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Generalized Fractional Integral Operators and Special Functions

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Abstract

In the present paper, we established some composition formulas Marichev-Saigo-Maeda (MSM Fractional Calculus Operator & MSM Fractional Differential Operators), then studied about some functions like Lerch Transcendent Function, Bessel Functions from various research papers and applied them to these integral operators and get some new results.

Keywords: MSM Fractional Operator; Bessel Function; Lerch Transcendent Function.
2020 Mathematics Subject Classification: 33E99, 26A33.

1. Introduction and Preliminaries

MSM fractional operators (Marichev–Saigo–Maeda operators) are generalized fractional integral operators used in studying fractional differential equations and in obtaining generalized results for many special functions.

1.1 MSM Fractional Calculus Operators

Marichev [1] introduced and researched fractional calculus operators, which are an extension of the Saigo operators, which were later developed by Saigo and Maeda [2]. For $\xi, \xi', \epsilon, \epsilon', \zeta \in C$ and $x \in R^+$ with $R(\xi) > 0$, The left-hand and right-hand sided MSM fractional integral and derivative operators associated with the third Appell function F3, as defined by S. Chandak, Biniyam Shimelis, Nigussie Abeye, and A. Padma in 2021, are given by [8]

$$\left(I_{O+}^{\xi,\xi',\epsilon,\epsilon',\zeta} f\right)(x) = \frac{x^{(-\xi)}}{\xi(\zeta)} \int_0^x \frac{(x-t)^{\zeta-1}}{t^{(\xi')}} F_3\left(\xi,\xi',\epsilon,\epsilon',\zeta; 1-\frac{t}{x}, 1-\frac{x}{t}\right) f(t)\, dt \tag{1}$$

$$\left(I_{-}^{\xi,\xi',\epsilon,\epsilon',\zeta} f\right)(x) = \frac{x^{(-\xi')}}{\xi(\zeta)} \int_0^x \frac{(t-x)^{\zeta-1}}{t^{(\xi)}} F_3\left(\xi,\xi',\epsilon,\epsilon',\zeta; 1-\frac{x}{t}, 1-\frac{t}{x}\right) f(t)\, dt \tag{2}$$

$$(D_{O+}^{\xi,\xi',\epsilon,\epsilon',\zeta} f)(x) = \left(\frac{d}{dx}\right)^q \, (I_{o+}^{-\xi,-\xi',\epsilon,-\epsilon'+q,\zeta+q} f)(x) \tag{3}$$

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$$(D_{-}^{\xi,\xi',\epsilon,\epsilon',\zeta}f)(x)=\left(\frac{-d}{dx}\right)^{q}\ (I_{-}^{-\xi,-\xi',\epsilon,-\epsilon'+q,\zeta+q}f)(x) \tag{4}$$

respectively, where $q=[R(\zeta)]+1$ and the third Appell function is defined by

$$F_3(\xi,\xi',\epsilon,\epsilon',\zeta;x,y)=\sum_{n=0}^{\infty}\frac{(\xi)_m\ (\xi')_n(\epsilon)_m(\epsilon')_n}{\zeta_{m+n}}\ \frac{x^{(m)}y^{(m)}}{(m)!(n)!} \tag{5}$$

where, $\max(|x|,|y|)<1$. Consider the left sided generalized k- fractional integral operator. Since

$$\left[I_{0,y}^{a,b,c}\ g(t)\right](y)=\frac{y^{-a-b}}{\xi(a)}\int_0^u(y-t)^{a-1}\ {}_2F_{1,k}\left(a+b,-c:a:1-\frac{t}{y}\right)g(t)dt \tag{6}$$

$$\left(I_{y,\infty}^{d,e,f}\ g(t)\right)(y)=\frac{1}{\xi(d)}\int_y^{\infty}(t-y)^{d-1}(t)^{-d-e}\ {}_2F_1\left(d+e,-f,d,1-\frac{y}{t}\right)g(t)dt \tag{7}$$

By Hyper-geometric k-function [3]

$${}_2F_{1,k}\left[(\zeta',k),(\tau',k),(-\eta',k);t\right]=\sum_{m=0}^{\infty}\frac{(\zeta')_m,k(\tau')_{m,k}\ t^m}{(\eta')_{m,k}(m)!}\quad(k>0) \tag{8}$$

or

$${}_2F_{1,k}\left[(\zeta',k),(\tau',k),(-\eta',k);1\right]=\frac{\xi_k(\eta')\xi_k(\eta'-\tau'-\zeta')}{\xi_k(\eta'-\zeta')\xi_k(\eta'-\tau')} \tag{9}$$

1.2 MSM Fractional Differential Operator

The Left Hand Side MSM fractional differential operator $[D_{O+}]$ is [4]

$$[D_{O+}^{\zeta,\zeta',\tau,\tau',\xi}\ t^{\rho-1}](t)\ =\ \frac{\xi(\rho)\xi(-\tau+\zeta+\rho)\xi(\zeta+\zeta'+\tau'-\xi+\rho)}{\xi(-\tau+\rho)\xi(\zeta+\zeta'-\xi+\rho)\xi(\zeta+\tau'-\xi+\rho)}\ t^{\zeta+\zeta'-\xi+\rho-1} \tag{10}$$

The Right Hand Side MSM fractional differential operator $[D_{-}]$ is [4]

$$[D_{-}^{\zeta,\zeta',\tau,\tau',\xi}\ t^{-\rho}](t)\ =\ \frac{\xi(\tau'+\rho)\xi(-\zeta-\zeta'+\xi+\rho)\xi(-\zeta'-\tau+\xi+\rho)}{\xi(\rho)\xi(-\zeta'+\tau'+\rho)\xi(-\zeta-\zeta'-\tau+\xi+\rho)}\ t^{\zeta+\zeta'-\xi-\rho} \tag{11}$$

