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Some Algorithms for Solving Higher Order Mixed Variational-like Inequalities

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Abstract

In this paper, we introduce a new class of variational-like inequalities, which is called higher order mixed variational-like inequality. We proved that the minimum of higher order strongly preinvex function can be characterized by this class of variational-like inequalities. Also, some iterative methods for solving higher order mixed variational-like inequalities are suggested.

Keywords: strongly preinvex function; variational-like inequality; iterative methods.
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1. Introduction

The variational inequality theory which was introduced by Stampacchia [15] is an interesting branch of mathematics with many applications in engineering, economics, optimizations and, operation research. This theory provides a natural and elegant framework to study a wide class of nonlinear problems, see, for example [1,4–6,8]. It is known that the concept of convexity plays an important role in the study of variational inequalities. This concept has been generalized in many directions see, for example [2,3,7,16]. An important generalization of convex functions is preinvex functions. It has been proved that the minimum of preinvex functions on the invex sets can be characterized by a class of variational inequalities, known as variational-like inequalities, see, for example [14]. Many extensions and generalizations of the variational-like inequality theory have been considered, in the last years, see, for example [9–11]. Recently, Noor [12] introduced the concept of higher order strongly convex functions and studied their properties. It has been shown that the optimality conditions of higher order strongly convex functions are characterized by a class of variational inequalities, see, for example [13]. Motivated and inspired by the above works, we introduce and study a new class of higher order variational-like inequalities. We use the auxiliary principle technique to suggest and analyze some iterative schemes for solving this type of variational-like inequality.

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2. Preliminaries

Let H be a real Hilbert space, whose norm and inner product are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. Let K be a nonempty closed subset in H and $\eta\colon K\times K\to H$ be a continuous function. We now recall some basic definitions and concepts.

Definition 2.1 ([16])**.** *A set K is said to be an invex set, if there exists a function $\eta(\cdot,\cdot)$, such that*

$$u+t\eta(v,u)\in K,\quad \forall u,v\in K,\ t\in[0,1].$$

From now on, the set K is an invex set in H, unless otherwise specified.

Definition 2.2. ([16]) A function $F\colon K\to\mathbb{R}$ is said to be a preinvex function, if there exists a function $\eta(\cdot,\cdot)$, such that

$$F(u+t\eta(v,u))\leq(1-t)F(u)+tF(v),\quad \forall u,v\in K,\ t\in[0,1].$$

It follows that a minimum of a differentiable preinvex function F on the invex set K can be characterized by the variational-like inequality

$$\langle F'(u),\eta(v,u)\rangle\geq 0,\quad \forall v\in K.$$

Definition 2.3. ([10]) A function $F\colon K\to\mathbb{R}$ is said to be strongly preinvex function, if there exists a function $\eta(\cdot,\cdot)$ and a constant $\mu>0$, such that

$$F(u+t\eta(v,u))\leq(1-t)F(u)+tF(v)-\mu(1-t)t\,\|\eta(v,u)\|^2,\quad \forall u,v\in K,\ t\in[0,1].$$

We note that the differentiable strongly preinvex function is a strongly invex function, i.e.,

$$F(v)-F(u)\geq\langle F'(u),\eta(v,u)\rangle+\mu\,\|\eta(v,u)\|^2.$$

Definition 2.4. ([13]) The function $F\colon K\to\mathbb{R}$ is said to be higher order strongly preinvex function, if there exists a function $\eta(\cdot,\cdot)$ and a constant $\mu>0$, such that

$$F(u+t\eta(v,u))\leq(1-t)F(u)+tF(v)-\mu\{t^p(1-t)+t(1-t)^p\}\,\|\eta(v,u)\|^p,$$
$$\forall u,v\in K,\ t\in[0,1],\ p\geq 1.$$

Theorem 2.5. ([13]) Let F be a differentiable higher order strongly preinvex function with modulus $\mu>0$. If $u\in K$ is the minimum of the function F, then

$$F(v)-F(u)\geq\langle F'(u),\eta(v,u)\rangle+\mu\,\|\eta(v,u)\|^p,\quad \forall u,v\in K. \tag{1}$$

Remark 2.6. It is clear that, for $\eta(v,u) = v - u$, the invex set K is a convex set, the preinvex functions are convex functions and strongly preinvex functions reduce to strongly convex functions, but the converse is not true. Moreover, every higher order strongly preinvex function is a higher order strongly convex function.

