefore, the automorphism group of $spl\,(K_{1,n})$ must be the direct product of S_n and S_n. Hence, $Aut\,[spl\,(K_{1,n})] \cong S_n \times S_n$. □

Theorem 2.8. *The distinguishing number of $D_2\,(P_n)$ is 2, where P_n is a path.*

Proof. Let $D_2\,(P_n)$ be the shadow graph of a path P_n (**Theorem 2.1**). We have $|Aut\,[D_2\,(P_n)]\,| = 4$ and $Aut\,[D_2\,(P_n)] \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. We assign the vertices $u_1, u_2, \ldots, u_{\lfloor \frac{n+1}{2} \rfloor}$ with label 1 and the vertices $u_{\lfloor \frac{n+1}{2} \rfloor+1}, u_{\lfloor \frac{n+1}{2} \rfloor+2}, \ldots, u_n$ with label 2. We assign the vertices $v_1, v_2, \ldots, v_{\lfloor \frac{n+1}{2} \rfloor}, v_{\lfloor \frac{n+1}{2} \rfloor+1}, v_{\lfloor \frac{n+1}{2} \rfloor+2}, \ldots, v_n$ with label 1, 2 alternate. This assignment of labels is distinguishing because there is no non-trivial automorphism of $D_2\,(P_n)$ fix all the labels. Hence, $D\,[D_2\,(P_n)] = 2$. □

Illustration 2.9. *The shadow graph $D_2\,(P_4)$ of path p_4 is obtained by taking two copies of path graph P_4, namely P_4' and P_4'', join each vertex u_i in P_4' to the neighbors of the corresponding vertex v_i in P_4'', for $i = 1, 2, 3, 4$. The vertex labeling and distinguishing labeling for the shadow graph $D_2\,(P_4)$ of the path graph P_4 are demonstrated in Figure 1.*

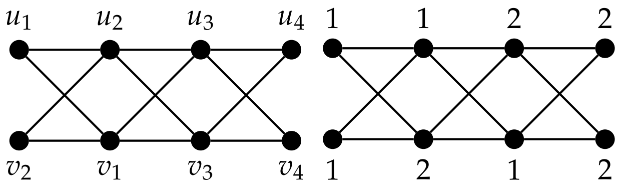

Figure 1: The vertex labeling and distinguishing labeling for the shadow graph $D_2\left(P_4\right)$ of the path P_4.

Theorem 2.10. *The distinguishing number of $D_2\left(C_n\right)$ is 2 where C_n is a cycle.*

Proof. Let $D_2\left(C_n\right)$ be the shadow graph of a cycle C_n (**Theorem 2.2**). We have $\left|Aut\left[D_2\left(C_n\right)\right]\right| = 2n$ and $Aut\left[D_2\left(C_n\right)\right] \cong D_n$. We assign the vertex u_1 with label 1 and the vertices $u_2, \ldots, u_n$ with label 2. Let ϕ be a non-trivial automorphism of $D_2\left(C_n\right)$. We have the following possible cases for ϕ:

1. If ϕ is a rotation, then ϕ can map u_1 to u_i for $2 \leq i \leq n$. Since, u_1 and u_i have different labels, so ϕ can not fix all the labels of vertices $u_1, u_2, ..., u_n$.

2. If ϕ is a reflection, then ϕ can not fix all the labels of vertices $u_1, u_2, \ldots, u_n$ except ϕ is a reflection which fix u_1. So, we assign the vertex v_n with label 2 and $v_1, v_2, \ldots, v_{n-1}$ with label 1. So, this reflection can not fix all the labels of vertices $v_1, v_2, \ldots, v_n$.

3. If ϕ maps u_i to v_i, for all $1 \leq i \leq n$. As we have assigned u_2 with label 2 and v_2 with label 1, so ϕ can not fix the labels of u_i to v_i, for all $1 \leq i \leq n$.