Then

$${}_p\psi_q(t)={}_p\psi_q\left[\begin{array}{c|c}(a_i,\zeta_i)_{1,p} & \\ & t\\ (b_j,\tau_j)_{1,q} & \end{array}\right]=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\xi(a_i+n\zeta_i)}{\prod_{j=1}^{q}\xi(b_j+n\tau_j)}\frac{z^n}{(n)!} \tag{12}$$

1.3 Lerch Transcendent Function

The Lerch transcendent is a generalization of the Hurwitz zeta function and the polylogarithm function. It's a special function that has many applications in various fields of mathematics and physics. It is classically defined by [5]

$$\phi(t,s,a) = \sum_{n=0}^{\infty} \frac{t^n}{(n+a)^s} \quad for \quad |t| < 1 \; and \; (a)! = 0, -1, ... \tag{13}$$

It is implemented in this form as Hurwitz Lerch $\phi(t,s,a)$ in the Wolfram Language.

1.4 Bessel Functions

Bessel functions of first kind, denoted as $J_\nu(t)$, are solutions of Bessel's differential equation. For integer or positive ν, Bessel functions of first kind are finite at the origin $(t = 0)$; while for non negative-integer ν, the Bessel Functions of the first kind diverge as t approaches zero. The Bessel function can be defined by its series expansion about $t = 0$, which can be found by applying the Frobenius method. The series expansion is given by [6]

$$J_v\,(t,s,a) = \sum_{m=1}^{\infty} \frac{(-1)^m}{\xi(v+m+1)(m)!} \left(\frac{t}{2}\right)^{2m+v} \tag{14}$$

1.5 Beta Function

The beta function formula is defined as follows [7]

$$\tau(m,n) = \int_0^1 (1-x)^{n-1} x^{m-1}\, dx$$

Where $m, n > 0$

Relation between k-Beta and k-Gamma Function:

The k-Beta function can be written in the form of k-Gamma function as follows:

$$\tau_k(m,n) = \frac{\xi_k(m)\xi_k(n)}{\xi_k(m+n)} \tag{15}$$

2. Main Results

Theorem 2.1. *For* $\zeta', \tau', \eta' \in C; y \in \Re+;$ *with* $\Re(\zeta') > 0$, $\Re(\tau') > 0$ *and* $\Re(\eta') > 0$, $Max[0, \Re(\tau' - \eta')]$; $(k > 0)$; $\Re(\rho r - \eta) > 0$ *then*

$$\left[I_{0,y}^{\zeta',\tau',\eta'} \phi(t,s,a)\right]_k (y) = \frac{y^{\frac{-\tau'}{k}}}{k} \frac{\xi_k(\eta' - \tau')}{\xi_k(\zeta' - \zeta' - \tau')\xi_k(\zeta' + \eta')} \phi(y,s,a) \frac{\xi(n+1)\xi(\frac{\zeta'}{k}+s)}{\xi(n+1+\frac{\zeta'}{k}+s)}$$

Proof. By using equation (6) and (13), we get

$$\left[I_{0,y}^{\zeta',\tau',\eta'} \phi(t,s,a)\right]_k (y) = \frac{y^{\frac{-\zeta'-\tau'}{k}}}{k\xi_k(\zeta')} \int_0^y (y-t)^{\frac{\zeta'}{k}-1}\, {}_2F_{1,k}\left[(\zeta'+\tau',k), (-\eta',k), (\zeta',k); 1 - \frac{t}{y}\right]$$

$$\times \sum_{n=0}^{\infty} \frac{t^n}{(n+a)^s} \tag{16}$$

$$= \frac{y^{\frac{-\zeta'-\tau'}{k}}}{k\xi_k(\zeta')} \sum_{s=0}^{\infty} \frac{(\zeta'+\tau')_{s,k}(-\eta)_{s,k}}{(\zeta')_{s,k}(s)!} \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \int_0^y (y-t)^{\frac{\zeta'}{k}-1}\left(1-\frac{t}{y}\right)^s t^n dt \tag{17}$$

on solving integral part we get, let

$$A = \int_0^y (y-t)^{\frac{\zeta'}{k}-1}\left(1-\frac{t}{y}\right)^s t^n dt$$

Let $t = yj \;\Rightarrow dt = ydj$ and put in integral

$$A = \int_0^1 (y-yj)^{\frac{\zeta'}{k}-1}(1-j)^s (yj)^n ydj$$

$$A = (y)^{\frac{\zeta'}{k}-1+n+1} \int_0^1 (1-j)^{\frac{\zeta'}{k}+s-1}(j)^{n+1-1}dj \tag{18}$$

put the equation (18) in equation (17) we get

$$= \frac{y^{\frac{-\tau'}{k}}}{k\xi_k(\zeta')} \sum_{s=0}^{\infty} \frac{(\zeta'+\tau')_{s,k}(-\eta)_{s,k}}{(\zeta')_{s,k}(s)!} \sum_{n=0}^{\infty} \frac{t^n}{(n+a)^s} \int_0^1 (1-j)^{\frac{\zeta'}{k}+s-1}(j)^{n+1-1}dj \tag{19}$$

Using the Definition of τ-function from (15)

$$= \frac{y^{\frac{-\tau'}{k}}}{k\xi_k(\zeta')} \sum_{s=0}^{\infty} \frac{(\zeta'+\tau')_{s,k}(-\eta)_{s,k}}{(\zeta')_{s,k}(s)!} \sum_{n=0}^{\infty} \frac{y^n}{(n+a)^s} \frac{\xi(n+1)\xi(\frac{\zeta'}{k}+s)}{\xi(n+1+\frac{\zeta'}{k}+s)} \tag{20}$$

