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A Class of Lightlike Hypersurface of Meta-metallic Pseudo-Riemannian Manifolds

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Abstract

The purpose of the present paper is to focus hypersurfaces using the meta-metallic-chi ratio. We study some properties of a structure induced on a hypersurface by a meta-metallic structure. We revisit the invariant hypersurfaces of a meta-metallic pseudo-Riemannian manifold, we deduce some properties of this type of hypersurface and we close with an example.

Keywords: Lightlike hypersurface; meta-metallic structure; pseudo-Riemannian manifolds.
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1. Introduction

The golden ratio often noted φ, is one of the most famous irrational numbers in mathematics. It is renowned for its aesthetic properties and its appearances in art, nature. But this number is not the only number of interest, for example the super golden ratio $\varphi_s = 1.465571231$ which is a solution of the equation $x^3 = x^2 + 1$, appears in contexts related to proportions that require a generalization of the golden ratio beyond the quadratic relationship. It is particularly relevant in the design of space and objects that seek to harmonize complex proportions, while maintaining an elegant and balanced geometric structure.
However, the metallic number forms a family of irrational numbers that generalizes the golden ratio. Each of them is defined by an equation of the form:

$$M_n = \frac{n + \sqrt{n^2 + 4}}{2},$$

with n a positive natural integer, the numbers corresponding to the case where $n = 1$, $n = 2$ and $n = 3$ are respectively the golden, silver and bronze ratios. In [1] Barllet proved that an important class of

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logarithmic spirals yields the golden meta-chi ratio. In this sense he constructs the meta-chi golden number given by:

$$\chi = \frac{1+\sqrt{4\phi+5}}{2\phi},$$

with $\phi = \frac{1+\sqrt{5}}{2}$ the golden number. The credit goes to Sahin [8] for introducing these new types of manifolds, constructed using the meta-golden-chi ratio and called them meta-golden Riemannian manifolds. Properly, Riemannian manifolds have important applications. One of such recent application is related to the modeling of the COVID pandemic, an approach in [2] developed by SBDIEM, which models similar epidemics. A metallic pseudo-Riemannian manifold is a differential manifold with a non-zero tensor field of type (1.1) that satisfies a number of conditions and is compatible with the pseudo-Riemannian metric. The growing importance of light-like geometry is motivated by its extensive use in mathematical physics, in general relativity.

In [7] a new type of manifold was introduced, inspired by the meta-metallic-Chi ratio defined as

$$\chi = \frac{q}{\wp} + \frac{p}{\chi},$$

with $\wp$ the metallic number and, called it meta-metallic manifolds. Then a meta-metallic pseudo-Riemannian manifold is a metallic pseudo-Riemannian manifold equipped with another tensor field of type (1.1) verifying a certain number of conditions and is compatible with the pseudo-Riemannian metric. These new manifolds open a great line towards other types of new manifolds. We can cite among others the meta-gold, meta-silver, meta-nickel, meta-alminium, meta-metallic-tangent, meta-metallic-complex, etc.

This work is divided into forth sections. The first one is devoted to this introduction and the second to the full of fundamental notions around meta-metallic-Chi ratio. The third section closes the properties of a lightlike hypersurface induced by a meta-metallic structure. Finally, the fourth one speaks of invariant light-type hypersurface.

2. Preliminary

Let the quadratic equation resulting from the meta-metallic-Chi ratio be:

$$\chi^2 - \frac{q}{\wp}\chi - p = 0, \tag{1}$$

with $\wp$ the metallic number. The equation (1) admits two solutions:

$$\chi_1 = \frac{q+\sqrt{4p\wp+p^2}}{2\wp} \quad \text{and} \quad \chi_2 = \frac{q-\sqrt{4p\wp+p^2}}{2\wp}.$$

This solutions satisfy the following relations:

$$\chi_1^2 = p + \frac{q}{\wp}\chi_1, \qquad \chi_2 = \frac{q}{\wp} - \chi_1, \qquad \chi_2^2 = p + \frac{q}{\wp}\chi_2.$$

Definition 2.1. *A nearly meta-metallic differential manifold is the give of a triplet $(\mathcal{M}^*, \mathcal{J}, \zeta)$ where $\mathcal{M}^*$ is a differential manifold, $\mathcal{J}$ a metallic structure and ζ an endomorphism of the tangent space, at any point p of the manifold $\mathcal{M}^*$. The endomorphism ζ satisfies the following relation:*

$$\mathcal{J}^2\mathbb{X} = p\mathcal{J}\mathbb{X} + q\mathbb{X}, \quad \forall\, \mathbb{X} \in \chi(\mathcal{M}^*). \tag{2}$$

Theorem 2.2. *The endomorphism ζ on the metallic manifold $(\mathcal{M}^*, \mathcal{J})$ is an almost meta-metallic structure if and only if:*

$$\zeta^2 = \mathcal{J}\zeta - p\zeta + pI \tag{3}$$

Proof. Suppose that the endomorphism ζ is a meta-metallic structure. Then, for all $\mathbb{X} \in \chi(\mathcal{M}^*)$, applying $\mathcal{J}$ to equation (2) one has:

$$\mathcal{J}^2\zeta^2\mathbb{X} = p\mathcal{J}^2\mathbb{X} + q\mathcal{J}\zeta\mathbb{X} \Longrightarrow p\mathcal{J}\zeta^2\mathbb{X} = p^2\mathcal{J}\mathbb{X} + pq\mathbb{X} + q\mathcal{J}\zeta\mathbb{X} \quad \text{(because } \mathcal{J} \text{ is metallic).}$$

From (2) one has:

$$\begin{aligned} p^2\mathcal{J}\mathbb{X} + pq\zeta\mathbb{X} + q\zeta^2\mathbb{X} &= p^2\mathcal{J}\mathbb{X} + pq\mathbb{X} + q\mathcal{J}\zeta\mathbb{X} \\ \zeta^2\mathbb{X} &= \mathcal{J}\zeta\mathbb{X} - p\zeta\mathbb{X} + p\mathbb{X}. \end{aligned}$$

Conversely, let us suppose that the endomorphism ζ verifies the relation (3). Then by applying $\mathcal{J}$ to (3) we have:

$$\mathcal{J}\zeta^2\mathbb{X} = \mathcal{J}^2\zeta\mathbb{X} - p\mathcal{J}\zeta\mathbb{X} + p\mathcal{J}\mathbb{X} \Longrightarrow \mathcal{J}\zeta^2\mathbb{X} = p\mathcal{J}\mathbb{X} + q\zeta\mathbb{X} \quad \text{(because } \mathcal{J} \text{ is metallic).}$$

□

Definition 2.3. *Let g^* be a pseudo-Riemannian metric on $\mathcal{M}^*$, if g^* is compatible with ζ ie:*

$$g^*(\zeta\mathbb{X}, \mathbb{Y}) = g^*(\mathbb{X}, \zeta\mathbb{Y}) \Longleftrightarrow g^*(\zeta\mathbb{X}, \zeta\mathbb{Y}) = g^*(\mathcal{J}\mathbb{X}, \zeta\mathbb{Y}) - pg^*(\mathbb{X}, \zeta\mathbb{Y}) + pg^*(\mathbb{X}, \mathbb{Y}),$$

then the quadruplet $(\mathcal{M}^*, \mathcal{J}, \zeta, g^*)$ is called almost meta-metallic Riemannian (or pseudo-Riemannian) manifold.

Definition 2.4. *Let $\mathcal{M}^*$ be a pseudo-Riemannian manifold of dimension $n+2$ with index q $(0 < q < n+1)$*

and $\mathcal{H}$ be a hypersurface of $\mathcal{M}^*$ whose induced metric is

$$\mathrm{g} = \mathrm{g}^*|_{\mathcal{H}}.$$

Then $\mathcal{H}$ is called a lighttype hypersurface of $\mathcal{M}^*$ if the induced metric on the hypersurface is of rank n and the orthogonal supplementary $(\mathcal{TM}^*)^{\perp}$ of $(\mathcal{TM}^*)$ is given by:

$$(\mathcal{TM}^*)^{\perp} = \bigcup_{x \in M^*} \{\mathbb{X}_x \in (\mathcal{T}_{\S}\mathcal{M}^*) : \mathrm{g}_x(\mathbb{X}_x, \mathbb{Y}_x) = 0, \forall\, \mathbb{Y}_x \in \Gamma((\mathcal{T}_{\S}\mathcal{M}^*))\}$$

is a rank one distribution on $\mathcal{M}^*$.

