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Balancing Terms and Polynomial Order in the Trial Equation with Correlation of Different Ansatz in HBM

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Abstract

The Homogeneous Balance Method (HBM) is a reliable method solving Non-linear partial differential equations (NLPDEs). Within the framework of HBM, this manuscript investigates the degree of transformation that connects various Ansatz formulations with the polynomial order in the trial equation. Furthermore, a link is established between the trial equations under consideration and the Ansatz from different HBM-based approaches. It has been shown that different methods based on HBM are mirror images of each other. This equivalence can be achieved by appropriately adjusting the specific values of the balancing term and selecting the order of the Riccati equations.

Keywords: Trial Equation; Homogeneous Balance Method; Non-linear partial differential equations.

1. Introduction

One effective analytical method for obtaining exact solutions to Non-linear Partial differential equations (NPDEs) is the homogeneous balancing method (HBM). This technique has been widely used in mathematical physics and engineering since it was initially presented to systematically derive soliton solutions [1]. Many researchers later adopted the same approach for many NPDEs and systems of NPDEs, including: [2–4]. Now, it is crucial to understand the fundamentals of this HBM in order to comprehend the motivation behind this manuscript.

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1.1 Methodology

A general nth-order nonlinear partial differential equation can be written as:

$$V\left(u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial^2 u}{\partial x_i \partial x_j}, \ldots, \frac{\partial^n u}{\partial x_1^{p_1} \partial x_2^{p_2} \ldots \partial x_m^{p_m}}, \ldots\right) = 0. \tag{1}$$

Where, $V = u(t, x_1, x_2, \ldots, x_m)$ is the dependent variable., $t, x_1, x_2, \ldots, x_m$ are independent variables, $\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}$ are first-order partial derivatives, $\frac{\partial^2 u}{\partial x_i \partial x_j}$ represents second-order mixed derivatives, $\frac{\partial^n u}{\partial x_1^{p_1} \partial x_2^{p_2} \ldots \partial x_m^{p_m}}$ represents the highest-order mixed derivative, where $p_1 + p_2 + \cdots + p_m = n$. Assuming that $\xi = \sum_i k_i x_i - ct$, we look for wave solutions of the type $u = u(\xi)$, and hence the given NPDEs into an ordinary differential equation (ODE). Now there are following steps for getting a solution of this ODE by HBM.

Step 1 - Selection of Ansatz : or trial solution in series form, is the first step in solving a transformed ODE (see Table 1.).

Methods	Ansatz
Direct Algebraic Method (DAM), Auxiliary Equation Method (AEM), [6] Simplest Equation Method (SIEM) $\frac{G'}{G}$-expansion Method [8,9] F-Expension method (FEM) [6,12], Kudrashov Method (KM), Sub-ODE Method	$u(\xi) = \sum_{i=0}^{j} a_i \phi^i(\xi)$
EXP$-\phi(\xi)$ Expansion (EM) [6]	$u(\xi) = \sum_{i=0}^{j} a_i (\exp(-\phi(\xi)))^i$
Modified Simple Equation method (MSEM) [7]	$u(\xi) = \sum_{i=0}^{j} a_i \left(\frac{\phi'(\xi)}{\phi(\xi)}\right)^i$
Irrational trial equation Method (ITEM) [10]	$u = \sum_{i=0}^{k_1} a_i u_i + \left(\sum_{i=0}^{k_2} b_i u_i\right)\sqrt{\sum_{i=0}^{k_3} c_i u_i}$
EXP-function method (EXFM) [11]	$u(\xi) = \frac{\sum_{i=-a}^{b} a_i e^{(a\xi)}}{\sum_{i=-c}^{d} a_i e^{(b\xi)}}$

Table 1: Various Ansatz for exact solution for NLPDEs on HBM.

Here, the constants a_i, b_i, and c_i are used to determine and, and the balance term j will be determined by balancing detremine by Theorem 1, in accordance with the information provided in the following section.

Step 2 - Balancing of Linear and Non-Linear Terms:

Theorem 1.1. *For a non-linear ordinary differential equation, the balance between the highest-order derivative term and the non-linear term is given by:*

$$j = \frac{r - p}{q}. \tag{2}$$

Where r is the order of the highest derivative term, p is the derivative order in the non-linear term, and q is the exponent of u.

Proof.

- Nonlinear term: $u^q \left(\frac{d^p u}{d\xi^p}\right)$ has order $(q+1)j + p$,

- Derivative term: $\frac{d^r u}{d\xi^r}$ has order $j + r$,
- Balance the orders: $(q+1)j + p = j + r$, yielding $j = \frac{r-p}{q}$.