Use (8) and (13) in (20)

$$= \frac{y^{\frac{-\tau'}{k}}}{k\xi_k(\zeta')}\; {}_2F_{1,k}\left[(\zeta'+\tau',k),(-\eta',k),(\zeta',k);1\right]\phi(y,s,a)\frac{\xi(n+1)\xi(\frac{\zeta'}{k}+s)}{\xi(n+1+\frac{\zeta'}{k}+s)} \tag{21}$$

Using the value from (9) and put in (21), we get

$$\left[I_{0,y}^{\zeta',\tau',\eta'}\phi(t,s,a)\right]_k(y) = \frac{y^{\frac{-\tau'}{k}}}{k}\frac{\xi_k(\eta'-\tau')}{\xi_k(\zeta'-\zeta'-\tau')\xi_k(\zeta'+\eta')}\phi(y,s,a)\frac{\xi(n+1)\xi(\frac{\zeta'}{k}+s)}{\xi(n+1+\frac{\zeta'}{k}+s)} \tag{22}$$

□

Theorem 2.2. *For $\zeta',\tau',\eta' \in C; y \in \Re+$; with $\Re(\zeta') > 0$, $\Re(\tau') > 0$ and $\Re(\eta') > 0$, $Max[0,\Re(\tau'-\eta')]$; $(k > 0)$; $\Re(\rho r - \eta) > 0$ then*

$$\left[I_{y,\infty}^{\zeta',\tau',\eta'}\phi(t,s,a)\right]_k(y) = \frac{y^{\frac{-\tau'}{k}}}{k}\frac{\xi_k(\eta'-\tau')}{\xi_k(\zeta'-\zeta'-\tau')\xi_k(\zeta'+\eta')}\phi(y,s,a)\;\frac{\xi(\frac{\tau'}{k}+n)\xi(\frac{\zeta'}{k}+s)}{\xi(\frac{\tau'}{k}+n+\frac{\zeta'}{k}+s)}$$

Proof. By using equation (7) and (13) we get

$$\left[I_{y,\infty}^{\zeta',\tau',\eta'}\phi(t,s,a)\right]_k(y) = \frac{1}{k\xi_k(\zeta')}\int_y^\infty (t-y)^{\frac{\zeta'}{k}-1}(t)^{\frac{-\zeta'-\tau'}{k}} \times {}_2F_{1,k}\left[(\zeta'+\tau',k),(-\eta',k),(\zeta',k);1-\frac{y}{t}\right]\sum_{n=0}^{\infty}\frac{t^n}{(n+a)^s} \tag{23}$$

$$= \frac{1}{k\xi_k(\zeta')}\sum_{s=0}^{\infty}\frac{(\zeta'+\tau')_{s,k}(-\eta')_{s,k}}{(\zeta')_{s,k}(s)!}\sum_{n=0}^{\infty}\frac{1}{(n+a)^s}\int_y^\infty (t-y)^{\frac{\zeta'}{k}-1}(t)^{\frac{-\zeta'-\tau'}{k}}\left(1-\frac{y}{t}\right)^s t^n dt \tag{24}$$

On solving integral part we get, let

$$A = \int_y^\infty (t-y)^{\frac{\zeta'}{k}-1}(t)^{\frac{-\zeta'-\tau'}{k}}\left(1-\frac{y}{t}\right)^s t^n dt$$

put $j=\frac{y}{t} \Rightarrow t=\frac{y}{j} \Rightarrow \quad dt=\frac{-y}{j^2}dj$ in integral

$$A = \int_1^0 \left(\frac{y}{j}-y\right)^{\frac{\zeta'}{k}-1}\left(\frac{y}{j}\right)^{\frac{-\zeta'-\tau'}{k}}(1-j)^s\left(\frac{y}{j}\right)^n\left(\frac{-y}{j^2}\right)dj$$

$$A = (y)^{-\frac{\tau'}{k}+n}\int_0^1 (1-j)^{\frac{\zeta'}{k}+s-1}(j)^{\frac{\tau'}{k}-n-1}dj \tag{25}$$

Put (25) in (24)

$$\left[I_{y,\infty}^{\zeta',\tau',\eta'}\phi(t,s,a)\right]_k(y) = \frac{(y)^{-\frac{\tau'}{k}+n}}{k\xi_k(\zeta')}\sum_{s=0}^{\infty}\frac{(\zeta'+\tau')_{s,k}(-\eta')_{s,k}}{(\zeta')_{s,k}(s)!} \times \sum_{n=0}^{\infty}\frac{1}{(n+a)^s}\int_0^1 (1-j)^{\frac{\zeta'}{k}+s-1}(j)^{\frac{\tau'}{k}-n-1}dj \tag{26}$$

Using the definition of τ-function from (15)

$$\left[I_{y,\infty}^{\zeta',\tau',\eta'}\phi(t,s,a)\right]_k(y) = \frac{(y)^{-\frac{\tau'}{k}}}{k\xi_k(\zeta')}\sum_{s=0}^{\infty}\frac{(\zeta'+\tau')_{s,k}(-\eta')_{s,k}}{(\zeta')_{s,k}(s)!}\sum_{n=0}^{\infty}\frac{(y)^n}{(n+a)^s}\frac{\xi(\frac{\tau'}{k}+n)\xi(\frac{\zeta'}{k}+s)}{\xi(\frac{\tau'}{k}+n+\frac{\zeta'}{k}+s)} \tag{27}$$