Theorem 2.5. *([5]) Let $(\mathcal{H}, \mathrm{g}, S(\mathcal{TH}))$ be a lightlike hypersurface of a pseudo-Riemannian manifold $\mathcal{M}^*$. Then, there exists a unique distribution $ltr(\mathcal{TH})$ of $(\mathcal{TM}^*)$ such that, for any vector field η of $Rad(\mathcal{TH})$ on a neighborhood $\mathcal{U}$ continuous in $\mathcal{H}$, there exists a vector field $\mathbb{N}$ of $ltr(\mathcal{TH})$ on $\mathcal{U}$ satisfying:*

$$\mathrm{g}^*(\mathbb{N}, \mathbb{W}) = 0 = \mathrm{g}^*(\mathbb{N}, \mathbb{N}), \quad \mathrm{g}^*(\mathbb{N}, \eta) = 1,$$

for all $\mathbb{W} \in \Gamma(\mathcal{TH})|_{\mathcal{U}}$.

Thus, we have the following decompositions:

$$\mathcal{TH} = S(\mathcal{TH}) \perp Rad(\mathcal{TH}), \quad (\mathcal{TM}^*) = \mathcal{TH} \oplus ltr(\mathcal{TH}) = S(\mathcal{TH})) \perp \big(S(\mathcal{TH}) \perp Rad(\mathcal{TH})\big).$$

The radical space is give by

$$Rad(\mathcal{TH}_x) = \mathcal{TH}_x \cap \mathcal{TH}_x^{\perp}.$$

and the distribution $ltr(\mathcal{TH})$ is called the lightlike transverse distribution.
Let ∇^* be the Levi-civita connection on $\mathcal{M}^*$, the Gauss-Weingarten formulas are respectively given by:

$$\nabla^*_{\mathbb{X}}\mathbb{Y} = \nabla_{\mathbb{X}}\mathbb{Y} + \mathbb{H}(\mathbb{X}, \mathbb{Y}), \tag{4}$$

$$\nabla^*_{\mathbb{X}}\mathbb{N} = -\mathbb{A}_{\mathbb{N}}\mathbb{X} + \nabla^t_{\mathbb{X}}\mathbb{N}, \tag{5}$$

for all vector fields $\mathbb{X},\ \mathbb{Y} \in \Gamma(\mathcal{TM}^*)$ and the vector field $\mathbb{N} \in \Gamma(ltr(\mathcal{TM}^*))$.
Let $(\mathcal{M}^*, \mathcal{J}, \zeta, \mathrm{g}^*)$ be a meta-metallic pseudo-Riemannian manifold and $\mathcal{H}$ be a lightlike hypersurface of $\mathcal{M}^*$. Consider a vector field $\sharp$ of type (1.1) and $\flat$ a 1-form on $\mathcal{M}^*$. For all $\mathbb{X} \in \Gamma((\mathcal{TM}^*))$ we have:

$$\zeta\mathbb{X} = \sharp\mathbb{X} + \flat(\mathbb{X})\mathbb{N}, \qquad \mathcal{J}\mathbb{X} = P\mathbb{X} + u(\mathbb{X})\mathbb{N}, \tag{6}$$

and

$$\zeta\mathbb{N} = \mathbb{V} + \flat(\mathbb{N})\mathbb{N}, \qquad \mathcal{J}\mathbb{N} = \mathbb{U} + u(\mathbb{N})\mathbb{N}, \tag{7}$$

with $\mathbb{U}, \mathbb{V} \in \Gamma((\mathcal{T}\mathcal{M}^*))$, $\mathbb{N} \in \Gamma(ltr((\mathcal{T}\mathcal{M}^*))$, $\flat(\cdot) = \mathrm{g}^*(\cdot, \zeta\eta)$, $u(\cdot) = \mathrm{g}^*(\cdot, \mathcal{J}\eta)$ and

$$\begin{aligned} \sharp : \Gamma((\mathcal{T}\mathcal{M}^*)) &\longrightarrow \Gamma((\mathcal{T}\mathcal{M}^*)) \\ \mathbb{X} &\longmapsto \sharp\mathbb{X} = (\zeta\mathbb{X})^{\top}. \end{aligned}$$

In this case, if we assume that in the second parts of equations (6) and (7) that $u(\mathbb{X}) = 0$ and $\mathbb{U} = 0$ then $\mathcal{J}\mathbb{X} = P\mathbb{X}$.

Consider the Nijenhuis tensor given by:

$$N_{\zeta}(\mathbb{X}, \mathbb{Y}) = \mathcal{J}\zeta[\mathbb{X}, \mathbb{Y}] - p\zeta[\mathbb{X}, \mathbb{Y}] + p[\mathbb{X}, \mathbb{Y}] + [\zeta\mathbb{X}, \zeta\mathbb{Y}] + \zeta[\zeta\mathbb{X}, \mathbb{Y}] - \zeta[\mathbb{X}, \zeta\mathbb{Y}].$$

If N_{ζ} is identically null, then the structure ζ is integrable $(\nabla = 0 \Longrightarrow \nabla\mathcal{J} = 0)$ (see [7] Theorem 3.1.1). Thus, the almost meta-metallic pseudo-Riemannian manifold $\mathcal{M}^*$ will simply be called a meta-metallic pseudo-Riemannian manifold.

3. Lightlike Hypersurfaces of a Meta-metallic Pseudo-Riemannian Manifold

In this part we determine the properties of the induced structure on a hypersurface by a meta-metallic pseudo-Riemannian structure.

Proposition 3.1. *Let $(\mathcal{M}^*, \mathcal{J}, \zeta, \mathrm{g}^*)$ be a meta-metallic pseudo-Riemannian manifold and $\mathcal{H}$ be a lightlike hypersurface of $\mathcal{M}^*$, $u(\mathbb{N}) = \wp$ or $u(\mathbb{N}) = p - \wp$.*

Proof. Applying $\mathcal{J}$ to the second equation of (7) we have:

$$\mathcal{J}^2\mathbb{N} = \mathcal{J}\mathbb{U} + \mathcal{J}(u(\mathbb{N})\mathbb{N}.$$

Since $\mathcal{J}$ is a metallic structure, then:

$$\begin{aligned} (p\mathcal{J} + q)\mathbb{N} &= P\mathbb{U} + u(\mathbb{U})\mathbb{N} + u(\mathbb{N})\big(\mathbb{U} + u(\mathbb{N})\mathbb{N}\big), \\ p\mathbb{U} + pu(\mathbb{N})\mathbb{N} + q\mathbb{N} &= P\mathbb{U} + u(\mathbb{U})\mathbb{N} + u(\mathbb{N})\mathbb{U} + u^2(\mathbb{N})\mathbb{N}. \end{aligned}$$

It is clear that from this last relation, taking the normal part one has:

$$pu(\mathbb{N}) + q = u^2(\mathbb{N}).$$

□

It is well known that (see [3]), a structure induced on a hypersurface by a metallic structure verifies the following theorem.

