International Journal of
Mathematics And its Applications
ISSN: 2347-1557

Int. J. Math. And Appl., **13(3)**(2025), 99–107
Available Online: http://ijmaa.in

On Degree of Approximation of Equi - Convergent Series Associated With Fourier Series

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Abstract

This article studies the rate of convergence of equi - convergence series for functions belonging to $Lip(\omega, p)$, $p > 1$ class. Previously Salem and Zygmond have studied the equi - convergence of the series associated with certain integrals for functions of bounded variation class of monotonic type and which is belonging to Lipschitz class.

Keywords: Fourier series; Equi - Convergence; $Lip(\omega, p)$ class; Monotonic Type 1.
2020 Mathematics Subject Classification: 41A25, 42A24.

1. Introduction

In the extensive area of approximation theory, the study of trigonometric approximation theory is both practically and mathematically significant. It is crucial both mathematical sciences like approximation of functions and solution of integral equations, as well as in engineering sciences such as signal processing, image processing, and computer-aided geometric design. The degree of approximation of functions are studied by many researchers [1-6, 8, 9], where the functions belonging to Lip α, Lip(α, p) class or BV class generalized Lipschitz class using various means through trigonometric Fourier approximation. Salem and Zygmond [7] have studied the equi - convergence of the series associated with certain integrals for functions of bounded variation class of monotonic type and which is belonging to Lipschitz class.

2. Definitions and Notations

Suppose that the function $f \in L_p[0, 2\pi]$ for $p \geq 1$ and is 2π periodic then the Fourier series of f at x is defined as follows:

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) = \sum_{k=1}^{\infty} B_k(x) \tag{1}$$

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and its corresponding conjugate series is

$$\sum_{k=1}^{\infty} (a_k \sin kx - b_k \cos kx) = \sum_{k=1}^{\infty} A_k(x) \tag{2}$$

We can write

$$\varphi(x,t) = f(x+t) + f(x-t) - 2f(x)$$
$$\Psi(x,t) = f(x+t) - f(x-t)$$

3. Modulus of Continuity

Suppose ω is the modulus of continuity if it satisfies the following

$$\|f(.+t) - f(.)\|_p = \begin{cases} O(\omega(|t|)) \\ o(\omega(|t|)) \end{cases}$$

Then for the first case $f \in \text{Lip}(\omega, p)$, $p \geq 1$. In second $f \in \text{lip}(\omega, p), p \geq 1$. For $0 < \alpha \leq 1$, and $p \geq 1$, if $\omega(t) = t^\alpha$ then Lip (ω, p) class reduces to Lip (α, p) class.

Monotonic Type: A function f is said to be monotonic type [7] if for some constant C the Function $f(x) + Cx$ is either monotonic increasing or monotonic decreasing.

$$g(x,t,u) = \Psi(x,t+u) + \Psi(x,t-u) - 2\Psi(x,t) \tag{3}$$

$$m = \left[\frac{\log n}{\log 2}\right], \eta = \frac{1}{2^m} \tag{4}$$

$$D_n(t) = \frac{1}{2} + \sum_{k=1}^{n} \cos kt = \frac{\sin\left(n + \frac{1}{2}\right)t}{2\sin(t/2)} \tag{5}$$

From [10], we have

$$\Psi(x,t) \sim -2\sum_{k=1}^{n} B_k(x) \sin kt \tag{6}$$

$$S_n(x,t) = -\sum_{k=1}^{n} 2\sin kt B_k(x) \tag{7}$$

For $0 < \delta < 1$ and $0 < \theta < \infty$

$$I_\delta(x;\theta) = -\frac{1}{\pi}\Gamma(\delta+1)\cos\frac{\pi\delta}{2}\int_\theta^\infty \frac{\Psi(x,t}{t^{1+\delta}}dt$$

Suppose that $f \in C_{2\pi}$ and the function be bounded variation of monotonic type

$$\rho_n(.) = I_\delta\left(x; \frac{1}{n}\right) - \sum_{k=1}^{n} k^\delta B_k(x) \tag{8}$$