Use (8) and (13) in (27)

$$\left[I_{y,\infty}^{\zeta',\tau',\eta'}\phi(t,s,a)\right]_k(y) = \frac{y^{\frac{-\tau'}{k}}}{k\xi_k(\zeta')}\,{}_2F_1\left[(\zeta'+\tau',k),(-\eta',k),(\zeta',k);1\right] \times \phi(y,s,a)\frac{\xi(\frac{\tau'}{k}+n)\xi(\frac{\zeta'}{k}+s)}{\xi(\frac{\tau'}{k}+n+\frac{\zeta'}{k}+s)} \tag{28}$$

Using the value from (9) and put in (28), we get

$$\left[I_{y,\infty}^{\zeta',\tau',\eta'}\phi(t,s,a)\right]_k(y) = \frac{y^{\frac{-\tau'}{k}}}{k}\frac{\xi_k(\eta'-\tau')}{\xi_k(\zeta'-\zeta'-\tau')\xi_k(\zeta'+\eta')}\phi(y,s,a)\,\frac{\xi(\frac{\tau'}{k}+n)\xi(\frac{\zeta'}{k}+s)}{\xi(\frac{\tau'}{k}+n+\frac{\zeta'}{k}+s)} \tag{29}$$

□

Theorem 2.3. *For $\zeta', \tau', \eta' \in C; y \in \Re+$; with $\Re(\zeta') > 0$, $\Re(\tau') > 0$ and $\Re(\eta') > 0$, $Max[0, \Re(\tau' - \eta')]$; $(k > 0)$; $\Re(\rho r - \eta) > 0$ then*

$$\left[I_{0,y}^{\zeta',\tau',\eta'} J_v(t,s,a)\right]_k (y) = \frac{y^{\frac{-\tau'}{k}}}{k} \frac{\xi_k(\eta' - \tau')}{\xi_k(\zeta' - \zeta' - \tau')\xi_k(\zeta' + \eta')} J_v(y,s,a) \frac{\xi(2m+v+1)\xi(\frac{\zeta'}{k}+s)}{\xi(2m+v+1+\frac{\zeta'}{k}+s)}$$

Proof. By using equation (6) and (14) we get

$$\begin{aligned}\left[I_{0,y}^{\zeta',\tau',\eta'} J_v(t,s,a)\right]_k (y) &= \frac{y^{\frac{-\zeta'-\tau'}{k}}}{k\xi_k(\zeta')} \int_0^y (y-t)^{\frac{\zeta'}{k}-1} {}_2F_{1,k}\left[(\zeta'+\tau',k),(-\eta',k),(\zeta',k);1-\frac{t}{y}\right] \\ &\times \sum_{m=1}^{\infty} \frac{(-1)^m}{\xi(v+m+1)(m)!}\left(\frac{t}{2}\right)^{2m+v} \qquad (30)\\ &= \frac{y^{\frac{-\zeta'-\tau'}{k}}}{k\xi_k(\zeta')} \sum_{s=0}^{\infty} \frac{(\zeta'+\tau')_{s,k}(-\eta)_{s,k}}{(\zeta')_{s,k}(s)!} \sum_{m=1}^{\infty} \frac{(-1)^m}{\xi(v+m+1)(m)!}\left(\frac{1}{2}\right)^{2m+v} \\ &\times \int_0^y (y-t)^{\frac{\zeta'}{k}-1}\left(1-\frac{t}{y}\right)^s (t)^{2m+v} dt \qquad (31)\end{aligned}$$

On solving integral part we get, let

$$A = \int_0^y (y-t)^{\frac{\zeta'}{k}-1}\left(1-\frac{t}{y}\right)^s (t)^{2m+v} dt$$

Put $\frac{t}{y} = j \Rightarrow t = yj \Rightarrow dt = ydj$ in integral

$$\begin{aligned}A &= \int_0^1 (y-yj)^{\frac{\zeta'}{k}-1}(1-j)^s (yj)^{2m+v} ydj \qquad (32)\\ &= (y)^{\left(\frac{\zeta'}{k}-1+2m+v+1\right)} \int_0^1 (1-j)^{\frac{\zeta'}{k}+s-1}(j)^{2m+v+1-1} dj \qquad (33)\end{aligned}$$

Using (33) in (31) we get

$$\begin{aligned}\left[I_{0,y}^{\zeta',\tau',\eta'} J_v(t,s,a)\right]_k (y) &= \frac{y^{\left(\frac{-\zeta'}{k}-\frac{\tau'}{k}+\frac{\zeta'}{k}+2m+v\right)}}{k\xi_k(\zeta')} \sum_{s=0}^{\infty} \frac{(\zeta'+\tau')_{s,k}(-\eta')_{s,k}}{(\zeta')_{s,k}(s)!} \\ &\times \sum_{m=1}^{\infty} \frac{(-1)^m}{\xi(v+m+1)(m)!}\left(\frac{1}{2}\right)^{2m+v} \int_0^1 (1-j)^{\frac{\zeta'}{k}+s-1}(j)^{2m+v+1-1} dj \qquad (34)\\ &= \frac{y^{-\frac{\tau'}{k}}}{k\xi_k(\zeta')} \sum_{s=0}^{\infty} \frac{(\zeta'+\tau')_{s,k}(-\eta')_{s,k}}{(\zeta')_{s,k}(s)!} \sum_{m=1}^{\infty} \frac{(-1)^m}{\xi(v+m+1)(m)!}\left(\frac{y}{2}\right)^{2m+v} \\ &\times \int_0^1 (1-j)^{\frac{\zeta'}{k}+s-1}(j)^{2m+v+1-1} dj \qquad (35)\end{aligned}$$