Thus, the assignment of above mentioned labeling is distinguishing because any non-trivial automorphism of $D_2\left(C_n\right)$ can not fix all the vertices of $D_2\left(C_n\right)$. Hence, $D\left[D_2\left(C_n\right)\right] = 2$. □

Illustration 2.11. *The shadow graph $D_2\left(C_4\right)$ of a cycle C_n is obtained by taking two copies of cycle graph C_4, namely $C_4^{'}$ and $C_4^{''}$, join each vertex u_i in $C_4^{'}$ to the neighbors of the corresponding vertex v_i in $C_4^{''}$, for $i = 1, 2, 3, 4$. The vertex labeling and distinguishing labeling for the shadow graph $D_2\left(C_4\right)$ of the cycle graph C_4 are demonstrated in Figure 2.*

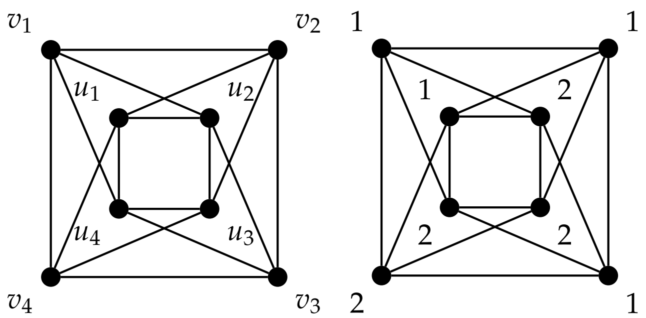

Figure 2: The vertex labeling and distinguishing labeling for the shadow graph $D_2\left(C_4\right)$ of the cycle graph C_4.

Theorem 2.12. *The distinguishing number of $D_2(K_{1,n})$ is $2n$ where $K_{1,n}$ is a star graph.*

Proof. Let $D_2(K_{1,n})$ be the shadow graph of a path $K_{1,n}$ (**Theorem 2.3**). We have $|Aut[D_2(K_{1,n})]| = 2.(2n)!$ and $Aut[D_2(K_{1,n})] \cong \mathbb{Z}_2 \times \mathbb{S}_{2n}$. We have $\deg(u_i) = \deg(v_i) = 2$, for all $i = 1, 2, 3, ..., n$, and the neighbor of all u_i and v_i are u and v in $D_2(K_{1,n})$. So, ϕ can send the vertices $u_1, u_2, ..., u_n, v_1, v_2, ..., v_n$ to each other independently. So, we have to assign the vertices $u_1, u_2, ..., u_n, v_1, v_2, ..., v_n$ with different labels. We assign the vertices $u_1, u_2, ..., u_n, v_1, v_2, ..., v_n$ with labels $1, 2, \ldots, 2n$, respectively. We assign the vertices u, v with labels $1, 2$, respectively. Hence, $D(D_2(K_{1,n})) = 2n$. □

Illustration 2.13. *The shadow graph $D_2(K_{1,4})$ of a star graph $K_{1,4}$ is obtained by taking two copies of star graph $K_{1,4}$, namely $K'_{1,4}$ and $K''_{1,4}$, join each vertex u_i in $K'_{1,4}$ to the neighbors of the corresponding vertex v_i in $K''_{1,4}$, for $i = 1, 2, 3, 4,$. The vertex labeling and distinguishing labeling for the shadow graph $D_2(K_{1,4})$ of the star graph $K_{1,4}$ are demonstrated in Figure 3.*

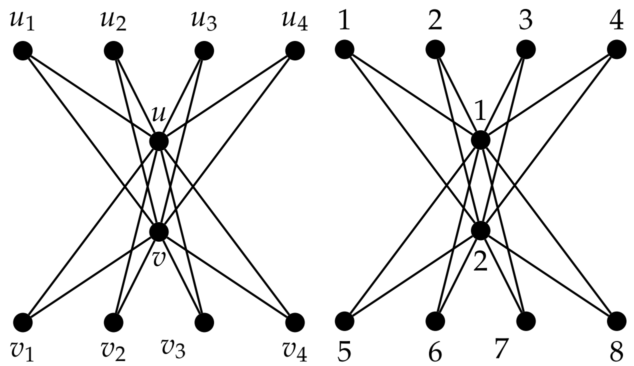

Figure 3: The vertex labeling and distinguishing labeling for the shadow graph $D_2(K_{1,4})$ of the star graph $K_{1,4}$.