Assumption 2.7. The function $\eta\colon H \times H \to H$ satisfies the condition:

$$\eta(u,v) = \eta(u,w) + \eta(w,v), \quad \forall u,v,w \in H.$$

It follows that $\eta(u,u) = 0$ and $\eta(v,u) = -\eta(u,v), \quad \forall u,v \in H.$

Definition 2.8. An operator $A\colon H \to H$ is said to be:

(a) strongly η-monotone, if there exists a constant $\alpha > 0$, such that

$$\langle Av - Au, \eta(v,u)\rangle \geq \alpha \left\|\eta(v,u)\right\|^2, \quad \forall u,v \in H. \tag{2}$$

(b) uniformly η-monotone, if

$$\langle Av - Au, \eta(v,u)\rangle \geq a\big(\left\|\eta(v,u)\right\|\big) \left\|\eta(v,u)\right\|, \quad \forall u,v \in H. \tag{3}$$

where the continuous function $a\colon \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing with $a(0) = 0$ and $a(t) \to \infty$ as $n \to \infty$. We may choose $a(t) = \alpha\left|t\right|^{p-1}$ with $p > 1$ and $\alpha > 0$. In this case, we have

$$\langle Av - Au, \eta(v,u)\rangle \geq \alpha \left\|\eta(v,u)\right\|^p, \quad \forall u,v \in H. \tag{4}$$

(c) monotone, if

$$\langle Av - Au, \eta(v,u)\rangle \geq 0, \quad \forall u,v \in H. \tag{5}$$

Note that, if $\eta(v,u) = v - u$, the definition 2.8 reduces to the standard definition of strongly monotonicity, uniformly monotonicity and the monotonicity of the nonlinear operator A.

3. Main Results

In this section, we consider a new class of higher order variational-like inequality and suggest some iterative algorithms for solving this type of problem. We show that the variational-like inequalities arise in the connection with minimization of differentiable higher order strongly preinvex functions. Now, we consider the functional G, defined as

$$G(w) = F(w) + \varphi(w), \quad \forall w \in K. \tag{6}$$

where F is differentiable higher order strongly preinvex function and φ is a nondifferentiable higher order strongly preinvex function. We show that the minimum of the functional $G(w)$ defined by (6) can be characterized by a class of variational-like inequalities.

Theorem 3.1. *Let K be a invex set in H and let F be a differentiable higher order strongly preinvex function with modulus $\mu > 0$ and φ be a nondifferentiable higher order strongly preinvex function with modulus $\rho > 0$. Then $u \in K$ is the minimum of the functional G if and only if $u \in K$ satisfies the inequality*

$$\langle F'(u), \eta(v,u)\rangle + \varphi(v) - \varphi(u) \geq \rho \|\eta(v,u)\|^p, \quad \forall v \in K,\ p \geq 1. \tag{7}$$

Proof. Let $u \in K$ be a minimum of G on invex set K. Then

$$G(u) \leq G(w), \quad \forall w \in K. \tag{8}$$

Since K is an invex set, we have $w_t = u + t\eta(v,u) \in K,\ \forall v \in K,\ t \in [0,1]$. Replacing w by w_t in (8), we get

$$G(u) \leq G(u + t\eta(v,u)), \quad \forall v \in K.$$

It follows that,

$$\begin{aligned} F(u) + \varphi(u) &\leq F(u + t\eta(v,u)) + \varphi(u + t\eta(v,u)) \\ &\leq F(u + t\eta(v,u)) + \varphi(u) + t(\varphi(v) - \varphi(u)) - \rho\{t^p(1-t) + t(1-t)^p\} \|\eta(v,u)\|^p. \end{aligned}$$

Dividing the inequality by t and taking the limit as $t \to 0$, we have

$$\langle F'(u), \eta(v,u)\rangle + \varphi(v) - \varphi(u) \geq \rho \|\eta(v,u)\|^p, \quad \forall v \in K.$$

Conversely, let $u \in K$ satisfying (7). Let $w \in K$ arbitrary, then using (1), we have

$$\begin{aligned} G(w) - G(u) &= F(w) - F(u) + \varphi(w) - \varphi(u) \\ &\geq \langle F'(u), \eta(w,u)\rangle + \mu \|\eta(w,u)\|^p + \varphi(w) - \varphi(u) \\ &\geq \mu \|\eta(w,u)\|^p + \rho \|\eta(w,u)\|^p \\ &= (\mu + \rho) \|\eta(w,u)\|^p \\ &\geq 0. \end{aligned}$$

This shows that u is the minimum of the functional G, defined by (6). □

Now, we introduce and analyze a more general variational-like inequality. Let $A\colon H \to H$ be a continuous nonlinear operator and $\varphi\colon H \to \mathbb{R} \cup \{\infty\}$ be a nondifferentiable higher order strongly preinvex function with modulus $\rho > 0$. We consider the problem of finding $u \in K$, such that

$$\langle Au, \eta(v,u) \rangle + \varphi(v) - \varphi(u) \geq \rho \left\| \eta(v,u) \right\|^p, \quad \forall v \in K, \ p \geq 1. \tag{9}$$

which is called the higher order mixed variational-like inequality. If $Au = F'(u)$ is the differential of the differentiable higher order strongly preinvex function F, then the problem (9) reduce to the problem (7). Now we discuss some special cases of the problem (9):

I. For $\rho = 0$, the problem (9) is equivalent to finding $u \in K$, such that

$$\langle Au, \eta(v,u) \rangle + \varphi(v) - \varphi(u) \geq 0, \quad \forall v \in K, \tag{10}$$

which is known as the mixed variational-like inequality, studied by Noor [9].