□

Step 3 - Choice of Trial Equation : In some HBM-based techniques, a single trial equation—an ordinary differential equation with an order lower than the transformed ODE and whose exact solution is known—will be employed. Any nonlinear ordinary differential equation of lower order than the transformed ODE of the original NLPDEs that has a known general solution is referred to as the simplest equation, and all associated techniques are called the **simplest equation method (SIEM)**.
The most simple and common trial equation is in the form of equation:

$$\phi'(\xi) = \sum_{i=0}^{k} f_i \phi^i(\xi) \tag{3}$$

Which is known as the Riccati equation for different values of k. Also, The classical Riccati equation typically refers to the case where $k = 2$, but higher-order versions are sometimes studied in nonlinear differential equations and mathematical physics. As k increases, the equation remains nonlinear and becomes more difficult to solve analytically. Special techniques, such as Lie symmetries, numerical approximations, or transformations to simpler forms, may be necessary. where the highest exponent of y is 3, making it more nonlinear than the Riccati equation.
Now, a different approach falls under HBM based on the selection of trial equations, as shown in the equations below (see Table 2).

Method	Trial Equation
DAM, EM	$\phi'(\xi) = \sum_{i=0}^{k} b_i \phi^i(\xi)$
AEM	$(\phi'(\xi))^2 = \sum_{i=0}^{k} b_i \phi^i(\xi)$
Sub Equation Method (SUEM)	$\phi'(\xi) = b_0 + (\phi(\xi))^2$
Extended simple equation method (ESEM)	$\phi'(\xi) = b_0 + b_1\phi(\xi) + b_2(\phi(\xi))^2$
Bernoulli Sub-ODE Method	$\phi'(\xi) + \lambda\phi(\xi) = \mu\phi^2(\xi)$
FEM	$(\phi'(\xi))^2 = b_1 + b_2\phi^2(\xi) + b_4\phi^4(\xi)$
$\frac{G'}{G}$-expansion Method [8,9]	$G'' + \lambda(G') + \mu G = 0$
EXP$-\phi(\xi)$ Expansion Method [6]	$\phi'(\xi) + b_0 e^{\phi(\xi)} + b_2 e^{-\phi(\xi)} = 0$

Table 2: Trial Equation for Various Methods for SIEM Based on HBM

2. Relation Between Balancing Term & Transformation Degree in Nonlinear PDE and Degree of Polynomial if Trial Equation

The majority of researchers have randomly selected various polynomial degrees in the trial equation, and the balancing terms in the majority of methods based on the Homogeneous Balance Method

(HBM) are derived from Theorem 1. In conclusion, no particular rule has been developed to illustrate the connection between the degree of transformation of the polynomial in the trial equation and the balancing terms. As a result, we found a relationship between the trial equation's balancing term and polynomial degree (see Theorem 2.1).

Theorem 2.1. *For positive integers, r, q, p, and s, where, r and q are the highest order linear and nonlinear terms, p is the order of an additional term in the nonlinear term, and s is the degree of the term with degree p in a given PDE, there exists a linear relation between the balancing* $\deg(V)$ *and the degree of the transformation polynomial=* $\deg(\phi)$, *given by:*

$$k = \deg(\phi) = j\left(\frac{s+q-1}{r-sp}\right) + 1, \tag{4}$$

Where $j = \deg(V)$ is the balancing degree of V, and $k = \deg(\phi)$ is the degree of the transformation polynomial ϕ.

Proof. Consider Anstaz for Direct Algebraic Method:

$$u(\xi) = \sum_{i=0}^{n} a_i \phi^i(\xi), \tag{5}$$

where f_i are constants and $\phi(\xi)$ is the transformation polynomial. The $\mathcal{D}(\phi)$ in $u(\xi)$ is j.
The $\mathcal{D}(\phi)$ in ϕ' is k. Then:

$$\mathcal{D}(\phi) \text{ in } u(\xi)' = (j-1)\mathcal{D}(\phi) \text{ in } \phi' = j + k - 1. \tag{6}$$

Similarly, for second derivatives:

$$\mathcal{D}(\phi) \text{ in } u(\xi)'' = j + 2k - 2. \tag{7}$$

Hence, for r^{th} derivative:

$$\mathcal{D}(\phi) \text{ in } u(\xi)^r = j + r(k-1). \tag{8}$$

For the nonlinear term $\left(u(\xi)^{(p)}\right)^s u(\xi)^q$, we have:

$$\mathcal{D}(\phi) \text{ in } (u(\xi)^{(p)})^s = [j + p(k-1)] \cdot s, \tag{9}$$

$$\mathcal{D}(\phi) \text{ in } (u(\xi)^q = j \cdot q. \tag{10}$$

Balancing the terms $\mathcal{D}(u(\xi)^{(r)})$ and $\mathcal{D}\left(u(\xi)^{(p)}\right)^s u(\xi)^q$ is given by:

$$k = 1 + j(\frac{s+q-1}{r-sp}). \tag{11}$$

One should remember $j \in \mathbb{N}$ and $k \in \mathbb{N} - \{1\}$. □

Example 2.2 (Korteweg-de Vries (KdV) Equation)**.** *Consider the case for the Korteweg-de Vries (KdV)* *equation, where we set $r = 2$, $q = 2$, and $s = 0$. Substituting these values into the above formula, we obtain:*

$$k = 1 + j \cdot \frac{1}{2}. \tag{12}$$

A few solution pairs (k, j) for this equation are:

$$\{(2,2), (3,4), (4,6), (5,8), (6,10), \dots\}.$$

Now, let's explore these solutions in more detail:

Sub case: $(k, j) = (2, 2)$

For this specific pair, the solution is given by:

$$V(\xi) = f_0 + f_1\phi + f_2\phi^2, \tag{13}$$

$$\phi' = a_0 + a_1\phi + a_2\phi^2. \tag{14}$$

In this case, the transformation polynomial ϕ' becomes a Riccati algebraic equation, whose solutions are well-known and have been used earlier in similar problems.

Subcase: $(k, j) = (3, 4)$

For the pair $(3, 4)$, the solution expands as:

$$V(\xi) = f_0 + f_1\phi + f_2\phi^2 + f_3\phi^3 + f_4\phi^4, \tag{15}$$

$$\phi' = a_0 + a_1\phi + a_2\phi^2 + a_3\phi^3. \tag{16}$$

Again, a similar transformation is applied, leading to a more complex algebraic equation, where solutions can be derived following the same method.

Theorem 2.3. *Solutions of the equation extended to the k^{th} degree are the $(k-1)^{th}$ roots of the algebraic Riccati equation (having $k = 2$), whose constant term is zero and the remaining coefficients are chosen appropriately.*

Proof.

$$\phi' = \sum_{i=0}^{k} b_i\phi^i$$

The above equation also represents a non-linear equation. So now, balancing ϕ^k and ϕ'.
Let m be the balancing degree here.

$$m + 1 = km \quad \Rightarrow \quad m = \frac{1}{k-1}$$

Put

$$\phi = V^{\frac{1}{k-1}}$$

Then,

$$\phi' = \frac{1}{k-1}\frac{V'}{V^{\frac{k-2}{k-1}}}$$

Substituting gives,

$$V' = (k-1)V^{\frac{k-2}{k-1}}\sum_{i=0}^{k} b_i V^{\frac{i}{k-1}}$$
$$V' = \left(\sum_{i=0}^{k} b_i V^{\frac{k+i-2}{k-1}}\right)\cdot(k-1)$$

Choosing b_i as:

$$b_i = \begin{cases} b_1 & i=1 \\ b_k & i=k \\ 0 & \text{otherwise} \end{cases}$$

Now,

$$V' = (k-1)\left(b_k V^2 + b_1 V\right)$$
$$V' = (k-1)b_k V^2 + (k-1)b_1 V$$

The above represents a Riccati equation with the constant coefficient as zero. So, the equation would give solutions. Thus, the solution of ϕ is:

$$\text{Solution of } \phi = (\text{Solution of } V)^{\left(\frac{1}{k-1}\right)}$$

□

3. Corelation of DifferentDirect Algebraic Approach Based on HBM

3.1 Equivalence of $e^{-\phi(\xi)}$ Expansion Method with Direct Algebraic Approach

In this case put $\phi(\xi) = e^{-\phi(\xi)}$, in ansatz for the the direct algebraic method, we get

$$u(\xi) = \sum_{i=0}^{j} \alpha_i (e^{-\phi(\xi)})^i. \tag{17}$$

Now substituting value of expression of ϕ in

$$\phi'(\xi) = \sum_{i=0}^{k} b_i \phi^i(\xi)$$

we get

$$-\phi'(\xi) = \sum_{i=0}^{k} b_i (e^{-\phi(\xi)})^{i-1}. \tag{18}$$

Case 1: For $k = 2$ and $b_1 = 0$ [7]

$$\phi'(\xi) + b_0 e^{\phi(\xi)} + b_2 e^{-\phi(\xi)} = 0.$$

3.2 Equivalence of G'/G Expansion Method with Direct Algebraic Approach

In this case put $\phi(\xi) = \frac{G'}{G}$, in ansatz for direct algebraic method:

$$u(\xi) = \sum_{i=0}^{j} b_i \Big(\frac{G'}{G}\Big)^i. \tag{19}$$

Now substituting value of ϕ in from general Riccati equation :

$$\Big(\frac{G'}{G}\Big)' = \sum_{i=0}^{k} b_i \Big(\frac{G'}{G}\Big)^i \Rightarrow \frac{GG'' - (G')^2}{G^2} = \sum_{i=0}^{k} b_i \Big(\frac{G'}{G}\Big)^i. \tag{20}$$

For $k = 2$,

$$\frac{GG'' - (G')^2}{G^2} = b_0 + b_1 \Big(\frac{G'}{G}\Big) + b_2 \Big(\frac{G'}{G}\Big)^2. \tag{21}$$

Taking $b_2 = -1$:

$$G'' - b_1(G') - b_0 G = 0 \;\text{ or }\; G'' + \lambda(G') + \mu G = 0. \tag{22}$$

Where, $\lambda = -b_1$ and $\mu = -b_0$ [8,9].