By the definition of τ-function from (15)

$$\left[I_{0,y}^{\zeta',\tau',\eta'} J_v(t,s,a)\right]_k (y) = \frac{y^{-\frac{\tau'}{k}}}{k\xi_k(\zeta')} \sum_{s=0}^{\infty} \frac{(\zeta'+\tau')_{s,k}(-\eta')_{s,k}}{(\zeta')_{s,k}(s)!} \sum_{m=1}^{\infty} \frac{(-1)^m}{\xi(v+m+1)(m)!}\left(\frac{y}{2}\right)^{2m+v}$$

$$\times \frac{\xi(2m+v+1)\xi(\frac{\zeta'}{k}+s)}{\xi(2m+v+1+\frac{\zeta'}{k}+s)} \tag{36}$$

Using (8) and (14) in (36)

$$\left[I_{0,y}^{\zeta',\tau',\eta'} J_v(t,s,a)\right]_k (y) = \frac{y^{-\frac{\tau'}{k}}}{k\xi_k(\zeta')} \, {}_2F_{1,k}\left[(\zeta'+\tau',k),(-\eta',k),(\zeta',k);1\right]$$
$$\times J_v(y,s,a) \, \frac{\xi(2m+v+1)\xi(\frac{\zeta'}{k}+s)}{\xi(2m+v+1+\frac{\zeta'}{k}+s)} \tag{37}$$

Using the value from (9) and put in (37), we get

$$\left[I_{0,y}^{\zeta',\tau',\eta'} J_v(t,s,a)\right]_k (y) = \frac{y^{\frac{-\tau'}{k}}}{k\xi_k(\zeta')} \, \frac{\xi_k(\zeta')\xi_k(\zeta'-\zeta'-\tau'+\eta')}{\xi_k(\zeta'-\zeta'-\tau')\xi_k(\zeta'+\eta')} \, J_v(y,s,a) \, \frac{\xi(2m+v+1)\xi(\frac{\zeta'}{k}+s)}{\xi(2m+v+1+\frac{\zeta'}{k}+s)}$$
$$\left[I_{0,y}^{\zeta',\tau',\eta'} J_v(t,s,a)\right]_k (y) = \frac{y^{\frac{-\tau'}{k}}}{k} \frac{\xi_k(\eta'-\tau')}{\xi_k(\zeta'-\zeta'-\tau')\xi_k(\zeta'+\eta')} \, J_v(y,s,a) \, \frac{\xi(2m+v+1)\xi(\frac{\zeta'}{k}+s)}{\xi(2m+v+1+\frac{\zeta'}{k}+s)} \tag{38}$$

□

Theorem 2.4. *For $\zeta',\tau',\eta' \in C; y \in \Re+$; with $\Re(\zeta') > 0$, $\Re(\tau') > 0$ and $\Re(\eta') > 0$, $Max[0,\Re(\tau'-\eta')]$; $(k > 0)$; $\Re(\rho r - \eta) > 0$ then*

$$\left[I_{y,\infty}^{\zeta',\tau',\eta'} J_v(t,s,a)\right]_k (y) = \frac{y^{\frac{-\tau'}{k}}}{k} \frac{\xi_k(\eta'-\tau')}{\xi_k(\zeta'-\zeta'-\tau')\xi_k(\zeta'+\eta')} \times J_v(y,s,a) \frac{\xi(\frac{\tau'}{k}-2m-v)\xi(\frac{\zeta'}{k}+s)}{\xi(\frac{\tau'}{k}-2m-v+\frac{\zeta'}{k}+s)}$$

Proof. By using equation (7) and (14), we get

$$\left[I_{y,\infty}^{\zeta',\tau',\eta'} J_v(t,s,a)\right]_k (y) = \frac{1}{k\xi_k(\zeta')} \int_y^{\infty} (t-y)^{\frac{\zeta'}{k}-1} (t)^{\frac{-\zeta'-\tau'}{k}}$$
$$\times \, {}_2F_{1,k}\left[(\zeta'+\tau',k),(-\eta',k),(\zeta',k);1-\frac{y}{t}\right] \sum_{m=1}^{\infty} \frac{(-1)^m}{\xi(v+m+1)(m)!} \left(\frac{t}{2}\right)^{2m+v} \tag{39}$$

$$= \frac{1}{k\xi_k(\zeta')} \sum_{s=0}^{\infty} \frac{(\zeta'+\tau')_{s,k}(-\eta)_{s,k}}{(\zeta')_{s,k}(s)!} \sum_{m=1}^{\infty} \frac{(-1)^m}{\xi(v+m+1)(m)!} \left(\frac{1}{2}\right)^{2m+v}$$
$$\times \int_y^{\infty} (t-y)^{\frac{\zeta'}{k}-1} (t)^{\frac{-\zeta'-\tau'}{k}} \left(1-\frac{y}{t}\right)^s (t)^{2m+v} dt \tag{40}$$

On solving integral part, let

$$A = \int_y^{\infty} (t-y)^{\frac{\zeta'}{k}-1} (t)^{\frac{-\zeta'-\tau'}{k}} \left(1-\frac{y}{t}\right)^s (t)^{2m+v} \, dt$$

Put $\frac{y}{t} = j \Rightarrow t = \frac{y}{j} \Rightarrow dt = \frac{-y}{j^2} dj$ in integral

$$A = \int_1^0 \left(\frac{y}{j}-y\right)^{\frac{\zeta'}{k}-1} \left(\frac{y}{j}\right)^{\frac{-\zeta'-\tau'}{k}} (1-j)^s \left(\frac{y}{j}\right)^{2m+v} \left(\frac{-y}{j^2}\right) dj$$

$$= (y)^{-\frac{\tau'}{k}+2m+v} \int_0^1 (1-j)^{\frac{\zeta'}{k}+s-1} (j)^{\frac{\tau'}{k}-2m-v-1} dj \tag{41}$$

