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On Tri-Edge $\mathbb{Z}_t-$graphs

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Abstract

In this paper, we define a new graph, Tri-Edge $\mathbb{Z}_t-$Graph using triangular numbers and the cyclic group $\mathbb{Z}_n$. We introduce the concept of the Tri-Edge Index, $\Omega(G)$, and determine $\Omega(G)$ of some classes of graphs G. Moreover, we establish a bound for $\Omega(G)$. We listed all non-isomorphic Tri-Edge $\mathbb{Z}_{\Omega_q}-$Graphs, $G = (p, q)$ for $1 \leq q \leq 7$. Furthermore, we introduce the concept of Weak Tri-Edge $\mathbb{Z}_t$-Graph.

Keywords: Triangular Number; Tri-Edge $\mathbb{Z}_t-$Graph; Weak Tri-Edge $\mathbb{Z}_t-$Graph; Tri-Edge Index $\Omega(G)$.

2020 Mathematics Subject Classification: 05C25, 05C78.

1. Introduction

Labeling is an attracting field within Graph Theory that has a wide-ranging applications in the disciplines such as computer science, chemistry, and social sciences. At its core, graph labeling involves assigning labels, often numerical or alphanumeric, to the vertices of a graph or edges of it according to specific rules or constraints. These labeled graphs are then analyzed to extract valuable insights, unveiling essential properties or relationships within the underlying structures. Graph labeling, which entails assigning integers to vertices or edges based on defined conditions, first emerged in the mid-1960s. This development is largely attributed to the work of Alexander Rosa in 1967, marking a foundational milestone for various graph labeling techniques. Information on graph labeling can be found at the wonderful site founded and maintained by J. Gallian, in [6]. Alexander Rosa's $\beta - valuation$ [9] gives a universal acceptance to Graph labelings. The study of graph labeling encompasses a wide range of topics, each offering unique insights and applications. Some common types of graph labeling include vertex labeling, edge labeling, and total labeling. In vertex labeling, we assign labels to a graph's vertices, while in edge labeling we assign labels to the edges. Total labeling

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combines vertex and edge labeling by assigning distinct labels to both vertices and edges, thereby providing a more comprehensive representation of the graph. One of the key motivations behind the study of graph labeling is its applicability to real-world problems. For example, in computer science, graph labeling is used in network routing algorithms, where labels are assigned to vertices or edges to facilitate efficient routing of data packets. In chemistry, graph labeling is used to represent molecular structures, where atoms are labeled based on their types and connections. In social sciences, graph labeling is used to analyze networks in the society, where we represent individuals by vertices and relationships between them by edges. In this paper we define and study a new graph, we call Tri-Edge $\mathbb{Z}_t-$Graph. The Triangular Number in the n^{th} position [4] is denoted by T_n. That is, $T_n = 1+2+3+...+n$. The Triangular Numbers are $1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, ...$

2. Tri-Edge $\mathbb{Z}_t-$Graph

Definition 2.1. *Suppose $G = (p, q)$ is a simple graph which is connected. G is a Tri-Edge $\mathbb{Z}_t-$Graph, if there exists a mapping $f : V(G) \to \mathbb{Z}_t - \{0\}$ and f gives a bijection $f^* : E(G) \to \Delta_q$ as $f^*(uv) = f(u)f(v)$, for each edge $e = uv$ in G. Such a mapping of a graph G is called Tri-Edge $\mathbb{Z}_t-$labeling.*

Definition 2.2. *The least t such that G is a Tri-Edge $\mathbb{Z}_t-$Graph is the Tri-Edge Index and is denoted by $\Omega(G)$.*

Theorem 2.3. *For all Tri-Edge $\mathbb{Z}_t-$Graph $G = (p, q)$, $q + 1 \leq \Omega(G) \leq T_{p-1} + 1$.*