Theorem 3.2. *Let $(\mathcal{M}^*, \mathcal{J}, \zeta, \mathrm{g}^*)$ be a meta-metallic pseudo-Riemannian manifold and $\mathcal{H}$ be a lightlike hypersurface of $\mathcal{M}^*$. The structure $\Sigma = (P, g, u, \mathbb{U})$ induced by the metallic structure $\mathcal{J}$ satisfies the following conditions:*

$$\begin{aligned}
\mathcal{J}^2\mathbb{X} &= pP\mathbb{X} + q, \\
u(\mathcal{J}\mathbb{U}) &= 0, \\
\mathcal{J}\mathbb{U} &= 0, \\
u^2(\mathbb{N}) &= pu(\mathbb{N}) + q, \\
\mathrm{g}(\mathcal{J}\mathbb{X}, \mathcal{J}\mathbb{Y}) &= p\mathrm{g}(\mathcal{J}\mathbb{X}, \mathbb{Y}) + q\mathrm{g}(\mathbb{X}, \mathbb{Y}).
\end{aligned}$$

The structure $\mathcal{K}$ analague to Σ induecd on the hypersurface satisfy the same properties given in the following theorem.

Theorem 3.3. *Let $(\mathcal{M}^*, \mathcal{J}, \zeta, \mathrm{g}^*)$ be a meta-metallic pseudo-Riemannian manifold and $\mathcal{H}$ a lightlike hypersurface of $\mathcal{M}^*$. The structure $\mathcal{K} = (\sharp, P, g, \flat, \mathbb{V})$ satisfies the following conditions:*

$$\begin{aligned}
\sharp^2\mathbb{X} &= P\sharp\mathbb{X} - p\sharp\mathbb{X} + p\mathbb{X} - \flat(\mathbb{X})\mathbb{V}, \\
\flat(\sharp\mathbb{X}) &= \big(u(\mathbb{N}) - pI - \flat(\mathbb{N})\big)\flat(\mathbb{X}), \\
\sharp\mathbb{V} &= P\mathbb{V} - p\mathbb{V} - \flat(\mathbb{N})\mathbb{V}, \\
\flat^2(\mathbb{V})\mathbb{N} &= (u(\mathbb{N}) - p)\flat(\mathbb{N}) + pI - \flat(\mathbb{V}) \\
\mathrm{g}(\sharp\mathbb{X}, \mathbb{Y}) &= \mathrm{g}(\mathbb{X}, \sharp\mathbb{Y}) + \flat(\mathbb{X})\tau(\mathbb{X}) - \flat(\mathbb{Y})\tau(\mathbb{X}) \\
\mathrm{g}(\sharp\mathbb{X}, \sharp\mathbb{Y}) &= \mathrm{g}(P\mathbb{X}, \sharp\mathbb{Y}) - p\mathrm{g}(\mathbb{X}, \sharp\mathbb{Y}) + p\mathrm{g}(\mathbb{X}, \mathbb{Y}) + \flat(P\mathbb{Y})\tau(\mathbb{X}) - p\flat(\mathbb{Y})\tau(\mathbb{X}) - \flat(\mathbb{Y})\theta(\sharp\mathbb{X}) - \flat(\mathbb{X})\theta(\sharp\mathbb{Y})
\end{aligned}$$

Proof. Applying ζ to the first part of equation (6) one has:

$$\begin{aligned}
\zeta^2\mathbb{X} &= \mathcal{J}\big(\sharp\mathbb{X} + \flat(\mathbb{X})\mathbb{N}\big) - p\big(\sharp\mathbb{X} + \flat(\mathbb{X})\mathbb{N}\big) + p\mathbb{X} \\
&= \sharp(P\mathbb{X} + u(\mathbb{X})\mathbb{N}) + \flat(\mathbb{X})(\mathbb{U} + u(\mathbb{N})\mathbb{N}) - p\big(\sharp\mathbb{X} + \flat(\mathbb{X})\mathbb{N}\big) + p\mathbb{X} \\
&= \sharp P\mathbb{X} + \flat(\mathbb{X})u(\mathbb{N})\mathbb{N} - p\big(\sharp\mathbb{X} + \flat(\mathbb{X})\mathbb{N}\big) + p\mathbb{X} \\
\sharp\zeta\mathbb{X} + \flat(\mathbb{X})\zeta\mathbb{N} &= \sharp\big(\sharp\mathbb{X} + \flat(\mathbb{X})\mathbb{N}\big) + \flat(\mathbb{X})\mathbb{V} + \flat(\mathbb{N})\flat(\mathbb{X})\mathbb{N} \\
&= \big(\sharp^2\mathbb{X} + \flat(\sharp\mathbb{X})\mathbb{N}\big) + \flat(\mathbb{X})\mathbb{V} + \flat(\mathbb{N})\flat(\mathbb{X})\mathbb{N} \\
&= \big(\sharp^2\mathbb{X} + \flat(\sharp\mathbb{X})\mathbb{N}\big) + \flat(\mathbb{X})\mathbb{V} + \flat(\mathbb{N})\flat(\mathbb{X})\mathbb{N}.
\end{aligned}$$

Applying ζ to the first part of equation (6) , one has:

$$\begin{cases} \sharp^2\mathbb{X} &= P\sharp\mathbb{X} - p\sharp\mathbb{X} + p\mathbb{X} - \flat(\mathbb{X})\mathbb{V} \\ \flat(\sharp\mathbb{X})\mathbb{N} &= \big(u(\mathbb{N}) - pI - \flat(\mathbb{N})\big)\flat(\mathbb{X})\mathbb{N}. \end{cases}$$

Applying ζ to the first equation of the relation (7) one has also:

$$\begin{aligned} \zeta\mathbb{V} + \flat(\mathbb{N})\zeta\mathbb{N} &= \sharp\mathbb{V} + \flat(\mathbb{V})\mathbb{N} + \flat(\mathbb{N})\big(\mathbb{V} + \flat(\mathbb{N})\big)\mathbb{N} \\ &= \sharp\mathbb{V} + \flat(\mathbb{V})\mathbb{N} + \flat(\mathbb{N})\mathbb{V} + \flat^2(\mathbb{N})\mathbb{N}. \\ \zeta^2\mathbb{V} &= \mathcal{J}\zeta\mathbb{N} - p\zeta\mathbb{N} + p\mathbb{N} \\ &= \mathcal{J}\big(\mathbb{V} + \flat(\mathbb{N})\big) - p\big(\mathbb{V} + \flat(\mathbb{N})\big) + p\mathbb{N} \\ &= \mathcal{J}\mathbb{V} + \flat(\mathbb{N})\mathcal{J}\mathbb{N} - p\big(\mathbb{V} + \flat(\mathbb{N})\big) + p\mathbb{N} \\ &= P\mathbb{V} + u(\mathbb{V})\mathbb{N} + \flat(\mathbb{N})\big(\mathbb{U} + u(\mathbb{N})\mathbb{N}\big) - p\mathbb{V} - p\flat(\mathbb{N})\mathbb{N} + p\mathbb{N} \\ &= P\mathbb{V} + u(\mathbb{V})\mathbb{N} + \flat(\mathbb{N})u(\mathbb{N})\mathbb{N} - p\mathbb{V} - p\flat(\mathbb{N})\mathbb{N} + p\mathbb{N}. \end{aligned}$$

Applying ζ to the first equation of the relation (7) then one has:

$$\begin{cases} P\mathbb{V} - p\mathbb{V} &= \sharp\mathbb{V} + \flat(\mathbb{N})\mathbb{V} \\ \flat^2(\mathbb{N})\mathbb{N} &= \flat(\mathbb{N})\big(u(\mathbb{N}) - p\big)\mathbb{N} + p\mathbb{N}\flat(\mathbb{V})\mathbb{N}. \end{cases}$$

On one hand, one has:

$$\begin{aligned} \mathrm{g}(\zeta\mathbb{X}, \mathbb{Y}) &= \mathrm{g}(\mathbb{X}, \zeta\mathbb{Y}), \\ \mathrm{g}(\sharp\mathbb{X} + \flat(\mathbb{X})\mathbb{N}, \mathbb{Y}) &= \mathrm{g}(\mathbb{X}, \sharp\mathbb{Y} + \flat(\mathbb{Y})\mathbb{N}), \\ \mathrm{g}(\sharp\mathbb{X}, \mathbb{Y}) &= \mathrm{g}(\mathbb{X}, \sharp\mathbb{Y}) + \flat(\mathbb{Y})\mathrm{g}(\mathbb{N}, \mathbb{X}) - \flat(\mathbb{X})\mathrm{g}(\mathbb{N}, \mathbb{Y}). \end{aligned}$$

