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F-uphill Index of Graphs

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Abstract

In this study, we introduce the F-uphill index and its corresponding polynomial of a graph. Furthermore, we compute this index for some standard graphs, wheel graphs, gear graphs and helm graphs.

Keywords: F-uphill index; F-uphill polynomial; graphs.
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1. Introduction

In this paper, G denotes a finite, simple, connected graph, $V(G)$ and $E(G)$ denote the vertex set and edge set of G. The degree $d_G(u)$ of a vertex u is the number of vertices adjacent to u. A topological index is a numerical parameter mathematically derived from the graph structure. Several topological indices were defined by using vertex degree concept [1]. The Zagreb, Nirmala, Gourava, Sombor, Revan, Sombor, delta indices are the most degree based graph indices in Chemical Graph Theory, see [2-37]. Topological indices have their applications in various disciplines in Science and Technology [38]. A $u-v$ path P in G is a sequence of vertices in G, starting with u and ending at v, such that consecutive vertices in P are adjacent, and no vertex is repeated. A path $\pi = v_1, v_2, \ldots, v_{k+1}$ in G is a downhill path if for every i, $1 \leq i \leq k$, $d_G(v_i) \geq d_G(v_{i+1})$. A vertex v is downhill dominates a vertex u if there exists a downhill path originated from u to v. The downhill neighborhood of a vertex v is denoted by $N_{dn}(v)$ and defined as: $N_{dn}(v) = \{u : v$ downhill dominates u$\}$. The downhill degree $d_{dn}(v)$ of a vertex v is the number of downhill neighbors of v, see [39]. Recently, some downhill indices were studied in [40-44]. The uphill domination is introduced by Deering in [45].

A $u-v$ path P in G is a sequence of vertices in G, starting with u and ending at v, such that consecutive vertices in P are adjacent, and no vertex is repeated. A path $\pi = v_1, v_2, \ldots, v_{k+1}$ in G is a uphill path if for every i, $1 \leq i \leq k$, $d_G(v_i) \leq d_G(v_{i+1})$. A vertex v is uphill dominates a vertex u if there exists an uphill path originated from u to v. The uphill neighborhood of a vertex v is denoted by $N_{up}(v)$ and

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defined as: $N_{up}(v) = \{u : v \text{ uphill dominates u}\}$. The uphill degree $d_{up}(v)$ of a vertex v is the number of uphill neighbors of v, see [46]. The F-index [47] of a graph G is defined as

$$F(G) = \sum_{uv \in E(G)} \left(d_G(u)^2 + d_G(v)^2 \right).$$

We introduce the F-uphill index of a graph and it is defined as

$$FU(G) = \sum_{uv \in E(G)} \left(d_{up}(u)^2 + d_{up}(v)^2 \right).$$

Considering the F-uphill index, we introduce the F-uphill polynomial of a graph G and it is defined as

$$FU(G, x) = \sum_{uv \in E(G)} x^{d_{up}(u)^2 + d_{up}(v)^2}.$$

In this paper, the F-uphill index and its corresponding polynomial of certain graphs are computed.

2. Results for Some Standard Graphs

Proposition 2.1. *Let G be r-regular with n vertices and $r \geq 2$. Then*

$$FU(G) = nr(n-1)^2.$$

Proof. Let G be an r-regular graph with n vertices and $r \geq 2$ and $\frac{nr}{2}$ edges. Then $d_{up}(v) = n - 1$ for every v in G.

$$\begin{aligned} FU(G) &= \sum_{uv \in E(G)} \left(d_{up}(u)^2 + d_{up}(v)^2 \right) \\ &= \frac{nr}{2} \left((n-1)^2 + (n-1)^2 \right) \\ &= nr(n-1)^2. \end{aligned}$$

□

Corollary 2.2. *Let C_n be a cycle with $n \geq 3$ vertices. Then $FU(C_n) = 2n(n-1)^2$.*

Corollary 2.3. *Let K_n be a complete graph with $n \geq 3$ vertices. Then $FU(K_n) = n(n-1)^3$.*

Proposition 2.4. *Let G be r-regular with n vertices and $r \geq 2$. Then*

$$FU(G) = \frac{nr}{2(n-1)}.$$

Proof. Let G be an r-regular graph with n vertices and $r \geq 2$ and $\frac{nr}{2}$ edges. Then $d_{up}(v) = n - 1$ for every v in G.

$$\begin{aligned} FU(G,x) &= \sum_{uv\in E(G)} x^{d_{up}(u)^2+d_{up}(v)^2} \\ &= \frac{nr}{2} x^{(n-1)^2+(n-1)^2} \\ &= \frac{nr}{2} x^{2(n-1)^2}. \end{aligned}$$