4. Main Theorem

Theorem 4.1. *Suppose that the function f is monotonic type and $f \in C_{2\pi}\,\mathrm{Lip}(\omega, p), p \geq 1$. For $0 < \delta \leq 1$*

$$\|\rho_n(.)\|_p = O(1)\frac{1}{n^{1-\delta}} \quad \int_{\frac{\pi}{n}}^{\pi} \frac{\omega(u)}{u^2}du$$

where

$$\rho_n(x) = I_\delta\left(x; \frac{1}{n}\right) - \sum_{k=1}^{n} k^\delta B_k(x)$$

The following lemmas are required to prove the theorem

Lemma 4.2. *If $f \in C_{2\pi} \cap \mathrm{Lip}(\omega, p), p \geq 1$ and $\Psi(x,t)$ and $g(x,t,u)$ are defined above then we get*

(i). $\|g(x,t,u)\|_p = O(\omega(u))$

(ii). $\|\psi(x,t+u) - \psi(x,t-u)\|_p = O(\omega(u))$

Proof.

(i). As $g(x,t,u) = \Psi(x,t+u) + \Psi(x,t-u) - 2\Psi(x,t)$. Applying Minkowski's inequality in above inequality we get,

$$\begin{aligned}
\|g(x,t,u)\|_p &= \|\Psi(x,t+u) + \Psi(x,t-u) - 2\Psi(x,t)\|_p \\
&\leq \|f(x+t+u) - f(x+t)\|_p - \|f(x-t) - f(x-t+u)\|_p \\
&+ \|f(x+t-u) - f(x+t)\|_p - \|f(x-t) - f(x-t+u)\|_p \\
&= O(\omega(u)) \qquad (3.1)
\end{aligned}$$

(ii). In the similar manner applying Minkowski's inequality

$$\begin{aligned}
\|\Psi(x,t+u) + \Psi(x,t-u)\|_p &\leq \|f(x+t+u) - f(x+t)\|_p - \|f(x-t) - f(x-t+u)\|_p \\
&+ \|f(x+t) - f(x+t-u)\|_p - \|f(x-t+u) - f(x-t)\|_p \\
&= O(\omega(u))
\end{aligned}$$

This completes the proof of Lemma 4.2.

□

Lemma 4.3. *Suppose the hypothesis of Theorem 4.1 holds. Then we can prove the following*

$$\|S_n(.;t) - \psi(.;t)\|_p = O(1)\left(\frac{1}{n}\right)\int_{\frac{\pi}{n}}^{\pi} \frac{\omega(u)}{u^2}du$$

Proof.

$$\begin{aligned}
S_n(x;t) &= -\sum_{k=1}^{n} 2\sin kt \left(-\frac{1}{2\pi}\int_{-\pi}^{\pi}\Psi(x,u)\sin kudu\right) \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\Psi(x,u)\sum_{k=1}^{n}[\cos k(u-t)-\cos(u+t)]du \\
&= \frac{1}{\pi}\int_{0}^{\pi}[\Psi(x,t+u)+\Psi(x,t-u)]D_n(u)du \\
&= \frac{1}{\pi}\int_{0}^{\pi} g(x,t,u)D_n(u)du+\Psi(x,t)
\end{aligned}$$

which follows that

$$\begin{aligned}
S_n(x;t)-\psi(x,t) &= \frac{1}{\pi}\left(\int_{0}^{\eta}+\int_{\eta}^{\pi}\right)\int_{0}^{\pi} g(x,t,u)D_n(u)du \\
&= \frac{1}{\pi}[I(x,t)+J(x,t)] \qquad (9)
\end{aligned}$$

For the first integral applying Lemma 4.2 (i), we get

$$\begin{aligned}
\|I(.,t)\|_p &= \int_{0}^{\eta} O(\omega(u))\,|D_n(u)|\,du \\
&= O(1)\omega(\eta)n\eta = O(1)\omega\left(\frac{1}{n}\right) \qquad (10)
\end{aligned}$$