By setting $\tau(\mathbb{X}) = \mathrm{g}(\mathbb{N}, \mathbb{X})$ then $\tau(\mathbb{Y}) = \mathrm{g}(\mathbb{N}, \mathbb{Y})$ and one has:

$$\mathrm{g}(\sharp\mathbb{X}, \mathbb{Y}) = \mathrm{g}(\mathbb{X}, \sharp\mathbb{Y}) + \flat(\mathbb{Y})\tau(\mathbb{X}) - \flat(\mathbb{X})\tau(\mathbb{Y}). \tag{8}$$

On the other hand,

$$\mathrm{g}(\zeta\mathbb{X}, \zeta\mathbb{Y}) = \mathrm{g}(\mathbb{X}, \zeta^2\mathbb{Y}). \tag{9}$$

Let's calculate:

$$\begin{aligned} \mathrm{g}(\zeta\mathbb{X}, \zeta\mathbb{Y}) &= \mathrm{g}\big(\sharp\mathbb{X} + \flat(\mathbb{X})\mathbb{N}, \sharp\mathbb{Y} + \flat(\mathbb{Y})\mathbb{N}\big) \\ &= g(\sharp\mathbb{X}, \sharp\mathbb{Y}) + \flat(\mathbb{Y})\mathrm{g}(\sharp\mathbb{X}, \mathbb{N}) + \flat(\mathbb{Y})\mathrm{g}(\sharp\mathbb{Y}, \mathbb{N}) + \flat(\mathbb{X})\flat(\mathbb{Y})\mathrm{g}.(\mathbb{N}, \mathbb{N}). \end{aligned}$$

By setting $\theta(\sharp(\mathbb{X})) = \mathrm{g}(\sharp\mathbb{X}, \mathbb{N})$ one has:

$$\mathrm{g}(\zeta\mathbb{X}, \zeta\mathbb{Y}) = g(\sharp\mathbb{X}, \sharp\mathbb{Y}) + \flat(\mathbb{X})\theta(\sharp(\mathbb{X})) + \flat(\mathbb{Y})\theta(\sharp(\mathbb{X})). \tag{10}$$

$$\begin{aligned}
\mathrm{g}(\mathbb{X}, \zeta^2\mathbb{Y}) &= \mathrm{g}(\mathbb{X}, (\mathcal{J}\zeta - p\zeta + pI)\mathbb{Y}) \\
&= \mathrm{g}(\mathbb{X}, \zeta\mathcal{J}\mathbb{Y}) - p\mathrm{g}(\mathbb{X}, \zeta\mathbb{Y}) + p\mathrm{g}(\mathbb{X}, \mathbb{Y}) \\
&= \mathrm{g}(\mathbb{X}, \zeta(P\mathbb{Y} + u(\mathbb{Y})\mathbb{N})) - p\mathrm{g}(\mathbb{X}, \sharp\mathbb{Y} + \flat(\mathbb{Y})\mathbb{N}) + p\mathrm{g}(\mathbb{X}, \mathbb{Y}) \\
&= \mathrm{g}(\mathbb{X}, P\zeta\mathbb{Y}) + \mathrm{g}(\mathbb{X}, u(\mathbb{Y})\zeta\mathbb{N}) - p\mathrm{g}(\mathbb{X}, \sharp\mathbb{Y}) - p\flat(\mathbb{Y})\mathrm{g}(\mathbb{X}, \mathbb{N}) + p\mathrm{g}(\mathbb{X}, \mathbb{Y}) \\
&= \mathrm{g}(\mathbb{X}, P(\sharp\mathbb{Y}\mathbb{N} + \flat(\mathbb{Y})) - p\mathrm{g}(\mathbb{X}, \sharp\mathbb{Y}) - p\flat(\mathbb{Y})\mathrm{g}(\mathbb{X}, \mathbb{N}) + p\mathrm{g}(\mathbb{X}, \mathbb{Y}) \\
&= \mathrm{g}(P\mathbb{X}, \sharp\mathbb{Y}) + \flat(P\mathbb{Y})\mathrm{g}(\mathbb{X}, \mathbb{N}) - p\mathrm{g}(\mathbb{X}, \sharp\mathbb{Y}) - p\flat(\mathbb{Y})\mathrm{g}(\mathbb{X}, \mathbb{N}) + p\mathrm{g}(\mathbb{X}, \mathbb{Y}).
\end{aligned}$$

$$\mathrm{g}(\mathbb{X}, \zeta^2\mathbb{Y}) = \mathrm{g}(P\mathbb{X}, \sharp\mathbb{Y}) + \flat(P\mathbb{Y})\tau(\mathbb{X}) - p\mathrm{g}(\mathbb{X}, \sharp\mathbb{Y}) - p\flat(\mathbb{Y})\tau(\mathbb{X}) + p\mathrm{g}(\mathbb{X}, \mathbb{Y}). \tag{11}$$

By equating the relation (10) with the relation (11) we have:

$$\begin{aligned}
\mathrm{g}(\sharp\mathbb{X}, \sharp\mathbb{Y}) &= \mathrm{g}(P\mathbb{X}, \sharp\mathbb{Y}) - p\mathrm{g}(\mathbb{X}, \sharp\mathbb{Y}) + p\mathrm{g}(\mathbb{X}, \mathbb{Y}) + \flat(P\mathbb{Y})\tau(\mathbb{X}) - p\flat(\mathbb{Y})\tau(\mathbb{X}) \\
&\quad -\flat(\mathbb{Y})\theta(\sharp\mathbb{X}) - \flat(\mathbb{X})\theta(\sharp\mathbb{Y}).
\end{aligned}$$

□

Proposition 3.4. *Let $(\mathcal{M}^*, \mathcal{J}, \zeta, \mathrm{g}^*)$ be a meta-metallic pseudo-Riemannian manifold and ∇^* the Levi-Civita connection on $\mathcal{M}^*$. For all $\mathbb{Y}\zeta \in \chi(\mathcal{M}^*)$ one has:*

$$(\nabla^*_{\mathbb{X}}\mathcal{J}\zeta) = 0.$$

Proof. A direct computation gives:

$$(\nabla^*_{\mathbb{X}}\mathcal{J}\zeta)\mathbb{Y} = \nabla^*_{\mathbb{X}}\mathcal{J}\zeta - \mathcal{J}\zeta\nabla^*_{\mathbb{X}}\mathbb{Y} = \mathcal{J}\nabla^*_{\mathbb{X}}\zeta - \mathcal{J}\zeta\nabla^*_{\mathbb{X}}\mathbb{Y}.$$

□

Theorem 3.5. *Let $\mathcal{H}$ be a hypersurface of a meta-metallic pseudo-Riemannian manifold $(\mathcal{M}^*, \mathcal{J}, \zeta, \mathrm{g}^*)$. Then one has:*

$$\begin{aligned}
(\nabla_{\mathbb{X}}\sharp)\mathbb{Y} &= \flat(\mathbb{Y})\mathbb{A}_{\mathbb{N}}\mathbb{X} + \mathbb{H}(\mathbb{X}, \mathbb{Y})\mathbb{V}, \\
(\nabla_{\mathbb{X}}\flat)\mathbb{Y} &= -\mathbb{H}(\mathbb{X}, \mathbb{Y})\flat(\mathbb{N}) + \mathbb{H}(\mathbb{X}, \sharp\mathbb{Y})\mathbb{V} + \flat(\mathbb{Y})\tau(\mathbb{X}), \\
\nabla_{\mathbb{X}}\mathbb{V} &= -\sharp\mathbb{A}_{\mathbb{N}}\mathbb{X} + \tau(\mathbb{X})\mathbb{V} + \flat(\mathbb{N})\mathbb{A}_{\mathbb{N}}\mathbb{X},
\end{aligned}$$

$$\mathbb{X}(\flat(\mathbb{N})) = -\mathbb{H}(\mathbb{X},\mathbb{V}) - \flat(\mathbb{A}_{\mathbb{N}}\mathbb{X}).$$