Note: For the $\frac{G'}{G}$ method, the only inherent constraint is that $G \neq 0$, but this corresponds to trivial solutions, which are typically disregarded. Therefore, constraints leading to trivial solutions can be effectively ignored in practical applications.

3.3 Subsidiary Ordinary Differential Equation method (sub-ODE method for short)

Instead of solving the transformed ODE directly, approximate its solution using a subsidiary equation, often a well-known simple and solvable ODE that is the sub-ODE. The Sub-ODE method can employ different types of auxiliary equations, including polynomial-type equations, Riccati equations, and more general solvable ODEs.

Simplest Equation Method:

The Simplest Equation Method is a specialized form of the Sub-ODE method that reduces the ODE to a known solvable equation, such as the Riccati or Bernoulli equation, assuming the solution can be expressed in terms of a simple known equation. The Sub-ODE method provides a broader framework that includes multiple approaches, whereas the Simplest Equation Method is a structured technique within this framework.

- **Sub Equation Method (SUEM)** $\phi'(\xi) = b_0 + (\phi(\xi))^2$.
- **Extended simple equation method (ESEM)** $\phi'(\xi) = b_0 + b_1\phi(\xi) + b_2(\phi(\xi))^2$.
- **Bernoulli Sub-ODE Method** $\phi'(\xi) + \lambda\phi(\xi) = \mu\phi^2(\xi)$.

Equivalence of F Expansion Method with DAM

In this case put $\phi(\xi) = \mathcal{F}(\xi)$, in ansatz for direct algebraic method:

$$V(\xi) = \sum_{i=0}^{j} a_i \mathcal{F}^i(\xi). \tag{23}$$

Now, substituting value of ϕ in

$$\mathcal{F}'(\xi) = \sum_{i=0}^{k} b_i \mathcal{F}^j(\xi). \tag{24}$$

Now take $k = 2$ and squaring both sides of above equation:

$$(\mathcal{F}'(\xi))^2 = b_2^2\mathcal{F}^4(\xi) + 2b_1b_2\mathcal{F}^3(\xi) + (2b_2b_0 + b_1^2)\mathcal{F}^2(\xi) + 2b_1b_0\mathcal{F}(\xi) + b_0^2. \tag{25}$$

Case 1 - Auxiliary Equation Method: When $b_0 = 0$ then,

$$(\mathcal{F}'(\xi))^2 = b_2^2\mathcal{F}^4(\xi) + 2b_1b_2\mathcal{F}^3(\xi) + b_1^2\mathcal{F}^2(\xi).$$

The above can be re-written as [5]:

$$(\mathcal{F}'(\xi))^2 = A\mathcal{F}^4(\xi) + B\mathcal{F}^3(\xi) + C\mathcal{F}^2(\xi). \tag{26}$$

Case 2 - Sardar sub equation method: When $b_1 = 0$ then

$$(\mathcal{F}'(\xi))^2 = b_2^2\mathcal{F}^4(\xi) + 2b_2b_0\mathcal{F}^2(\xi) + b_0^2$$

The above can be re-written as [12]:

$$\mathcal{F}'(\xi) = \pm\sqrt{A\mathcal{F}^4(\xi) + C\mathcal{F}^2(\xi) + E} \tag{27}$$

4. Concluding Remarks of Ansatz in HBM

One dependable technique for resolving non-linear partial differential equations (NLPDEs) is the homogeneous balance method (HBM). This manuscript explores the degree of transformation between different Ansatz formulations and the polynomial order in the trial equation within the context of

HBM. Additionally, a connection is made between the trial equations in question and the Ansatz from various HBM-based methodologies. Various HBM-based techniques have been demonstrated to be mirror versions of one another. By carefully choosing the order of the Riccati equations and modifying the precise values of the balancing term, this equivalency can be reached. Methods such as the exponential function approach provide for a wider variety of possible outcomes because the function $f(\xi)$ is left unbounded before the answer is obtained. On the other hand, an ansatz of the form $\log(f(\xi))$ naturally limits the range of possible solutions by imposing constraints on $f(\xi)$ even before it solves. Since it allows for more flexibility in deriving solutions, it is often better to choose an ansatz with less previous constraints.

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