Since the function is of monotonic type there exists a constant C so that $F(x) = f(x) + Cx$ is either never decreasing or increasing in $(-\infty, \infty)$. Now we assume that F is increasing in similar manner we can prove for the decreasing case. In the replacement of $f(x)$ by $F(x) - Cx$, we obtain

$$\begin{aligned}
g(x,t,u) &= [F(x+t+u)-F(x+t)]-[F(x-t+u)-F(x-t)] \\
&\quad +[F(x-t)-F(x-t-u)]-[F(x+t)-F(x+t-u)] \qquad (11)
\end{aligned}$$

Using above we get

$$\begin{aligned}
J(x,t) &= \int_{\eta}^{\pi}[F(x+t+u)-F(x+t)]D_n(u)du+\int_{\eta}^{\pi}[F(x-t)-F(x-t-u)]D_n(u)du \\
&\quad -\int_{\eta}^{\pi}[F(x-t+u)-F(x-t)]D_n(u)du-\int_{\eta}^{\pi}[F(x+t)-F(x+t-u)]D_n(u)du \\
&= J_1(x,t)+J_2(x,t)+J_3(x,t)+J_4(x,t) \qquad (12)
\end{aligned}$$

We can write

$$J_1(x,t) = \sum_{j=1}^{m}\int_{\pi/2^j}^{\pi/2^{j+1}}[F(x+t+u)-F(x+t)]D_n(u)du = \sum_{j=1}^{m} Q_j(x,t) \qquad (13)$$

where

$$Q_j(x,t) = \int_{\pi/2^j}^{\pi/2^{j+1}}[F(x+t+u)-F(x+t)]\frac{\sin\left(n+\frac{1}{2}\right)u}{2\sin u/2}du \qquad (14)$$

Since $F(x+t+u)-F(x+t)$ is increasing in u and $\frac{1}{2\sin(u/2)}$ is decreasing in u. Repeated application of Mean-Value Theorem, we get

$$Q_j(x,t) = \frac{1}{2\sin\frac{1}{2^{j+1}}}\int_{1/2^j}^{\xi}[F(x+t+u)-F(x+t)]du\sin\left(n+\frac{1}{2}\right)udu$$

where $\frac{1}{2^j} < \xi < \frac{1}{2^{j+1}}$

$$\begin{aligned}
&= \frac{1}{2\sin\frac{1}{2^{j+1}}}[F(x+t+\xi)-F(x+t)]\int_{\xi_1}^{\xi}\sin\left(n+\frac{1}{2}\right)udu, \frac{1}{2^j} < \xi_1 < \xi \\
&= \frac{[F(x+t+\xi)-F(x+t)]}{2\sin\frac{1}{2^{j+1}}}\left[\frac{\cos\left(n+\frac{1}{2}\right)\xi_1 - \cos\left(n+\frac{1}{2}\right)\xi}{\left(n+\frac{1}{2}\right)}\right]
\end{aligned}$$

From which it follows that for $j = 1, 2\ldots m$. and $\sin u \geq \frac{2u}{\pi}$

$$\begin{aligned}
\|Q_J(.,t)\|_p &\leq \frac{2}{\left(n+\frac{1}{2}\right)}\frac{\|F(.+t+\xi)-F(.+t)\|_p}{2.\frac{2}{\pi}\frac{1}{2^{j+1}}} \\
&= O(1)\frac{2^j}{n}\omega(\xi) = O(1)2^j\omega\left(\frac{1}{2^{j-1}}\right) \\
\|J_1(.,t)\|_p &\leq \sum_{j=1}^{m}\|Q_J(.,t)\|_p \\
&= O(1)\frac{1}{n}\sum_{j=1}^{m}\omega\left(\frac{1}{2^{j-1}}\right)2^j \\
&= O(1)\frac{1}{n}\int_{\frac{\pi}{n}}^{\pi}\frac{\omega(u)}{u^2}du \qquad (15)
\end{aligned}$$

Using the above we can prove that, for $k = 2, 3$ and 4

$$\|J_k(.,t)\|_p = O(1)\frac{1}{n}\int_{\frac{\pi}{n}}^{\pi}\frac{\omega(u)}{u^2}du$$

Combining (15) and (4), we get

$$\|J(.,t)\|_p \leq \sum_{k=1}^{4}\|J_k(.,t)\|_p = O(1)\frac{1}{n}\int_{\frac{\pi}{n}}^{\pi}\frac{\omega(u)}{u^2}du \qquad (16)$$