Proof. Let $\mathcal{H}$ be a hypersurface of a meta-metallic pseudo-Riemannian manifold $(\mathcal{M}^*,\mathcal{J},\zeta,g^*)$. One has:

$$\begin{aligned}
\nabla^*_{\mathbb{X}}\zeta\mathbb{Y} &= \nabla^*_{\mathbb{X}}(\sharp\mathbb{Y}+\flat(\mathbb{X})\mathbb{N})\\
&= \nabla^*_{\mathbb{X}}\sharp\mathbb{Y}+\nabla^*_{\mathbb{X}}\flat(\mathbb{X})\mathbb{N}\\
&= \nabla_{\mathbb{X}}\sharp\mathbb{Y}+\mathbb{H}(\mathbb{X},\sharp\mathbb{Y})+\mathbb{X}(\flat(\mathbb{Y}))\mathbb{N}+\flat(\mathbb{Y})(-\mathbb{A}_{\mathbb{N}}\mathbb{X}+\nabla^t_{\mathbb{X}}\mathbb{N})\\
&= \nabla_{\mathbb{X}}\sharp\mathbb{Y}+\mathbb{H}(\mathbb{X},\sharp\mathbb{Y})+\mathbb{X}(\flat(\mathbb{Y}))\mathbb{N}-\flat(\mathbb{Y})\mathbb{A}_{\mathbb{N}}\mathbb{X}+\flat(\mathbb{Y})\nabla^t_{\mathbb{X}}\mathbb{N}\\
&= \nabla_{\mathbb{X}}\sharp\mathbb{Y}+\mathbb{H}(\mathbb{X},\sharp\mathbb{Y})+\mathbb{X}(\flat(\mathbb{Y}))\mathbb{N}-\flat(\mathbb{Y})\mathbb{A}_{\mathbb{N}}\mathbb{X}+\flat(\mathbb{Y})\tau(\mathbb{X}),\\
\zeta\nabla^*_{\mathbb{X}}\mathbb{Y} &= \zeta(\nabla_{\mathbb{X}}\mathbb{X}+\mathbb{H}(\mathbb{X},\mathbb{Y}))\\
&= \zeta\nabla_{\mathbb{X}}\mathbb{Y}+\mathbb{H}(\mathbb{X},\mathbb{Y})\zeta\mathbb{N}\\
&= \sharp\nabla_{\mathbb{X}}\mathbb{Y}+\flat(\nabla_{\mathbb{X}}\mathbb{X})\mathbb{N}+\mathbb{H}(\mathbb{X},\mathbb{Y})(\mathbb{V}+\flat(\mathbb{N})\mathbb{N})\\
&= \sharp\nabla_{\mathbb{X}}\mathbb{Y}+\flat(\nabla_{\mathbb{X}}\mathbb{X})\mathbb{N}+\mathbb{H}(\mathbb{X},\mathbb{Y})\mathbb{V}+\mathbb{H}(\mathbb{X},\mathbb{Y})\flat(\mathbb{N})\mathbb{N}.
\end{aligned}$$

By equating these expressions and taking the tangential part and the normal part we find the first two equations.

Now let's calculate:

$$\begin{aligned}
\nabla^*_{\mathbb{X}}\zeta\mathbb{N} &= \nabla^*_{\mathbb{X}}(\mathbb{V}+\flat(\mathbb{N})\mathbb{N})\\
&= \nabla^*_{\mathbb{X}}\mathbb{V}+\nabla^*_{\mathbb{X}}\flat(\mathbb{N})\mathbb{N}\\
&= \nabla_{\mathbb{X}}\mathbb{V}+\mathbb{H}(\mathbb{X},\mathbb{V})+\mathbb{X}(\flat(\mathbb{N}))\mathbb{N}+\flat(\mathbb{N})(\mathbb{A}_{\mathbb{N}}\mathbb{X}+\nabla^t_{\mathbb{X}}\mathbb{N})\\
&= \nabla_{\mathbb{X}}\mathbb{V}+\mathbb{H}(\mathbb{X},\mathbb{V})+\mathbb{X}(\flat(\mathbb{N}))\mathbb{N}+\flat(\mathbb{N})\mathbb{A}_{\mathbb{N}}\mathbb{X}+\flat(\mathbb{N})\tau(\mathbb{X}).\\
\zeta\nabla^*_{\mathbb{X}}\mathbb{N} &= \zeta(-\mathbb{A}_{\mathbb{N}}\mathbb{X})+\zeta(\nabla^t_{\mathbb{X}}\mathbb{N}) = -\sharp\mathbb{A}_{\mathbb{N}}\mathbb{N}-\flat(\mathbb{A}_{\mathbb{N}}\mathbb{N})\mathbb{N}+\nabla^t_{\mathbb{X}}\mathbb{N}(\mathbb{V}-\flat(\mathbb{N})\mathbb{N})\\
&= -\sharp\mathbb{A}_{\mathbb{N}}\mathbb{N}-\flat(\mathbb{A}_{\mathbb{N}}\mathbb{N})\mathbb{N}+\tau(\mathbb{X})(\mathbb{V}-\flat(\mathbb{N})\mathbb{N})\\
&= -\sharp\mathbb{A}_{\mathbb{N}}\mathbb{N}-\flat(\mathbb{A}_{\mathbb{N}}\mathbb{N})\mathbb{N}+\tau(\mathbb{X})\mathbb{V}-\tau(\mathbb{X})\flat(\mathbb{N})\mathbb{N}.
\end{aligned}$$

Since $(\nabla^*_{\mathbb{X}}\zeta)\mathbb{N}=0 \Longrightarrow \nabla^*_{\mathbb{X}}\zeta\mathbb{N}=\zeta\nabla^*_{\mathbb{X}}\mathbb{N}$. Then by equating these expressions and taking the tangential part and the normal part one find the last two equations. □

Theorem 3.6. *Let $\mathcal{H}$ be a hypersurface of a meta-metallic pseudo-Riemannian manifold $(\mathcal{M}^*,\mathcal{J},\zeta,g^*)$. Then one has:*

$$\begin{aligned}
(\nabla_{\mathbb{X}}P)\mathbb{Y} &= 0,\\
\mathbb{H}(\mathbb{X},P\mathbb{Y}) &= \mathbb{H}(\mathbb{X},\mathbb{Y})u(\mathbb{N}),\\
\mathbb{X}(u(\mathbb{N})) &= 0,
\end{aligned}$$

$$P\mathbb{A}_{\mathbb{N}}\mathbb{X} \;=\; u(\mathbb{N})\mathbb{A}_{\mathbb{N}}\mathbb{X}.$$

Proof. For all $\mathbb{X},\ \mathbb{Y} \in \Gamma(\mathcal{TM}^*)$ $(\nabla^*_{\mathbb{X}}\mathcal{J})\mathbb{Y} = 0 \Longrightarrow \nabla^*_{\mathbb{X}}\mathcal{J}\mathbb{Y} = \mathcal{J}\nabla^*_{\mathbb{X}}\mathbb{Y}$. Then:

$$\begin{aligned}
(E_1)\ \nabla^*_{\mathbb{X}}\mathcal{J}\mathbb{Y} &= \nabla^*_{\mathbb{X}}(P\mathbb{Y} + u(\mathbb{Y})\mathbb{N}) = \nabla^*_{\mathbb{X}}P\mathbb{Y} = \nabla_{\mathbb{X}}P\mathbb{X} + \mathbb{H}(\mathbb{X}, P\mathbb{Y})\mathbb{N},\\
(E_2)\ \mathcal{J}\nabla^*_{\mathbb{X}}\mathbb{Y} &= \mathcal{J}(\nabla_{\mathbb{X}}\mathbb{Y} + \mathbb{H}(\mathbb{X},\mathbb{Y})\mathbb{N}) = \mathcal{J}\nabla_{\mathbb{X}}\mathbb{Y} + \mathbb{H}(\mathbb{X},\mathbb{Y})\mathcal{J}\mathbb{N}\\
&= P\nabla_{\mathbb{X}}\mathbb{Y} + u(\nabla_{\mathbb{X}}\mathbb{Y})\mathbb{N} + \mathbb{H}(\mathbb{X},\mathbb{Y})(\mathbb{U} + u(\mathbb{N})\mathbb{N}).
\end{aligned}$$