II. Moreover, if $\eta(v,u) = v - u$, then the problem (10) is equivalent to finding $u \in K$ such that

$$\langle Au, v - u \rangle + \varphi(v) - \varphi(u) \geq 0, \quad \forall v \in K, \tag{11}$$

which is known as mixed variational inequality, introduced by Lions and Stampacchia [8].

III. If $\varphi(v) = 0$, $\forall v \in K$ then the problem (10) is equivalent to finding $u \in K$, such that

$$\langle Au, \eta(v,u) \geq 0, \quad \forall v \in K, \tag{12}$$

which is known as the variational-like inequality, introduced by Parida et al. [14].

IV. For $\eta(v,u) = v - u$, then the problem (12) is equivalent to finding $u \in K$, such that

$$\langle Au, v - u \rangle \geq 0, \quad \forall v \in K, \tag{13}$$

which is known as variational inequality, introduced and studied by Stampacchia [15].

Now, we use the auxiliary principle technique of Glowinski, Lions and Tremolieres [4], to suggest and analyze some iterative algorithms for solving the higher order variational-like inequality (9). For a given $u \in K$, consider the problem of finding $w \in K$, satisfying the auxiliary variational-like inequality

$$\langle \lambda Aw + \Phi'(w) - \Phi'(u), \eta(v,w) \rangle + \lambda\varphi(v) - \lambda\varphi(w) \geq \lambda\rho \left\| \eta(v,w) \right\|^p, \forall v \in K, \ p \geq 1, \tag{14}$$

where $\lambda > 0$ is a constant and Φ' is the differential of a strongly preinvex function Φ. We note that, if $w = u$, then w is a solution of the problem (9). Next, we propose and analyze an iterative algorithm for solving the variational-like inequality (9).

Algorithm 3.2. *For a given $u_0 \in K$, compute the approximate solution u_{n+1} by the iterative scheme:*

$$\langle \lambda A u_{n+1} + \Phi'(u_{n+1}) - \Phi'(u_n), \eta(v, u_{n+1}) \rangle \geq \lambda\varphi(u_{n+1}) - \lambda\varphi(v) + \lambda\rho \left\| \eta(v, u_{n+1}) \right\|^p, \quad \forall v \in K, \ p \geq 1, \tag{15}$$

which is known as the proximal point algorithm for solving the higher order mixed variational-like inequality (9). If $\rho = 0$, the Algorithm 3.2 reduces to:

Algorithm 3.3. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme:*

$$\langle \lambda A u_{n+1} + \Phi'(u_{n+1}) - \Phi'(u_n), \eta(v, u_{n+1}) \rangle \geq \lambda\varphi(u_{n+1}) - \lambda\varphi(v), \quad \forall v \in K, \tag{16}$$

which is known as the proximal point algorithm for solving mixed variational-like inequality (10). In particular, if $\eta(v, u) = v - u$, the Algorithm 3.3 reduces to:

Algorithm 3.4. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme:*

$$\langle \lambda A u_{n+1} + \Phi'(u_{n+1}) - \Phi'(u_n), v - u_{n+1} \rangle \geq \lambda\varphi(u_{n+1}) - \lambda\varphi(v), \quad \forall v \in K, \tag{17}$$

which is known as the proximal method for solving mixed variational inequality (11).

Theorem 3.5. *Let A be uniformly η-monotone with constant $\alpha > 0$ and Φ be differentiable higher order strongly preinvex function with modulus $\mu > 0$. If Assumption 2.7 holds and φ be nondifferentiable higher order strongly preinvex function with modulus $\rho > 0$, then the solution $\{u_n\}$ generated by Algorithm 3.1 converges to a solution u of the problem (9).*

Proof. Let $u \in K$ be a solution of (9). Then, for $v = u_{n+1}$, we have

$$\langle \lambda A u, \eta(u_{n+1}, u) \rangle + \lambda\varphi(u_{n+1}) - \lambda\varphi(u) \geq \lambda\rho \left\| \eta(u_{n+1}, u) \right\|^p. \tag{18}$$