Incorporating (5) and (16) in (9), we obtain

$$\begin{aligned}
\|S_n(.;t)-\psi(.;t)\|_p &= O(1)\omega\left(\frac{1}{n}\right) + O(1)\int_{\frac{\pi}{n}}^{\pi}\frac{\omega(u)}{u^2}du \\
&= O(1)\left(\frac{1}{n}\right)\int_{\frac{\pi}{n}}^{\pi}\frac{\omega(u)}{u^2}du
\end{aligned}$$

According to the monotonicity of ω,

$$\int_{\frac{\pi}{n}}^{\pi}\frac{\omega(u)}{u^2}du \geq \omega\left(\frac{\pi}{n}\right)\int_{\frac{\pi}{n}}^{\pi}\frac{du}{u^2}$$

$$= \omega\left(\frac{\pi}{n}\right)\left(\frac{n-1}{\pi}\right)$$

□

Lemma 4.4. *Suppose f is monotonic type and $f \in C_{2\pi} \cap \mathrm{Lip}(\omega, p), p \geq 1$ then*

$$\|S_n(.;t)\|_p = \left\|\frac{\partial}{\partial t}\left(S_n(.;t)\right)\right\|_p = O(1)\int_{\frac{\pi}{n}}^{\pi} \frac{\omega(u)}{u^2} du$$

Proof.

$$\begin{aligned}
\frac{\partial}{\partial t}\left((S_n(x;t))\right) &= -\sum_{k=1}^{n} 2k\cos ktB_k(x) \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Psi(x,u)\sum_{k=1}^{n} k[\mathrm{sink}(u+t) - \sin k(u-t)]du \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Psi(x,u)\frac{\partial}{\partial u}D_n(u+t)du - \frac{1}{2\pi}\int_{-\pi}^{\pi} \Psi(x,u)\frac{\partial}{\partial u}D_n(u-t)du \\
&= -\frac{1}{\pi}\left[\int_0^{\eta} + \int_{\eta}^{\pi}\right][\Psi(x,t+u) + \Psi(x,t-u)]D_n(u)du \\
&= -\frac{1}{\pi}[P(x,t) + Q(x,t)] \qquad (17)
\end{aligned}$$

Applying Lemma 4.2 (ii), we obtain

$$\begin{aligned}
\|P(.,t)\|_p &\leq \int_0^{\eta} \|\Psi(x,t+u) + \Psi(x,t-u)\|_p \left|D_n(u)'\right| du \qquad (18) \\
&= O(1)\omega(u)\int_0^{\eta} n^2 d\mathrm{u} = O(1)n\omega\left(\frac{1}{n}\right) \qquad (19) \\
Q(x,t) &= \int_{\eta}^{\pi}[F(x+t+u) - F(x-t-u) + F(x-t+u) - F(x+t-u) - 4Cu]D_n'(u)du \\
&= \int_{\eta}^{\pi}[F(x+t+u) - F(x+t)]D_n'(u)du + \int_{\eta}^{\pi}[F(x-t+u) - F(x-t)]D_n'(u)du \\
&+ \int_{\eta}^{\pi}[F(x-t) - F(x-t-u)]D_n'(u)du + \int_{\eta}^{\pi}[F(x+t) - F(x+t-u)]D_n'(u)du \\
&- 4C\int_{\eta}^{\pi} uD_n{}'(u)du \\
&= \sum_{i=1}^{4} P_i(x,t) - 4C\int_{\eta}^{\pi} uD_n'(u)du \qquad (20)
\end{aligned}$$

We can write

$$P_i(x,t) = \sum_{j=1}^{m} K_j(x,t)$$

where

$$K_j(x,t) = \int_{\pi/2^j}^{\pi/2^{j-1}} [F(x+t+u) - F(x+t)]D_n'(u)du \qquad (21)$$

Since $F(x+t+u) - F(x+t)$ is increasing in u, applying Mean-Value Theorem, we have, for $\pi/2^j <$