By equating (E_1), (E_2) and taking the normal part and the tangential part one has:

$$\nabla_{\mathbb{X}}P\mathbb{Y} - P\nabla_{\mathbb{X}}\mathbb{Y} = (\nabla_{\mathbb{X}}P)\mathbb{Y} = 0 \quad \text{and} \quad \mathbb{H}(\mathbb{X}, P\mathbb{Y})\mathbb{N} = \mathbb{H}(\mathbb{X},\mathbb{Y})u(\mathbb{N})\mathbb{N}.$$

Now for all $\mathbb{X}, \in \Gamma(\mathcal{TM}^*)$, $\mathbb{N} \in \Gamma(ltr(\mathcal{TM}^*))$, $(\nabla^*_{\mathbb{X}}\mathcal{J})\mathbb{N} = 0 \Longrightarrow \nabla^*_{\mathbb{X}}\mathcal{J}\mathbb{N} = \mathcal{J}\nabla^*_{\mathbb{X}}\mathbb{N}$. So :

$$\begin{aligned}
(E_1')\ \nabla^*_{\mathbb{X}}\mathcal{J}\mathbb{N} &= (\mathbb{U} + u(\mathbb{N})\mathbb{N}) = \nabla^*_{\mathbb{X}}u(\mathbb{N})\mathbb{N} = \mathbb{X}(u(\mathbb{N}))\mathbb{N} + u(\mathbb{N})(-\mathbb{A}_{\mathbb{N}}\mathbb{X} + \tau(\mathbb{X}))\\
&= \mathbb{X}(u(\mathbb{N}))\mathbb{N} - u(\mathbb{N})\mathbb{A}_{\mathbb{N}}\mathbb{X} + u(\mathbb{N})\tau(\mathbb{X}),\\
(E_2')\ \mathcal{J}\nabla^*_{\mathbb{X}}\mathbb{N} &= \mathcal{J}(-\mathbb{A}_{\mathbb{N}}\mathbb{X} + \nabla^t_{\mathbb{X}}\mathbb{N}) = -\mathcal{J}\mathbb{A}_{\mathbb{N}}\mathbb{X} + (\nabla^t_{\mathbb{X}}\mathbb{N})\mathcal{J}\mathbb{N} = -P\mathbb{A}_{\mathbb{N}}\mathbb{X} + \tau(\mathbb{X})(\mathbb{U} + u(\mathbb{N})\mathbb{N})\\
&= -P\mathbb{A}_{\mathbb{N}}\mathbb{X} + \tau(\mathbb{X})u(\mathbb{N})\mathbb{N}.
\end{aligned}$$

By equating (E_1') with (E_2') and taking the normal part and the tangential part one has:

$$\begin{aligned}
\mathbb{X}(u(\mathbb{N})) &= 0,\\
P\mathbb{A}_{\mathbb{N}}\mathbb{X} &= u(\mathbb{N})\mathbb{A}_{\mathbb{N}}\mathbb{X}.
\end{aligned}$$

□

Theorem 3.7. *Let $\mathcal{H}$ be a hypersurface of a meta-metallic pseudo-Riemannian manifold $(\mathcal{M}^*, \mathcal{J}, \zeta, g^*)$. Then:*

$$\begin{aligned}
(\nabla_{\mathbb{X}}\sharp P)\mathbb{Y} &= \mathbb{H}(\mathbb{X},\mathbb{Y})\mathbb{V} + \flat(\mathbb{N})u(\mathbb{N})\tau(\mathbb{X}),\\
\flat(\nabla_{\mathbb{X}}\mathbb{Y})u(\mathbb{N}) &= \mathbb{H}(\mathbb{X},\mathbb{Y}) + \mathbb{X}(\flat(\mathbb{N}))u(\mathbb{N}) + \flat(\mathbb{N})u(\mathbb{N})\tau(\mathbb{X}) - \mathbb{H}(\mathbb{X},\mathbb{Y})\flat(\mathbb{N})u(\mathbb{N}),\\
\nabla_{\mathbb{X}}P\mathbb{V} &= \flat(\mathbb{N})u(\mathbb{N})\mathbb{N} - \sharp P\mathbb{A}_{\mathbb{N}}\mathbb{X} + \tau(\mathbb{X})P\mathbb{V},\\
\mathbb{X}(\flat(\mathbb{N}))u(\mathbb{N}) &= -\flat(\mathbb{A}_{\mathbb{N}}\mathbb{X})u(\mathbb{N}) - \mathbb{H}(\mathbb{X}, P\mathbb{V}).
\end{aligned}$$

Proof. For all $\mathbb{X},\ \mathbb{Y} \in \Gamma(\mathcal{TM}^*)$, $(\nabla^*_{\mathbb{X}}\mathcal{J}\zeta)\mathbb{Y} = 0 \Longrightarrow \nabla^*_{\mathbb{X}}\mathcal{J}\zeta\mathbb{Y} = \mathcal{J}\zeta\nabla^*_{\mathbb{X}}\mathbb{Y}$. Then:

$$\begin{aligned}
\nabla^*_{\mathbb{X}}\mathcal{J}\zeta\mathbb{Y} &= \nabla^*_{\mathbb{X}}\mathcal{J}(\sharp\mathbb{Y} + \flat(\mathbb{Y})\mathbb{N}) = \nabla^*_{\mathbb{X}}(\sharp\mathcal{J}\mathbb{Y} + \flat(\mathbb{Y})\mathcal{J}\mathbb{N})\\
&= \nabla^*_{\mathbb{X}}(\sharp(P\mathbb{Y} + u(\mathbb{Y})\mathbb{N})) + \flat(\mathbb{Y})(\mathbb{U} + u(\mathbb{N})\mathbb{N})) = \nabla^*_{\mathbb{X}}(\sharp P\mathbb{Y} + \flat(\mathbb{Y})u(\mathbb{N})\mathbb{N})\\
&= \nabla^*_{\mathbb{X}}\sharp P\mathbb{Y} + \nabla^*_{\mathbb{X}}\flat(\mathbb{Y})u(\mathbb{N})\mathbb{N}
\end{aligned}$$