□

Corollary 2.5. *Let C_n be a cycle with $n \geq 3$ vertices. Then $FU(C_n, x) = nx^{2(n-1)^2}$.*

Corollary 2.6. *Let K_n be a complete graph with $n \geq 3$ vertices. Then $FU(K_n) = \frac{n(n-1)}{2} x^{2(n-1)^2}$.*

Proposition 2.7. *Let P be a path with $n \geq 3$ vertices. Then*

$$FU(P_n) = 2\left(2n^2 - 10n + 13\right) + 2(n-3)^3.$$

Proof. Let P be a path with $n \geq 3$ vertices. We obtain two partitions of the edge set of P as follows:

$$E_1 = \{uv \in E(P) | d_{up}(u) = n-2, d_{up}(v) = n-3\}, \quad |E_1| = 2.$$

$$E_2 = \{uv \in E(P) | d_{up}(u) = d_{up}(v) = n-3\}, \quad |E_2| = n-3.$$

$$\begin{aligned} FU(P_n) &= \sum_{uv\in E(P_n)} \left(d_{up}(u)^2 + d_{up}(v)^2\right) \\ &= 2\left((n-2)^2 + (n-3)^2\right) + (n-3)\left((n-3)^2 + (n-3)^2\right) \\ &= 2\left(2n^2 - 10n + 13\right) + 2(n-3)^3. \end{aligned}$$

□

Proposition 2.8. *Let P_n be a path with $n \geq 3$ vertices. Then*

$$FU(P_n, x) = 2x^{2n^2-10n+13} + (n-3)\, x^{2(n-3)^2}.$$

Proof. We obtain

$$\begin{aligned} FU(P_n,x) &= \sum_{uv\in E(P_n)} x^{d_{up}(u)^2+d_{up}(v)^2} \\ &= 2x^{(n-2)^2+(n-3)^2} + (n-3)\, x^{(n-3)^2+(n-3)^2} \\ &= 2x^{2n^2-10n+13} + (n-3)\, x^{2(n-3)^2}. \end{aligned}$$

□

3. Results for Wheel Graphs

The wheel W_n is the join of C_n and K_1. Clearly W_n has $n+1$ vertices and $2n$ edges. Then W_n has two types of edges based on the uphill degree of the vertices of each edge as follows:

$$E_1 = \{uv \in E(W_n) | d_{up}(u) = 0, d_{up}(v) = n\}, \quad |E_1| = n.$$
$$E_2 = \{uv \in E(W_n) | d_{up}(u) = d_{up}(v) = n\}, \qquad |E_2| = n.$$

Theorem 3.1. *Let W_n be a wheel graph with $n+1$ vertices and $2n$ edges, $n \geq 4$. Then the F-uphill index of W_n is $FU(W_n) = 3n^3$.*

Proof. We deduce

$$\begin{aligned} FU(W_n) &= \sum_{uv \in E(W_n)} \left(d_{up}(u)^2 + d_{up}(v)^2\right) \\ &= n\left(0^2 + n^2\right) + n\left(n^2 + n^2\right) \\ &= 3n^3. \end{aligned}$$

□

Theorem 3.2. *Let W_n be a wheel graph with $n+1$ vertices, $n \geq 4$. Then the F-uphill polynomial of W_n is $FU(W_n, x) = nx^{n^2} + nx^{2n^2}$.*

Proof. We obtain

$$\begin{aligned} FU(W_n, x) &= \sum_{uv \in E(W_n)} x^{d_{up}(u)^2 + d_{up}(v)^2} \\ &= nx^{0^2+n^2} + nx^{n^2+n^2} \\ &= nx^{n^2} + nx^{2n^2}. \end{aligned}$$

□

4. Results for Gear Graphs

A bipartite wheel graph is a graph obtained from W_n with $n+1$ vertices adding a vertex between each pair of adjacent rim vertices and this graph is denoted by G_n and also called as a gear graph. Clearly, $|V(G_n)| = 2n+1$ and $|E(G_n)| = 3n$. A gear graph G_n is depicted in Figure 1.