$\zeta < \pi/2^{j-1}$

$$K_j(x,t) = \int_{\pi/2^j}^{\zeta} [F(x+t+u) - F(x+t)] D_n'(u) du$$

After simplification we get

$$\begin{aligned}
\left\| K_j(.,t) \right\|_p &= O(1)\omega\left(\frac{1}{2^{j-1}}\right)\left| D_n(\zeta) - D_n\left(\frac{\pi}{2^j}\right)\right| \\
&= O(1) 2^j \omega\left(\frac{1}{2^{j-1}}\right) \\
\|P_i(.,t)\|_p \le \sum_{j=1}^{m} \left\| K_j(.,t) \right\|_p &= O(1) \sum_{j=1}^{m} 2^j \omega\left(\frac{1}{2^{j-1}}\right) \\
&= O(1) \int_{\pi/n}^{\pi} \frac{\omega(u)}{u^2} du \qquad (22)
\end{aligned}$$

In similar manner we can prove for $i = 2, 3, 4$

$$\|P_i(.,t)\|_p = O(1) \int_{\pi/n}^{\pi} \frac{\omega(u)}{u^2} du \qquad (23)$$

Integrating we obtain,

$$\begin{aligned}
4C \int_{\eta}^{\pi} u D_n'(u) du &= 4C\left[\pi\left(D_n(\pi) - \eta\right)\left(D_n(\eta)\right)\right] - \int_{\eta}^{\pi} D_n(u) du \\
&= O(1) \qquad (24)
\end{aligned}$$

Applying (23) and (24) in (20), we obtain

$$\|P_i(.,t)\|_p = O(1) \int_{\pi/n}^{\pi} \frac{\omega(u)}{u^2} du$$

Plugging (19) and (4) in (17)

$$\left\| \frac{\partial}{\partial t}\left(S_n(.;t)\right)\right\|_p = O(1) \int_{\frac{\pi}{n}}^{\pi} \frac{\omega(u)}{u^2} du$$

This completes the Lemma 4.4. □

Lemma 4.5. *Using [7], we get*

$$\int_0^{\infty} \frac{S_n(x,t)}{t^{1+\delta}} dt = \frac{\pi k^{\alpha}}{2\cos(\pi\alpha/2)\Gamma(\alpha+1)}$$

Proof of Theorem 4.1. We obtain from (11) and Lemma 4.5,

$$\begin{aligned}
\int_0^{\infty} \frac{S_n(x,t)}{t^{1+\delta}} dt &= -\sum_{k=1}^{n} 2B_k(x) \int_0^{\infty} \frac{\sin kt}{t^{1+\delta}} dt \\
&= -\frac{\pi}{\cos\frac{\pi\delta}{2}\Upsilon(\delta+1)} \sum_{k=1}^{n} k^{\delta} B_k(x) \qquad (25)
\end{aligned}$$

Incorporating (8) and (25)

$$I_\delta\left(x,\frac{1}{n}\right)-\sum_{k=1}^{n}k^\delta B_k(x)=\frac{-\Upsilon(\delta+1)\cos\frac{\pi\delta}{2}}{\pi}\left[\int_{\frac{1}{n}}^{\infty}\frac{\Psi(x,t)}{t^{1+\delta}}-\int_0^\infty\frac{S_n(x,t)}{t^{1+\delta}}\right]dt$$
$$-\frac{\Upsilon(\delta+1)\cos\frac{\pi\delta}{2}}{\pi}\left[\int_{\frac{1}{n}}^{\pi}\frac{S_n(x,t)-\Psi(x,t)}{t^{1+\delta}}dt-\int_0^{\frac{1}{n}}\frac{S_n(x,t)}{t^{1+\delta}}\right]dt \tag{26}$$

Applying Lemma 4.3, we have

$$\left\|\int_{\frac{1}{n}}^{\infty}\frac{S_n(.,t)-\Psi(.,t)}{t^{1+\delta}}dt\right\|_p\le\int_{\frac{1}{n}}^{\infty}\frac{\|S_n(.,t)-\Psi(.,t)\|_p}{t^{1+\delta}}dt$$
$$=O(1)\frac{1}{n^{1-\delta}}\int_{\frac{\pi}{n}}^{\pi}\frac{\omega(u)}{u^2}du \tag{27}$$