$$\begin{aligned}
&= \nabla_{\mathbb{X}}\sharp P\mathbb{Y} + \mathbb{H}(\mathbb{X}, \sharp P\mathbb{Y})\mathbb{N} + \mathbb{X}[\flat(\mathbb{Y})u(\mathbb{N})]\mathbb{N} + \flat(\mathbb{Y})u(\mathbb{N})(-\mathbb{A}_{\mathbb{N}}\mathbb{X} + \tau(\mathbb{X}))\\
&= \nabla_{\mathbb{X}}\sharp P\mathbb{Y} + \mathbb{H}(\mathbb{X}, \sharp P\mathbb{Y})\mathbb{N} + [\mathbb{X}\flat(\mathbb{Y})u(\mathbb{N}) + \flat(\mathbb{Y})\mathbb{X}u(\mathbb{N})]\mathbb{N} - \flat(\mathbb{Y})u(\mathbb{N})\mathbb{A}_{\mathbb{N}}\mathbb{X}\\
&\quad +\flat(\mathbb{Y})u(\mathbb{N})\tau(\mathbb{X}).\\
\mathcal{J}\zeta\nabla^*_{\mathbb{X}}\mathbb{Y} &= \mathcal{J}\zeta(\nabla_{\mathbb{X}}\mathbb{Y} + \mathbb{H}(\mathbb{X}, \mathbb{Y})\mathbb{N}) = \mathcal{J}(\zeta\nabla_{\mathbb{X}}\mathbb{Y} + \mathbb{H}(\mathbb{X}, \mathbb{Y})\zeta\mathbb{N})\\
&= \mathcal{J}(\sharp\nabla_{\mathbb{X}}\mathbb{Y} + \flat(\nabla_{\mathbb{X}}\mathbb{Y})\mathbb{N} + \mathbb{H}(\mathbb{X}, \mathbb{Y})\mathbb{V} + \mathbb{H}(\mathbb{X}, \mathbb{Y})\flat(\mathbb{N})\mathbb{N})\\
&= \sharp\mathcal{J}\nabla_{\mathbb{X}}\mathbb{Y} + \flat(\nabla_{\mathbb{X}}\mathbb{Y})\mathcal{J}\mathbb{N} + \mathbb{H}(\mathbb{X}, \mathbb{Y})\mathcal{J}\mathbb{V} + \mathbb{H}(\mathbb{X}, \mathbb{Y})\flat(\mathbb{N})\mathcal{J}\mathbb{N}\\
&= \sharp(P\nabla_{\mathbb{X}}\mathbb{Y} + u(\nabla_{\mathbb{X}}\mathbb{Y})) + \flat(\nabla_{\mathbb{X}}\mathbb{Y})(\mathbb{U} + u(\mathbb{N})\mathbb{N}) + \mathbb{H}(\mathbb{X}, \mathbb{Y})(P\mathbb{V} + \flat(\mathbb{V})\mathbb{N})\\
&+ \mathbb{H}(\mathbb{X}, \mathbb{Y})\flat(\mathbb{N})(\mathbb{U} + u(\mathbb{N})\mathbb{N})\\
&= \sharp P\nabla_{\mathbb{X}}\mathbb{Y} + \flat(\nabla_{\mathbb{X}}\mathbb{Y})u(\mathbb{N})\mathbb{N} + \mathbb{H}(\mathbb{X}, \mathbb{Y})P\mathbb{V} + \mathbb{H}(\mathbb{X}, \mathbb{Y})\flat(\mathbb{V})\mathbb{N} + \mathbb{H}(\mathbb{X}, \mathbb{Y})\flat(\mathbb{N})u(\mathbb{N})\mathbb{N}.
\end{aligned}$$

By equating the two relations and taking the tangential and normal part, one has:

$$\begin{aligned}
\nabla_{\mathbb{X}}\sharp P\mathbb{Y} - \sharp P\nabla_{\mathbb{X}}\mathbb{Y} &= \mathbb{H}(\mathbb{X}, \mathbb{Y})\mathbb{V} + \flat(\mathbb{N})u(\mathbb{N})\tau(\mathbb{X})\\
\flat(\nabla_{\mathbb{X}}\mathbb{Y})u(\mathbb{N}) &= \mathbb{H}(\mathbb{X}, \mathbb{Y}) + \mathbb{X}(\flat(\mathbb{N}))u(\mathbb{N}) + \flat(\mathbb{N})u(\mathbb{N})\tau(\mathbb{X}) - \mathbb{H}(\mathbb{X}, \mathbb{Y})\flat(\mathbb{N})u(\mathbb{N}).
\end{aligned}$$

For all $\mathbb{X} \in \Gamma(\mathcal{TM}^*)$, $\mathbb{N} \in \Gamma(ltr(\mathcal{TM}^*))$,$(\nabla^*_{\mathbb{X}}\mathcal{J}\zeta)\mathbb{N} = 0 \Longrightarrow \nabla^*_{\mathbb{X}}\mathcal{J}\zeta\mathbb{N} = \mathcal{J}\zeta\nabla^*_{\mathbb{X}}\mathbb{N}$. Then :

$$\begin{aligned}
\nabla^*_{\mathbb{X}}\mathcal{J}\zeta\mathbb{N} &= \nabla^*_{\mathbb{X}}\mathcal{J}(\mathbb{V} + \flat(\mathbb{N})\mathbb{N})\\
&= \nabla^*_{\mathbb{X}}(\mathcal{J}\mathbb{V} + \flat(\mathbb{N})\mathcal{J}\mathbb{N}) = \nabla^*_{\mathbb{X}}(P\mathbb{V} + u(\mathbb{V})\mathbb{N}) + \nabla^*_{\mathbb{X}}\flat(\mathbb{N})(\mathbb{U} + u(\mathbb{N})\mathbb{N})\\
&= \nabla^*_{\mathbb{X}}P\mathbb{V} + \nabla^*_{\mathbb{X}}\flat(\mathbb{N})u(\mathbb{N})\mathbb{N}\\
&= \nabla_{\mathbb{X}}P\mathbb{V} + \mathbb{H}(\mathbb{X}, P\mathbb{V})\mathbb{N} + \mathbb{X}(\flat(\mathbb{N})u(\mathbb{N}))\mathbb{N} + \flat(\mathbb{N})u(\mathbb{N})(-\mathbb{A}_{\mathbb{N}}\mathbb{X} + \tau(\mathbb{X})\mathbb{N})\\
&= \nabla_{\mathbb{X}}P\mathbb{V} + \mathbb{H}(\mathbb{X}, P\mathbb{V})\mathbb{N} + (\mathbb{X}\flat(\mathbb{N})u(\mathbb{N}) + \flat(\mathbb{N})\mathbb{X}u(\mathbb{N}))\mathbb{N} - \flat(\mathbb{N})u(\mathbb{N})\mathbb{A}_{\mathbb{N}}\mathbb{X}\\
&\quad +\flat(\mathbb{N})u(\mathbb{N})\tau(\mathbb{X})\mathbb{N}\\
&= \nabla_{\mathbb{X}}P\mathbb{V} + \mathbb{H}(\mathbb{X}, P\mathbb{V})\mathbb{N} + \mathbb{X}\flat(\mathbb{N})u(\mathbb{N}) - \flat(\mathbb{N})u(\mathbb{N})\mathbb{A}_{\mathbb{N}}\mathbb{X} + \flat(\mathbb{N})u(\mathbb{N})\tau(\mathbb{X})\mathbb{N}.\\
\mathcal{J}\zeta\nabla^*_{\mathbb{X}}\mathbb{N} &= \mathcal{J}\zeta(\mathbb{A}_{\mathbb{N}}\mathbb{X} + \tau(\mathbb{X})\mathbb{N}) = \mathcal{J}(-\zeta\mathbb{A}_{\mathbb{N}}\mathbb{X} + \tau(\mathbb{X})\zeta\mathbb{N})\\
&= \mathcal{J}(-(\sharp\mathbb{A}_{\mathbb{N}}\mathbb{X} + \flat(\mathbb{A}_{\mathbb{N}}\mathbb{X})\mathbb{N} + \tau(\mathbb{X})(\mathbb{V} + \flat(\mathbb{N})\mathbb{N}))\\
&= -\sharp\mathcal{J}\mathbb{A}_{\mathbb{N}}\mathbb{X} - \flat(\mathbb{A}_{\mathbb{N}}\mathbb{X})\mathcal{J}\mathbb{N} + \tau(\mathbb{X})\mathcal{J}\mathbb{V} + \tau(\mathbb{X})\flat(\mathbb{N})\mathcal{J}\mathbb{N}\\
&= -\sharp(P\mathbb{A}_{\mathbb{N}}\mathbb{X} + u(\mathbb{A}_{\mathbb{N}}\mathbb{X})\mathbb{N}) - \flat(\mathbb{A}_{\mathbb{N}}\mathbb{X})(\mathbb{U} + u(\mathbb{N})\mathbb{N})\\
&+ \tau(\mathbb{X})(P\mathbb{V} + u(\mathbb{V})\mathbb{N}) + \tau(\mathbb{X})\flat(\mathbb{N})(\mathbb{U} + u(\mathbb{N})\mathbb{N})\\
&= -\sharp P\mathbb{A}_{\mathbb{N}}\mathbb{X} - \flat(\mathbb{A}_{\mathbb{N}}\mathbb{X})u(\mathbb{N})\mathbb{N} + \tau(\mathbb{X})P\mathbb{V} + \tau(\mathbb{X})\flat(\mathbb{N})u(\mathbb{N})\mathbb{N}.
\end{aligned}$$

By equating these two relations and taking the tangential and normal part one has:

$$\nabla_{\mathbb{X}}P\mathbb{V} = \flat(\mathbb{N})u(\mathbb{N})\mathbb{N} - \sharp P\mathbb{A}_{\mathbb{N}}\mathbb{X} + \tau(\mathbb{X})P\mathbb{V},$$

$$\mathbb{X}(\flat(\mathbb{N}))u(\mathbb{N}) = -\flat(\mathbb{A}_{\mathbb{N}}\mathbb{X})u(\mathbb{N}) - \mathbb{H}(\mathbb{X}, P\mathbb{V}).$$

□

4. Invariant Lightlike Hypersurface of a Meta-metallic Pseudo-Riemannian Manifold

In the following section we study the lightlike invariant hypersurface of a meta-metallic pseudo-Riemannian manifold.