Using (41) in (40)

$$\left[I_{y,\infty}^{\zeta',\tau',\eta'} J_v(t,s,a)\right]_k (y) = \frac{(y)^{-\frac{\tau'}{k}+2m+v}}{k\xi_k(\zeta')} \sum_{s=0}^{\infty} \frac{(\zeta'+\tau')_{s,k}(-\eta')_{s,k}}{(\zeta')_{s,k}(s)!}$$
$$\times \sum_{m=1}^{\infty} \frac{(-1)^m}{\xi(v+m+1)(m)!} \left(\frac{1}{2}\right)^{2m+v} \int_0^1 (1-j)^{\frac{\zeta'}{k}+s-1} (j)^{\frac{\tau'}{k}-2m-v-1} dj \tag{42}$$

By the definition of τ-function from (15)

$$\left[I_{y,\infty}^{\zeta',\tau',\eta'} J_v(t,s,a)\right]_k (y) = \frac{(y)^{-\frac{\tau'}{k}}}{k\xi_k(\zeta')} \sum_{s=0}^{\infty} \frac{(\zeta'+\tau')_{s,k}\,(-\eta')_{s,k}}{(\zeta')_{s,k}\,(s)!}$$
$$\times \sum_{m=1}^{\infty} \frac{(-1)^m}{\xi(v+m+1)(m)!} \left(\frac{y}{2}\right)^{2m+v} \frac{\xi(\frac{\tau'}{k}-2m-v)\xi(\frac{\zeta'}{k}+s)}{\xi(\frac{\tau'}{k}-2m-v+\frac{\zeta'}{k}+s)} \tag{43}$$

Using (8) and (14) in (43)

$$\left[I_{y,\infty}^{\zeta',\tau',\eta'} J_v(t,s,a)\right]_k (y) = \frac{(y)^{-\frac{\tau'}{k}}}{k\xi_k(\zeta')} \, {}_2F_{1,k}\left[(\zeta'+\tau',k),(-\eta',k),(\zeta',k);1\right]$$
$$\times J_v(y,s,a) \frac{\xi(\frac{\tau'}{k}-2m-v)\xi(\frac{\zeta'}{k}+s)}{\xi(\frac{\tau'}{k}-2m-v+\frac{\zeta'}{k}+s)} \tag{44}$$

Using the value from (9) and put in (44), we get

$$\left[I_{y,\infty}^{\zeta',\tau',\eta'} J_v(t,s,a)\right]_k (y) = \frac{y^{\frac{-\tau'}{k}}}{k\xi_k(\zeta')} \frac{\xi_k(\zeta')\xi_k(\zeta'-\zeta'-\tau'+\eta')}{\xi_k(\zeta'-\zeta'-\tau')\xi_k(\zeta'+\eta')} J_v(y,s,a) \frac{\xi(\frac{\tau'}{k}-2m-v)\xi(\frac{\zeta'}{k}+s)}{\xi(\frac{\tau'}{k}-2m-v+\frac{\zeta'}{k}+s)}$$
$$\left[I_{y,\infty}^{\zeta',\tau',\eta'} J_v(t,s,a)\right]_k (y) = \frac{y^{\frac{-\tau'}{k}}}{k} \frac{\xi_k(\eta'-\tau')}{\xi_k(\zeta'-\zeta'-\tau')\xi_k(\zeta'+\eta')} J_v(y,s,a) \frac{\xi(\frac{\tau'}{k}-2m-v)\xi(\frac{\zeta'}{k}+s)}{\xi(\frac{\tau'}{k}-2m-v+\frac{\zeta'}{k}+s)} \tag{45}$$

□

Theorem 2.5. *Let $\zeta,\zeta',\tau,\tau',\xi \in C$ and $Re(\rho r-\eta) > 0$ be such that $Re(\xi) > 0$, Also let $a \in C$ then for $z > 0$ then*

$$\left[D_{O+}^{\zeta,\zeta',\tau,\tau'\xi} \;\; \phi(t,s,a)\right](t) = t^{(\zeta+\zeta'-\xi-1)} \times_3 \psi_3 \left[\begin{array}{c} (0,1) \quad (-\tau+\zeta,1) \quad (\zeta+\zeta'+\tau'-\xi,1) \\ (-\tau,1) \;\; (\zeta+\zeta'-\xi,1) \;\; (\zeta+\tau'-\xi,1) \end{array} \middle| \, t^1 \right] \tag{46}$$

Proof.

$$\phi(t,s,a) = \sum_{n=0}^{\infty} \frac{t^n}{(n+a)^s} \tag{47}$$
$$= t^n \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}$$

$$= \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \left[D_{O+}^{\zeta,\zeta',\tau,\tau'\xi} \; t^{n-1} \right] \tag{48}$$

Using (10) in (48), we get

$$\begin{aligned}
\left[D_{O+}^{\zeta,\zeta',\tau,\tau'\xi} \; \phi(t,s,a) \right](t) &= \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \frac{\xi(n)}{\xi(-\tau+n)} \frac{\xi(-\tau+\zeta+n)}{\xi(\zeta+\zeta'-\xi+n)} \\
&\times \frac{\xi(\zeta+\zeta'+\tau'-\xi+n)}{\xi(\zeta+\tau'-\xi+n)} t^{(\zeta+\zeta'-\xi+n-1)} \qquad (49) \\
&= t^{(\zeta+\zeta'-\xi+n-1)} \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \frac{\xi(n)}{\xi(-\tau+n)} \frac{\xi(-\tau+\zeta+n)}{\xi(\zeta+\zeta'-\xi+n)} \frac{\xi(\zeta+\zeta'+\tau'-\xi+n)}{\xi(\zeta+\tau'-\xi+n)} \\
&= t^{(\zeta+\zeta'-\xi-1)} \sum_{n=0}^{\infty} \frac{t^n}{(n+a)^s} \frac{\xi(n)}{\xi(-\tau+n)} \frac{\xi(-\tau+\zeta+n)}{\xi(\zeta+\zeta'-\xi+n)} \frac{\xi(\zeta+\zeta'+\tau'-\xi+n)}{\xi(\zeta+\tau'-\xi+n)}
\end{aligned}$$