Theorem 2.14. *The distinguishing number of $spl(P_n)$ is 2, where P_n is a path graph.*

Proof. Let $spl(P_n)$ be the spilt graph of path P_n (**Theorem 2.5**). We have $|Aut[spl(P_n)]| = 2$ and $Aut[spl(P_n)] \cong \mathbb{Z}_2$. We assign the vertices $v_1, v_2, \ldots, v_{\lfloor \frac{n+1}{2} \rfloor}, v'_1, v'_2, \ldots, v'_{\lfloor \frac{n+1}{2} \rfloor}$ with label 1 and the vertices $v_{\lfloor \frac{n+1}{2} \rfloor+1}, v_{\lfloor \frac{n+1}{2} \rfloor+2}, \ldots, v_n, v'_{\lfloor \frac{n+1}{2} \rfloor+1}, v'_{\lfloor \frac{n+1}{2} \rfloor+2}, \ldots, v'_n$ with label 2. Hence, $D[spl(P_n)] = 2$. □

Illustration 2.15. *The vertex labeling and distinguishing labeling for the split graph $spl(P_4)$ of the path graph P_4 are demonstrated in Figure 4.*

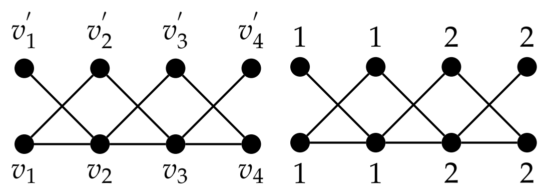

Figure 4: The vertex labeling and distinguishing labeling for the split graph $spl(P_4)$ of the path P_4.

Theorem 2.16. *The distinguishing number of $spl\left(C_n\right)$ is 2, where C_n is a cycle.*

Proof. Let $spl\left(C_n\right)$ be the spilt graph of a cycle C_n (**Theorem 2.6**). We have $\left|Aut\left[spl\left(C_n\right)\right]\right| = 2n$ and $Aut\left[spl\left(C_n\right)\right] \cong D_n$. We assign the vertex v_1 with label 1 and the vertices $v_2, \ldots, v_n$ with label 2. Let ϕ be a non-trivial automorphism of $spl\left(C_n\right)$. We have the following possible cases for ϕ:

1. If ϕ is a rotation, then ϕ can map v_1 to v_i for $2 \leq i \leq n$. Since, v_1 and v_i have different labels, so ϕ can not fix all the labels of vertices $v_1, v_2, ..., v_n$.

2. If ϕ is a reflection, then ϕ can not fix all the labels of vertices $v_1, v_2, \ldots, v_n$ except ϕ is a reflection which fix v_1. So, we assign the vertex $v_n^{'}$ with label 2 and $v_1^{'}, v_2^{'}, \ldots, v_{n-1}^{'}$ with label 1. So, this reflection can not fix all the labels of vertices $v_1^{'}, v_2^{'}, \ldots, v_n^{'}$.

Thus, the assignment of above mentioned labeling is distinguishing because any non-trivial automorphism of $D_2\left(C_n\right)$ can not fix all the vertices of $D_2\left(C_n\right)$. Hence, $D\left[D_2\left(C_n\right)\right] = 2$. □

Illustration 2.17. *The vertex labeling and distinguishing labeling for the split graph $spl\left(C_4\right)$ of the cycle C_4 are demonstrated in Figure 5.*

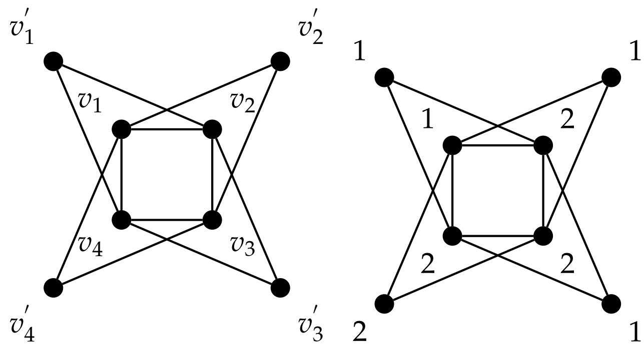

Figure 5: The vertex labeling and distinguishing labeling for the split graph $spl\left(C_4\right)$ of the cycle C_4.