Proof. Since G is a Tri-Edge $\mathbb{Z}_t-$Graph with q edges, the edge values are $T_1, T_2, \cdots, T_q$. If q is odd, the edge label $T_q = \frac{1}{2}q(q+1)$ is only obtained by the vertex labels q and $\frac{1}{2}(q+1)$. This is possible only if we are mapping V to $\mathbb{Z}_t$, for $t \geq q+1$. Similarly, if q is even, edge label $T_q = \frac{1}{2}q(q+1)$ is only obtained by the vertex labels $\frac{q}{2}$ and $q+1$. This is possible only if we are mapping V to $\mathbb{Z}_t$, for $t \geq q+2$. Therefore,

$$\Omega(G) \geq \begin{cases} q+1 & \text{if } q \text{ is odd,} \\ q+2 & \text{if } q \text{ is even.} \end{cases} \tag{1}$$

Hence $\Omega(G) \geq q+1$. Now, let $V = \{a_1, a_2, .., a_p\}$ be the vertex set of G. Since G is a Tri-Edge $\mathbb{Z}_t-$Graph, the vertex label 1 repeats twice. Therefore the maximum vertex label is T_{p-1}. Hence $\Omega(G) \leq T_{p-1} + 1$. □

Remark 2.4. *If G is a Tri-Edge $\mathbb{Z}_t-$Graph, then it is a Tri-Edge $\mathbb{Z}_s-$Graph, for all $s > t$.*

Remark 2.5. *Since the edge label $T_2 = 3$ is possible only when the vertices 1 and 3 are adjacent, in every Tri-Edge $\mathbb{Z}_t-$Graph, with $q \geq 2$, the vertices with values 1 and 3 are always adjacent.*

Remark 2.6. *Let $G = (p, q)$ be a Tri-Edge $\mathbb{Z}_t-$Graph. Then $\sum f^*(e) = \frac{1}{6}q(q+1)(q+2)$.*

Proof. Since $G = (p, q)$ is a Tri-Edge $\mathbb{Z}_t-$Graph, $f^*(e) = \{T_1, T_2, \cdots, T_q\}$. Therefore, $\sum f^*(e) = T_1 + T_2 + \cdots . + T_q = \frac{1}{6}q(q+1)(q+2)$. □

Theorem 2.7. *The path graph P_n is Tri-Edge $\mathbb{Z}_{n+1}-$Graph, and*

$$\Omega(P_n) = \begin{cases} n+1 & \text{if } n \text{ is odd,} \\ n & \text{if } n \text{ is even.} \end{cases}$$

Proof. Let the vertices of P_n be $V = \{b_1, b_2, \cdots, b_n\}$ and let $e_i = b_i b_{i+1}$, for $i = 1, 2, \cdots, n-1$.

Case (i): n is odd

Let $f : V \to \mathbb{Z}_{n+1} - \{0\}$ such that

$$f(v_i) = \begin{cases} i & \text{if } i \text{ is odd,} \\ \frac{i}{2} & \text{if } i \text{ is even.} \end{cases}$$

If i is odd, $f^*(e_i) = f^*(b_i b_{i+1}) = f(b_i) f(b_{i+1}) = i\frac{(i+1)}{2} = \frac{i(i+1)}{2} = T_i$. Now if i is even, $f^*(e_i) = f^*(b_i b_{i+1}) = f(b_i) f(b_{i+1}) = \frac{i}{2}(i+1) = \frac{i(i+1)}{2} = T_i$.

Case (ii): n is even

Let $f : V \to \mathbb{Z}_n - \{0\}$ such that

$$f(v_i) = \begin{cases} i & \text{if } i \text{ is odd,} \\ \frac{i}{2} & \text{if } i \text{ is even.} \end{cases}$$

If i is odd, $f^*(e_i) = f^*(b_i b_{i+1}) = f(b_i) f(b_{i+1}) = i\frac{(i+1)}{2} = \frac{i(i+1)}{2} = T_i$. Now if i is even, $f^*(e_i) = f^*(b_i b_{i+1}) = f(b_i) f(b_{i+1}) = \frac{i}{2}(i+1) = \frac{i(i+1)}{2} = T_i$. Since f is increasing on $V(G)$, f^* is also increasing. So f^* is injective and hence P_n is Tri-Edge $\mathbb{Z}_t-$Graph for either $t = n$ or $t = n+1$. Therefore clearly P_n is Tri-Edge $\mathbb{Z}_{n+1}-$Graph. Hence,

$$\Omega(P_n) \le \begin{cases} n+1 & \text{if } n \text{ is odd,} \\ n & \text{if } n \text{ is even.} \end{cases} \tag{2}$$