$$= t^{(\zeta+\zeta'-\xi-1)} \; {}_3\psi_3 \left[\begin{array}{ccc} (0,1) & (-\tau+\zeta,1) & (\zeta+\zeta'+\tau'-\xi,1) \\ (-\tau,1) & (\zeta+\zeta'-\xi,1) & (\zeta+\tau'-\xi,1) \end{array} \middle| t^1 \right] \tag{50}$$

□

Theorem 2.6. *Let $\zeta,\zeta',\tau,\tau',\xi \in C$ and $Re(\rho r-\eta)>0$ be such that $Re(\xi)>0$, Also let $a \in C$ then for $z>0$ then*

$$\left[D_{-}^{\zeta,\zeta',\tau,\tau'\xi} \; \phi(t,s,a) \right](t) = t^{(\zeta+\zeta'-\tau+\xi)} \times$$

$$\times {}_3\psi_3 \left[\begin{array}{ccc} (-\tau',-1) & (-\zeta-\zeta'+\xi,-1) & (-\zeta'-\tau+\xi,-1) \\ (0,-1) & (-\zeta'+\tau',-1) & (-\zeta-\zeta'-\tau+\xi,-1) \end{array} \middle| t^{(-1)} \right] \tag{51}$$

Proof.

$$\begin{aligned}
\phi(t,s,a) &= \sum_{n=0}^{\infty} \frac{t^n}{(n+a)^s} \qquad (52) \\
&= t^n \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \\
&= \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \left[D_{-}^{\zeta,\zeta',\tau,\tau'\xi} \; t^{-n} \right] \qquad (53)
\end{aligned}$$

Using (11) in (53), we get

$$\begin{aligned}
\left[D_{-}^{\zeta,\zeta',\tau,\tau'\xi} \; \phi(t,s,a) \right](t) &= \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \frac{\xi(\tau'-n)}{\xi(-n)} \frac{\xi(-\zeta-\zeta'+\xi+n)}{\xi(-\zeta'+\tau'-n)} \\
&\times \frac{\xi(-\zeta'-\tau+\xi-n)}{\xi(-\zeta-\zeta'-\tau+\xi-n)} t^{(\zeta+\zeta'-\tau+\xi-n)} \qquad (54) \\
&= t^{(\zeta+\zeta'-\tau+\xi-n)} \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \frac{\xi(\tau'-n)}{\xi(-n)} \frac{\xi(-\zeta-\zeta'+\xi+n)}{\xi(-\zeta'+\tau'-n)} \frac{\xi(-\zeta'-\tau+\xi-n)}{\xi(-\zeta-\zeta'-\tau+\xi-n)} \\
&= t^{(\zeta+\zeta'-\tau+\xi)} \sum_{n=0}^{\infty} \frac{t^{-n}}{(n+a)^s} \frac{\xi(\tau'-n)}{\xi(-n)} \frac{\xi(-\zeta-\zeta'+\xi+n)}{\xi(-\zeta'+\tau'-n)} \frac{\xi(-\zeta'-\tau+\xi-n)}{\xi(-\zeta-\zeta'-\tau+\xi-n)}
\end{aligned}$$

$$= t^{(\zeta+\zeta'-\tau+\xi)}\ {}_3\psi_3 \left[\begin{array}{c} (-\tau',-1) \quad (-\zeta-\zeta'+\xi,-1) \quad (-\zeta'-\tau+\xi,-1) \\ (0,-1) \quad (-\zeta'+\tau',-1) \quad (-\zeta-\zeta'-\tau+\xi,-1) \end{array} \middle| t^{(-1)} \right] \tag{55}$$

□

Theorem 2.7. *Let $\zeta, \zeta', \tau, \tau', \xi \in C$ and $Re(\rho r - \eta) > 0$ be such that $Re(\xi) > 0$, Also let $a \in C$then for $z > 0$ then*

$$\left[D_{o+}^{\zeta,\zeta',\tau,\tau',\xi}\ j_v(t,a)\right](t) = t^{(\zeta+\zeta'-\xi+v)} \times$$

$$\times {}_3\psi_4 \left[\begin{array}{c} (v,2) \quad (-\tau+\zeta+v,2) \quad (\zeta+\zeta'+\tau'-\xi+v,2) \\ (v+1,1) \quad (-\tau+v,2) \quad (\zeta+\zeta'-\xi+v,2) \quad (\zeta+\tau'-\xi+v,-2) \end{array} \middle| (-t)^{(2)} \right] \tag{56}$$

Proof.

$$J_v(t) = \sum_{m=1}^{\infty} \frac{(-1)^m}{\xi(v+m+1)(m)!} \left(\frac{t}{2}\right)^{(2m+v)} \tag{57}$$

$$= \left(\frac{t}{2}\right)^{(2m+v)} \sum_{m=1}^{\infty} \frac{(-1)^m}{\xi(v+m+1)(m)!}$$

$$= \sum_{m=1}^{\infty} \frac{(-1)^m}{\xi(v+m+1)(m)!} \left[D_{O+}^{\zeta,\zeta',\tau,\tau',\xi}\ (t)^{(2m+v-1)}\right] \tag{58}$$