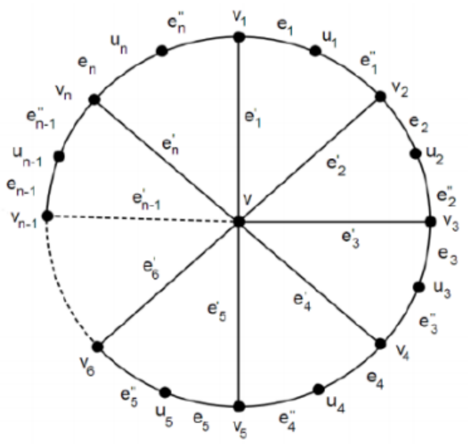

Figure 1: Gear graph G_n

Let G_n be a gear graph with $2n+1$ vertices, $3n$ edges, $n \geq 4$. Then G_n has two types of edges based on the uphill degree of the vertices of each edge as follows:

$$E_1 = \{u \in E(G_n) | d_{up}(u) = 1, d_{up}(v) = 0\}, \quad |E_1| = n$$
$$E_2 = \{u \in E(G_n) | d_{up}(u) = 1, d_{up}(v) = 3\}, \quad |E_2| = 2n.$$

Theorem 4.1. *Let G_n be a gear graph with $2n+1$ vertices, $3n$ edges, $n \geq 4$. Then the F-uphill index of G_n is $FU(G_n) = 21n$.*

Proof. We deduce

$$FU(G_n) = \sum_{uv \in E(G_n)} \left(d_{up}(u)^2 + d_{up}(v)^2\right)$$
$$= n\left(1^2 + 0^2\right) + 2n\left(1^2 + 3^2\right)$$
$$= 21n.$$

□

Theorem 4.2. *Let G_n be a gear graph with $2n+1$ vertices, $3n$ edges, $n \geq 4$. Then the F-uphill polynomial of G_n is $FU(G_n, x) = nx^1 + 2nx^{10}$.*

Proof. We deduce

$$FU(G_n, x) = \sum_{uv \in E(G_n)} x^{d_{up}(u)^2 + d_{up}(v)^2}$$
$$= nx^{1^2+0^2} + 2nx^{1^2+3^2}$$
$$= nx^1 + 2nx^{10}.$$

□

5. Results for Helm Graphs

The helm graph H_n is a graph obtained from W_n (with $n+1$ vertices) by attaching an end edge to each rim vertex of W_n. Clearly, $|V(H_n)| = 2n+1$ and $|E(H_n)| = 3n$. A graph H_n is shown in Figure 2.

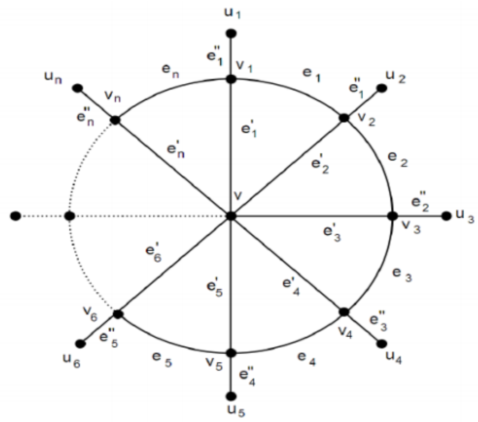

Figure 2: Helm graph H_n

Let H_n be a helm graph with $3n$ edges, $n \geq 5$. Then H_n has three types of edges based on the uphill degree of the vertices of each edge as follows:

$$E_1 = \{uv \in E(H_n) | d_{up}(u) = n+1, d_{up}(v) = n\}, \qquad |E_1| = n$$
$$E_2 = \{uv \in E(H_n) | d_{up}(u) = d_{up}(v) = n\}, \qquad |E_2| = n$$
$$E_3 = \{uv \in E(H_n) | d_{up}(u) = n, d_{up}(v) = 0\}, \qquad |E_3| = n$$

Theorem 5.1. *Let H_n be a helm graph with $2n+1$ vertices, $n \geq 5$. Then the F-uphill index of H_n is $FU(H_n) = 5n^3 + 2n^2 + n$.*

Proof. We obtain

$$\begin{aligned} FU(H_n) &= \sum_{uv \in E(H_n)} \left(d_{up}(u)^2 + d_{up}(v)^2 \right) \\ &= n\left((n+1)^2 + n^2 \right) + n\left(n^2 + n^2\right) + n\left(n^2 + 0^2\right) \\ &= 5n^3 + 2n^2 + n. \end{aligned}$$

□

Theorem 5.2. *Let H_n be a helm graph with $2n+1$ vertices, $3n$ edges, $n \geq 5$. Then the F-uphill polynomial of H_n is $FU(H_n, x) = nx^{2n^2+2n+1} + nx^{2n^2} + nx^{n^2}$.*

Proof. We deduce

$$FU(H_n, x) = \sum_{uv \in E(H_n)} x^{d_{up}(u)^2 + d_{up}(v)^2}$$

$$= nx^{(n+1)^2+n^2} + nx^{n^2+n^2} + nx^{n^2+0^2}$$
$$= nx^{2n^2+2n+1} + nx^{2n^2} + nx^{n^2}.$$

□

6. Conclusion

In this research work, the F-uphill index and its corresponding polynomial of a graph are defined. Also the F-uphill index and its corresponding polynomial of certain graphs are determined.

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