Since P_n has $n-1$ edges, by the proof of Theorem 2.3, we get

$$\Omega(P_n) \ge \begin{cases} n+1 & \text{if } n \text{ is odd,} \\ n & \text{if } n \text{ is even.} \end{cases} \tag{3}$$

From inequalities 2 and 3, we get the result. □

By Theorem 2.7, $\Omega(P_6) = 6$ and $\Omega(P_7) = 8$. Figure 1 shows such labeling of P_6 and P_7.

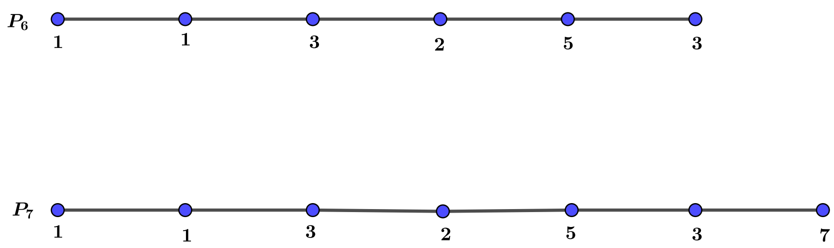


Figure 1: $\Omega(P_6) = 6$ and $\Omega(P_7) = 8$

Theorem 2.8. *Every star graph $K_{1,n}$ is a Tri-Edge $\mathbb{Z}_{T_n+1}-$Graph.*

Proof. Let u be the vertex which is adjacent to $v_1, v_2, v_3, \cdots, v_n$ in $K_{1,n}$. Define, $f : V \to \mathbb{Z}_{T_n+1}$ such that $f(u) = 1$ and $f(v_i) = T_i$, $\forall i = 1, 2, 3, \cdots, n$. Then the edge values are, $f^*(u, v_i) = T_i, \forall i = 1, 2, \cdots, n$. Since f is increasing on $V(G)$, f^* is also increasing. So f^* is injective. Hence $K_{1,n}$ is a Tri-Edge $\mathbb{Z}_{T_n+1}-$Graph, $\forall n$. $\square$

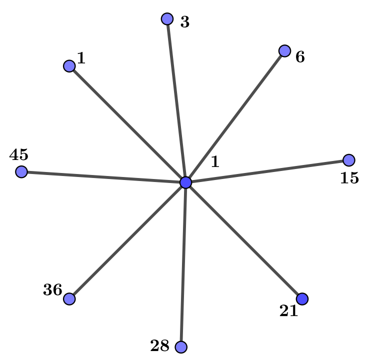


Figure 2: Tri-Edge $\mathbb{Z}_{48}-$labeling of $K_{1,8}$

Figure 2 shows a Tri-Edge $\mathbb{Z}_{48}-$labeling of $K_{1,8}$.

Theorem 2.9. *Bistar graph $B_{m,n}$ is a Tri-Edge $\mathbb{Z}_{T_{(m+n+1)}+1}-$Graph, $\forall m, n$.*

Proof. Let,

$$V(B_{m,n}) = \{u, v, u_i, v_j, 1 \leq i \leq m, 1 \leq j \leq n\} \text{ and}$$
$$E(B_{m,n}) = \{uv, uu_i, vv_j, 1 \leq i \leq m, 1 \leq j \leq n\}.$$