Theorem 2.18. *The distinguishing number of $spl\left(K_{1,n}\right)$ is n, where $K_{1,n}$ is a star graph.*

Proof. Let $spl\left(K_{1,n}\right)$ be the spilt graph of a star graph $K_{1,n}$ (**Theorem 2.7**). We have $\left|Aut\left[spl\left(K_{1,n}\right)\right]\right| = n!^2$ and $Aut\left[spl\left(K_{1,n}\right)\right] \cong S_n \times S_n$. We have $\deg\left(v_i\right) = 2$, for all $i = 1, 2, 3, ..., n$ and the neighbor of all v_i are u and v in $spl\left(K_{1,n}\right)$. So, ϕ can send the vertices $v_1, v_2, ..., v_n$ to each other independently. So, we have to assign the vertices $v_1, v_2, ..., v_n$ with different labels. We assign the vertices $v_1, v_2, ..., v_n$ with labels $1, 2, \ldots, n$, respectively. We assign the vertices u, v with labels $1, 2$, respectively. Moreover, we have $\deg\left(v_i^{'}\right) = 1$, for all $i = 1, 2, 3, ..., n$ and the neighbor of all $v_i^{'}$ is v in $spl\left(K_{1,n}\right)$. So, ϕ can send the vertices $v_1^{'}, v_2^{'}, ..., v_n^{'}$ to each other independently. So, we have to assign the vertices $v_1^{'}, v_2^{'}, ..., v_n^{'}$ with different labels. We assign the vertices $v_1^{'}, v_2^{'}, ..., v_n^{'}$ with labels $1, 2, \ldots, n$, respectively. We assign the vertices v and $v^{'}$ with label 1. Hence, $D\left(spl\left(K_{1,n}\right)\right) = n$. □

Illustration 2.19. *The vertex labeling and distinguishing labeling for the split graph spl* $(K_{1,4})$ *of the star graph* $K_{1,4}$ *are demonstrated in Figure 6.*

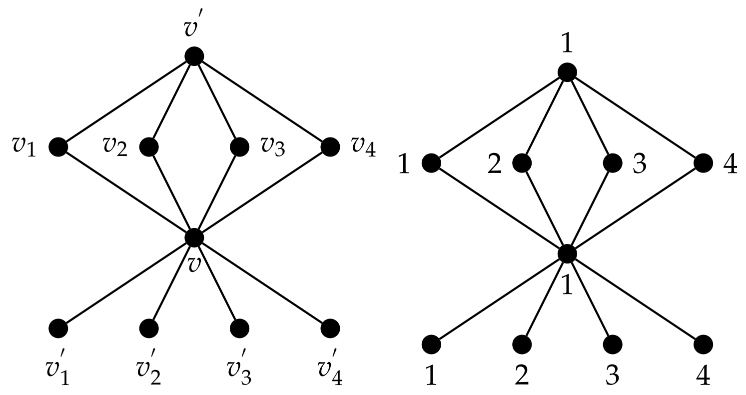

Figure 6: The vertex labeling and distinguishing labeling for the split graph $spl\ (K_{1,4})$ of the star graph $K_{1,4}$.

3. Conclusion

The automorphism group and distinguishing number of some standard graphs path P_n, cycle C_n and star graph $K_{1,n}$ available in the literature but we have investigated the automorphism group, also the distinguishing number of graph obtained from a given graph by graph operations. We have obtained results by considering two graph operations which are split and shadow graphs. This paper provide better insights about the automorphism group of shadow graph of path P_n is $\mathbb{Z}_2 \times \mathbb{Z}_2$, automorphism group of shadow graph of cycle C_n is $\mathbb{Z}_2 \times D_n$, automorphism group of shadow graph of star graph $K_{1,n}$ is $\mathbb{Z}_2 \times S_{2n}$. Moreover, the automorphism group of split graph of path P_n is $\mathbb{Z}_2$, automorphism group of split graph of cycle C_n is D_n and automorphism group of split graph of star graph $K_{1,n}$ is $S_n \times S_n$.

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