Using (10) in (58), we get

$$\left[D_{o+}^{\zeta,\zeta',\tau,\tau',\xi}\ j_v(t,a)\right](t) = \sum_{m=1}^{\infty} \frac{(-1)^m}{\xi(v+m+1)(m)!} \frac{\xi(2m+v)}{\xi(-\tau+2m+v)} \frac{\xi(-\tau+\zeta+2m+v)}{\xi(\zeta+\zeta'-\xi+2m+v)}$$
$$\times \frac{\xi(\zeta+\zeta'+\tau'-\xi+2m+v)}{\xi(\zeta+\tau'-\xi+2m+v)}\ t^{(\zeta+\zeta'-\xi+2m+v-1)} \tag{59}$$
$$= t^{(\zeta+\zeta'-\xi+v-1)} \sum_{m=1}^{\infty} \frac{(-1)^m t^{2m}}{\xi(v+m+1)(m)!} \frac{\xi(2m+v)}{\xi(-\tau+2m+v)} \frac{\xi(-\tau+\zeta+2m+v)}{\xi(\zeta+\zeta'-\xi+2m+v)}$$
$$\times \frac{\xi(\zeta+\zeta'+\tau'-\xi+2m+v)}{\xi(\zeta+\tau'-\xi+2m+v)}$$

$$= t^{(\zeta+\zeta'-\xi+v)}\ {}_3\psi_4 \left[\begin{array}{c} (v,2) \quad (-\tau+\zeta+v,2) \quad (\zeta+\zeta'+\tau'-\xi+v,2) \\ (v+1,1) \quad (-\tau+v,2) \quad (\zeta+\zeta'-\xi+v,2) \quad (\zeta+\tau'-\xi+v,-2) \end{array} \middle| -t^2 \right] \tag{60}$$

□

Theorem 2.8. *Let $\zeta, \zeta', \tau, \tau', \xi \in C$ and $Re(\rho r - \eta) > 0$ be such that $Re(\xi) > 0$, Also let $a \in C$then for $z > 0$ then*

$$\left[D_{-}^{\zeta,\zeta',\tau,\tau',\xi}\ j_v(t,a)\right](t) = t^{(\zeta+\zeta'-\xi-v)} \times$$

$$\times {}_3\psi_4 \left[\begin{array}{c} (-\tau'-v,-2) \quad (-\zeta-\zeta'+\xi-v,-2) \quad (-\zeta'-\tau+\xi-v,-2) \\ (v+1,1) \quad (-v,-2) \quad (-\zeta'+\tau'-\xi-v,-2) \quad (-\zeta-\zeta'-\tau+\xi-v,-2) \end{array} \middle| (-t)^{(-2)} \right] \tag{61}$$

Proof.

$$J_v(t) = \sum_{m=1}^{\infty} \frac{(-1)^m}{\xi(v+m+1)(m)!} \left(\frac{t}{2}\right)^{2m+v} \tag{62}$$

$$= \left(\frac{t}{2}\right)^{2m+v} \sum_{m=1}^{\infty} \frac{(-1)^m}{\xi(v+m+1)(m)!}$$

$$= \sum_{m=1}^{\infty} \frac{(-1)^m}{\xi(v+m+1)(m)!} \left[D_{-}^{\zeta,\zeta',\tau,\tau',\xi} \ (t)^{(-2m-v)}\right] \tag{63}$$

Using (11) in (63), we get

$$\left[D_{-}^{\zeta,\zeta',\tau,\tau',\xi} \ j_v(t,a)\right](t) = \sum_{m=1}^{\infty} \frac{(-1)^m}{\xi(v+m+1)(m)!} \frac{\xi(\tau'-2m-v)}{\xi(-2m-v)} \frac{\xi(-\zeta-\zeta'+\xi-2m-v)}{\xi(-\zeta'+\tau'-2m-v)}$$
$$\times \frac{\xi(-\zeta'-\tau'+\xi-2m-v)}{\xi(-\zeta-\zeta'-\tau+\xi-2m-v)} t^{(\zeta+\zeta'-\xi-2m-v)} \tag{64}$$
$$= t^{(\zeta+\zeta'-\xi-v)} \sum_{m=1}^{\infty} \frac{(-1)^m t^{-2m}}{\xi(v+m+1)(m)!} \frac{\xi(\tau'-2m-v)}{\xi(-2m-v)} \frac{\xi(-\zeta-\zeta'+\xi-2m-v)}{\xi(-\zeta'+\tau'-2m-v)}$$
$$\times \frac{\xi(-\zeta'-\tau'+\xi-2m-v)}{\xi(-\zeta-\zeta'-\tau+\xi-2m-v)} \tag{65}$$

$$= t^{(\zeta+\zeta'-\xi-v)} \, {}_3\psi_4 \left[\begin{matrix} (-\tau'-v,-2) \ (-\zeta-\zeta'+\xi-v,-2) \\ \\ (v+1,1) \ (-v,-2) \ (-\zeta'+\tau'-\xi-v,-2) \end{matrix} \right.$$
$$\left. \begin{matrix} (-\zeta'-\tau+\xi-v,-2) \\ \\ (-\zeta-\zeta'-\tau+\xi-v,-2) \end{matrix} \middle| (-t)^{(-2)} \right] \tag{66}$$

□

3. Conclusion

The methodology and technique that we have used here to develop our results can be further extended using different type of operators and special functions. The new results can be used in problems based on the field of science and engineering including turbulence medical imagining, robotics, electronics and electrical circuit theory. The new researcher can further use our findings to develop new results involving Pathway Fractional Integral operator, Marichev-Saigo Maeda (MSM) Fractional differential operators, Bessel Function, Lerch Transcendent, Integral Transforms to solve the problems based on Medical Sciences, Mathematical Science, life Sciences, Sciences and Engineering.

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