As $S_n(x,0)=0$, we can get $S_n(.,t)=S_n(.x,t)-S_n(x,0)=tS_n'(x,\lambda)$, where $0<\lambda<t$. Using Lemma 4.4, we obtain

$$\left\|\int_0^{1/n}\frac{S_n(.,t)}{t^{1+\delta}}dt\right\|_p\le\int_0^{1/n}\frac{\|S_n(.,t)\|_p}{t^{1+\delta}}dt$$
$$=O(1)\left(\int_{\frac{\pi}{n}}^{\pi}\frac{\omega(u)}{u^2}du\right)\int_0^{1/n}\frac{dt}{t^\delta}$$
$$=O(1)\frac{1}{n^{1-\delta}}\int_{\frac{\pi}{n}}^{\pi}\frac{\omega(u)}{u^2}du \tag{28}$$

Plugging (27) and (28) in (26), we obtain

$$I_\delta\left(x,\frac{1}{n}\right)-\sum_{k=1}^{n}k^\delta B_k(x)=O(1)\frac{1}{n^{1-\delta}}\int_{\frac{\pi}{n}}^{\pi}\frac{\omega(u)}{u^2}du$$

This completes the proof of Theorem. □

Taking $\omega(t)=t^\alpha, 0<\alpha\le 1$:

Corollary 4.6. *If f is monotonic type and $f\in C_{2\pi}\cap \mathrm{Lip}(\alpha,p)$, $p\ge 1$, for $0<\alpha\le 1$, then*

$$\|\rho_n(.)\|_p=O(1)\begin{cases}\frac{1}{n^{\alpha-\delta}}, 0<\delta\le\alpha<1\\ \frac{\log n}{n^{1-\delta}}, 0<\delta<\alpha=1\end{cases}$$

Putting $p=\infty$, in Corollary 4.6, we get

Corollary 4.7. *If f is monotonic type and $f\in C_{2\pi}\cap \mathrm{Lip}\,\alpha, 0<\alpha\le 1$, then*

$$\|\rho_n(.)\|_p=O(1)\begin{cases}\frac{1}{n^{\alpha-\delta}}, 0<\delta\le\alpha<1\\ \frac{\log n}{n^{1-\delta}}, 0<\delta<\alpha=1\end{cases}$$

References

[1] A. Ronald, Devore and George G. Lorentz, *Constructive Approximation*, Springer-Verlag, Berlin, Germany, (1993).

[2] G. Das, A. Nath and B. K. Ray, *An estimate of the rate of convergence of Fourier series in the Generalized Hölder metric*, Narosa Publishing House, New Delhi, India, (2002).

[3] G. Das, T. Ghosh and B. K. Ray, *Degree of approximation of functions by their Fourier series in the Generalized Holder metric*, Proc . Indian Acad. Sci, 106(2)(1996), 139-153.

[4] M. Bhuyan, G. Das and B. K. Ray, *Rate of Convergence of Fourier Series in the generalized Holder metric by Rogosinski Mean*, Journal of Odisha Mathematical Society, 27(1 & 2)(2008), 21-36.

[5] M. V. Singh, M. L. Mittal and B. E. Rhoades, *Approximation of functions in the generalized Zygmund class using Hausdorf means*, Journal of Inequalities and Applications, 2017(1)(2017).

[6] R. Bojanic, *An estimate of the rate of convergence of Fourier series of functions of bounded variation*, Inst. Math. (Beogard), 26(1979), 57-60.

[7] R. Salem and A. Zygmund, *Capacity of sets and Fourier series*, Transaction American Mathematical Society, 59(1946), 23-41.

[8] S. Beuria, G. Das and B. K. Ray, *An estimate of the rate of convergence of K^{λ} means of the Fourier series of functions of bounded variation*, Journal of Indian Math.Soc.

[9] S. Lal, *Approximation of functions belonging to the generalized Lipschitz class by C^1N_p Summability method of Fourier series*, Appl. Math. Comput., 209(2009), 340-350.

[10] A. Zygmund, *Trigonometric series*, Volume I and II combined, Cambridge University Press, New York, (1993).