Define,

$$f(u) = 1, f(v) = 1, f(u_i) = T_{i+1}, i = 1, 2, \cdots, m.$$

and

$$f(v_j) = T_{m+1+j}, \; j = 1, 2, \cdots, n.$$

Then, we get all the consecutive triangular numbers as the edge values. □

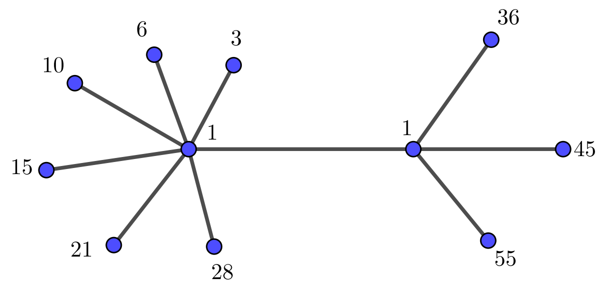


Figure 3: Tri-Edge $\mathbb{Z}_{56}-$ labeling of $B_{6,3}$.

Figure 3 shows a Tri-Edge $\mathbb{Z}_{56}-$labeling of the Bistar $B_{6,3}$.

Theorem 2.10. *The complete graph K_n is not a Tri-Edge $\mathbb{Z}_t-$ labeling, if $n \geq 3$.*

Proof. Let $v_1, v_2, \cdots, v_n$ be the vertices of K_n. Suppose K_n is a Tri-Edge $\mathbb{Z}_t-$Graph. Then, take $f(v_1) = 1, f(v_2) = 1, f(v_3) = 3$. If $n = 3$, the edge value 3 repeats, which is impossible. If $n \geq 4$, let $f(v_4) = x$, then since the vertex v_4 is adjacent to both v_1 and v_2, the edge label x repeats, this is impossible. Hence K_n is not a Tri-Edge $\mathbb{Z}_t-$Graph, if $n \geq 3$. □

2.1 Non-isomorphic Tri-Edge $\mathbb{Z}_t-$Graph

There are many non-isomorphic Tri-Edge $\mathbb{Z}_t-$Graph graphs $G = (p, q)$, for a given q, with different $\Omega(G)$. For example, P_6 and $K_{1,5}$ are two different graphs with same number of edges, $q = 5$; but $\Omega(P_5) = 6$ and $\Omega(K_{1,5}) = 16$. Let $\mathcal{G}$ be the collection of all non-isomorphic Tri-Edge $\mathbb{Z}_t-$Graph with q edges. Let $\Omega_q = min\{\Omega(G_i)/G_i \in \mathcal{G}\}$. Here, we list all non-isomorphic Tri-Edge $\mathbb{Z}_{\Omega_q}-$Graph, $G = (p, q)$ for $1 \leq q \leq 7$. The Table 1 and Table 2 exhibits such non-isomorphic graphs.

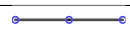
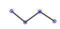
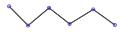
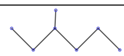
S. No.	q	Graph G	$\Omega(G)$
1	1		2
2	2		4
3	3		4
4	4		6
5	5		6
6	5		6
7	5		6
8	6		8
9	6		8

Table 1: Non-isomorphic Tri-Edge $\mathbb{Z}_{\Omega_q}-$Graphs with different q.

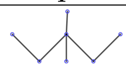
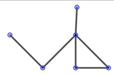
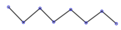
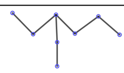
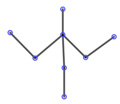
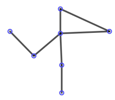
S. No.	q	Graph G	$\Omega(G)$
10	6		8
11	6		8
12	7		8
13	7		8
14	7		8
15	7		8

Table 2: Non-isomorphic Tri-Edge $\mathbb{Z}_{\Omega_q}-$Graph with different q.

q	$\omega(G)$	No. of Tri-Edge $\mathbb{Z}_{\Omega_q}-$Graph
1	2	1
2	3	1
3	4	1
4	6	1
5	6	3
6	8	4
7	8	4

Table 3: Number of Tri-Edge $\mathbb{Z}_{\Omega_q}-$Graph

In general, it is not easy to find a Tri-Edge $\mathbb{Z}_t-$labeling for a graph. So we look for a weaker condition.