Taking $v = u$ in the inequality (15), we get

$$\langle \lambda A u_{n+1} + \Phi'(u_{n+1}) - \Phi'(u_n), \eta(u, u_{n+1}) \rangle \geq \lambda\varphi(u_{n+1}) - \lambda\varphi(u) + \lambda\rho \left\| \eta(u, u_{n+1}) \right\|^p. \tag{19}$$

We consider the generalized Bregman function,

$$B(u, v) = \Phi(u) - \Phi(v) - \langle \Phi'(v), \eta(u, v) \rangle \geq \mu \left\| \eta(u, v) \right\|^p, \quad \forall\, v \in K, \tag{20}$$

associated with the differentiable higher order strongly preinvex function Φ. Using the Assumption 2.7, we have

$$\eta(u, u_n) = \eta(u, u_{n+1}) + \eta(u_{n+1}, u_n).$$

Now, combining (19), (20), it follows that

$$\begin{aligned} B(u,u_n)-B(u,u_{n+1}) &= \Phi(u_{n+1})-\Phi(u_n)-\langle\Phi'(u_n),\eta(u_{n+1},u_n)\rangle+\langle\Phi'(u_{n+1})-\Phi'(u_n),\eta(u,u_{n+1})\rangle, \\ &\geq \mu\left\|\eta(u_{n+1},u_n)\right\|^p+\langle\Phi'(u_{n+1})-\Phi'(u_n),\eta(u,u_{n+1})\rangle, \\ &\geq \mu\left\|\eta(u_{n+1},u_n)\right\|^p+\langle\lambda Au_{n+1},\eta(u_{n+1},u)\rangle \\ &\quad +\lambda\varphi(u_{n+1})-\lambda\varphi(u)+\lambda\rho\left\|\eta(u,u_{n+1})\right\|^p, \\ &= \mu\left\|\eta(u_{n+1},u_n)\right\|^p+\lambda\langle Au_{n+1}-Au,\eta(u_{n+1},u)\rangle+\lambda\langle Au,\eta(u_{n+1},u)\rangle \\ &\quad +\lambda\varphi(u_{n+1})-\lambda\varphi(u)+\lambda\rho\left\|\eta(u,u_{n+1})\right\|^p, \\ &= \mu\left\|\eta(u_{n+1},u_n)\right\|^p+Q. \end{aligned}$$

Since A is uniformly η - monotone, and using (18) we have

$$\begin{aligned} Q &= \lambda\langle Au_{n+1}-Au,\eta(u_{n+1},u)\rangle+\lambda\langle Au,\eta(u_{n+1},u)\rangle+\lambda\varphi(u_{n+1})-\lambda\varphi(u)+\lambda\rho\left\|\eta(u,u_{n+1})\right\|^p, \\ &\geq \lambda\alpha\left\|\eta(u_{n+1},u)\right\|^p+\lambda\varphi(u)-\lambda\varphi(u_{n+1})+\lambda\rho\left\|\eta(u_{n+1},u)\right\|^p \\ &\quad +\lambda\varphi(u_{n+1})-\lambda\varphi(u)+\lambda\rho\left\|\eta(u_{n+1},u)\right\|^p, \\ &= \lambda(\alpha+2\rho)\left\|\eta(u_{n+1},u)\right\|^p. \end{aligned}$$

Therefore,

$$B(u,u_n)-B(u,u_{n+1})\geq\mu\left\|\eta(u_{n+1},u_n)\right\|^p+\lambda(\alpha+2\rho)\left\|\eta(u_{n+1},u)\right\|^p.$$

If $u_{n+1}=u_n$, then clearly u_n is a solution of the problem (9). Otherwise, the sequence $B(u,u_n)-B(u,u_{n+1})$ is nonnegative, and we must have $\lim_{n\to\infty}\|\eta(u_{n+1},u_n)\|=0$. It follows that the sequence $\{u_n\}$ is bounded. Let $\widetilde{u}\in K$ be a cluster point of the $\{u_n\}$ and the subsequence $\{u_{n_k}\}$ converge to $\widetilde{u}$. Replacing u_n by u_{n_k} in (15) and taking the limit as $n_k\to\infty$, we conclude that $\widetilde{u}$ is a solution of the variational-like inequality (9). Thus it follow that the sequence $\{u_n\}$ has exactly one cluster point $\widetilde{u}$ and $\lim_{n\to\infty}u_n=\widetilde{u}$. □

4. Conclusion

In this paper, we have shown that the optimality conditions for higher order strongly preinvex functions can be characterized by higher order mixed variational-like inequalities. We proposed and analyzed some iterative methods, using the auxiliary principle technique and the generalized Bregman function in conjunction with higher order strongly preinvex functions. Some special cases are also discussed as applications of the main results.

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