Definition 2.11. *Let G be a simple graph with p vertices and q edges. Also, let G be a connected graph. G is said to be a Weak Tri-Edge $\mathbb{Z}_t-$Garph, if there exists a function $f : V(G) \to \mathbb{Z}_t - \{0\}$ such that f induces a* *surjection* $f^* : E(G) \to \Delta_k$, $k \leq q$ *defined by* $f^*(uv) = f(u)f(v)$, $\forall e = uv \in E(G)$. *Such labeling of a graph G is called Weak Tri-Edge* $\mathbb{Z}_t-$*Garph labeling.*

Theorem 2.12. *The comb graph with* $2n$ *vertices is a Weak Tri-Edge* $\mathbb{Z}_t-$*Garph,* $\forall n$, *where*

$$t = \begin{cases} n+1 & \text{if } n \text{ is odd,} \\ n+2 & \text{if } n \text{ is even.} \end{cases}$$

Proof. Let $P_n : u_1, u_2, \cdots, u_n$ be the path and let $w_i = u_iu_{i+1}$ $(1 \leq i \leq n-1)$ be the edges. Let $v_1, v_2, \cdots, v_n$ be the pendant vertices adjacent to $u_1, u_2, \cdots, u_n$ respectively and let $t_i = u_iv_i$, $1 \leq i \leq n$ be the another n edges. Now label the vertices $v_1, u_1, u_2, \cdots, u_n$, with the same rule as we do for path in Theorem 2.7 and label the vertices v_i, $i = 2, 3, \cdots, n$ as follows,

$$f(v_i) = f(u_{i-1})$$

Then, we get the consecutive triangular numbers as the edge values, but not different. □

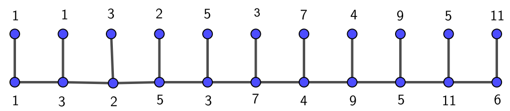


Figure 4: Comb with 22 vertices as a Weak Tri-Edge $\mathbb{Z}_{12}-$Garph.

Figure 4 shows that the comb graph with 22 vertices is a Weak Tri-Edge $\mathbb{Z}_{12}-$Garph.

Theorem 2.13. *The graph Double Comb with* $3n$ *vertices is a Weak Tri-Edge* $\mathbb{Z}_{T_{2n+1}+1}-$*Graph,* $\forall n$.

Proof. Let $P_n : u_1, u_2, \cdots, u_n$ be the path and let $e_i = u_iu_{i+1}$ $(1 \leq i \leq n-1)$ be the edges of the P_n. Let $v_1, v_2, \cdots, v_n$ be the pendant vertices adjacent to $u_1, u_2, \cdots, u_n$ respectively on one side and let $t_i = u_iv_i$, $1 \leq i \leq n$ be the edges. Let $w_1, w_2, \cdots, w_n$ be the pendant vertices adjacent to $u_1, u_2, \cdots, u_n$ respectively on the other side and let $s_i = u_iw_i$, $1 \leq i \leq n$ be the edges.
Now, label the vertices u_i for $i = 1, 2, \cdots, n$, with $T_1 = 1$, and label the vertices v_i and w_i, $i = 1, 2, 3, \cdots, n$ as follows:

$$f(v_i) = T_{2i+1},$$

and

$$f(w_i) = T_{2i}.$$

Then, we get all the consecutive triangular numbers as the edge values. □

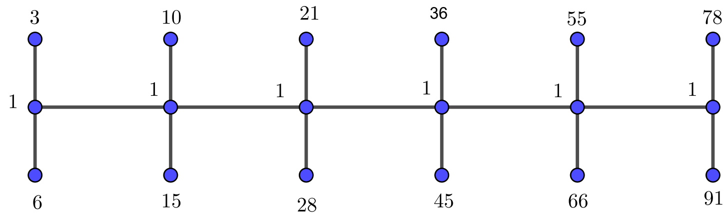


Figure 5: Double Comb with 18 vertices as a Weak Tri-Edge $\mathbb{Z}_{92}-$Garph.

Figure 5 shows that Double Comb with 18 vertices is a Weak Tri-Edge $\mathbb{Z}_{92}-$Garph.

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