Definition 4.1. *Let $(\mathcal{M}^*, \mathcal{J}, \zeta, g^*)$ be a meta-metallic pseudo-Riemannian manifold and $\mathcal{H}$ be a lightlike hypersurface. The hypersurface $\mathcal{H}$ is said to be invariant if:*

$$\mathcal{J}\zeta(\mathcal{TH}) \subset \mathcal{TH}.$$

This class of hypersurface can be characterized by the following theorem.

Theorem 4.2. *Let $(\mathcal{M}^*, \mathcal{J}, \zeta, g^*)$ be a meta-metallic pseudo-Riemannian manifold and $\mathcal{H}$ be a lightlike hypersurface, the following assertions are equivalent:*

1. *the hypersurface $\mathcal{H}$ is ζ-invariant ($\mathcal{J}\zeta$ is invariant),*
2. *the 1-form $\flat$ is zero on the hypersurface $\mathcal{H}$,*
3. *the tensor $\sharp$ is meta-metallic .*

Proof. If $\mathcal{H}$ is invariant then for all $\mathbb{X} \in \Gamma(\mathbb{TH})$ one can write: $\zeta\mathbb{X} = \sharp\mathbb{X}$. After the relation $\zeta\mathbb{X} = \sharp\mathbb{X} + \flat(\mathbb{X})\mathbb{N}$, one has $\flat(\mathbb{X}) = 0$.
Conversely, if $\flat(\mathbb{X}) = 0$ then the hypersurface $\mathcal{H}$ is invariant. Since the hypersurface $\mathcal{H}$ is ζ-invariant then:

$$\mathcal{J}\zeta\mathbb{X} = \sharp(P\mathbb{X} + u(\mathbb{X})\mathbb{N}) = \sharp P\mathbb{X} \quad \text{because} \quad u(\mathbb{X}) = 0.$$

The necessary and sufficient condition for $\mathcal{H}$ to be invariant is $\flat = 0$ and $\zeta\mathbb{X} = \sharp\mathbb{X}$, then :

$$\zeta^2\mathbb{X} = \sharp\zeta\mathbb{X} \Longrightarrow \mathcal{J}\zeta\mathbb{X} - p\zeta\mathbb{X} + p\mathbb{X} = \sharp\zeta\mathbb{X} \Longrightarrow \mathcal{J}\sharp\mathbb{X} - p\sharp\mathbb{X} + p\mathbb{X} = \sharp^2\mathbb{X}.$$

Moreover,

$$g(\zeta\mathbb{X}, \mathbb{Y}) = g(\sharp\mathbb{X}, \mathbb{Y}) = g(\mathbb{X}, \zeta\mathbb{Y}) = g(\mathbb{X}, \sharp\mathbb{Y}),$$

hence $\sharp$ is a meta-metallic structure. □

Theorem 4.3. *Let $(\mathcal{M}^*, \mathcal{J}, \zeta, g^*)$ be a meta-metallic pseudo-Riemannian manifold and $\mathcal{H}$ be an invariant lightlike hypersurface of $\mathcal{M}^*$. The structure $\mathcal{K} = (\sharp, P, g, \flat = 0, \mathbb{V})$ satisfies the following conditions:*

$$\sharp^2\mathbb{X} \quad = \quad P\sharp\mathbb{X} - p\sharp\mathbb{X} + q\mathbb{X},$$

$$\begin{aligned}
\flat(\sharp\mathbb{X}) &= 0,\\
\sharp\mathbb{V} &= P\mathbb{V} - p\mathbb{V} - \flat(\mathbb{N})\mathbb{V},\\
\flat^2(\mathbb{V})\mathbb{N} &= \flat(\mathbb{N})u(\mathbb{N})\mathbb{N} + p\mathbb{N} - (p+1)\flat(\mathbb{V})\mathbb{N} - \flat(\mathbb{V})\mathbb{N},\\
\mathrm{g}(\sharp\mathbb{X},\mathbb{Y}) &= \mathrm{g}(\mathbb{X},\sharp\mathbb{Y}),\\
\mathrm{g}(\sharp\mathbb{X},\sharp\mathbb{Y}) &= \mathrm{g}(P\mathbb{X},\sharp\mathbb{Y}) - p\mathrm{g}(\mathbb{X},\sharp\mathbb{Y}) + p\mathrm{g}(\mathbb{X},\mathbb{Y}).
\end{aligned}$$

Theorem 4.4. *Let $(\mathcal{M}^*, \mathcal{J}, \zeta, \mathrm{g}^*)$ be a meta-metallic pseudo-Riemannian manifold and $\mathcal{H}$ be an invariant lightlike hypersurface of $\mathcal{M}^*$. Then:*

$$\mathbb{H}(\sharp\mathbb{X},\mathbb{Y}) = \mathbb{H}(\mathbb{X},\sharp\mathbb{Y}) = \sharp\mathbb{H}(\mathbb{X},\mathbb{Y}), \tag{12}$$

$$\mathbb{H}(\sharp\mathbb{X},\sharp\mathbb{Y}) = \mathbb{H}(P\mathbb{X},\sharp\mathbb{Y}) - p\mathbb{H}(\sharp\mathbb{X},\mathbb{Y}) + p\mathbb{H}(\mathbb{X},\mathbb{Y}). \tag{13}$$

Proof. A direct computation gives:

$$\begin{aligned}
\nabla^*_{\mathbb{X}}\zeta\mathbb{Y} &= \nabla^*_{\mathbb{X}}(\sharp\mathbb{Y} + \flat(\mathbb{Y})\mathbb{N}) = \nabla_{\mathbb{X}}\sharp\mathbb{Y} + \mathbb{H}(\mathbb{X},\sharp\mathbb{Y}),\\
\zeta\nabla^*_{\mathbb{X}}\mathbb{Y} &= \sharp\nabla^*_{\mathbb{X}}\mathbb{Y} + \flat(\nabla^*_{\mathbb{X}}\mathbb{Y} = \sharp\nabla_{\mathbb{X}}\mathbb{Y} + \sharp\mathbb{H}(\mathbb{X},\mathbb{Y})
\end{aligned}$$

From the relation (12) one has

$$\begin{aligned}
\mathbb{H}(\zeta\mathbb{X},\zeta\mathbb{Y}) &= \mathbb{H}(\mathbb{X},\zeta^2\mathbb{Y})\\
\mathbb{H}(\sharp\mathbb{X},\sharp\mathbb{Y}) &= \mathbb{H}(\mathbb{X},\sharp P\mathbb{Y} + u(\sharp\mathbb{X})\mathbb{N})) - p\mathbb{H}(\mathbb{X},\sharp\mathbb{Y} + \flat(\mathbb{Y})\mathbb{N})) + p\mathbb{H}(\mathbb{X},\mathbb{Y}).
\end{aligned}$$

□

Example 4.5. *Let $\mathcal{M}^* = \mathbb{R}^5$ be a manifold equipped with a pseudo-Riemannian metric g^* of signature $(-,+,+,+,+)$. We define respectively the metallic $\mathcal{J}$ structure and the endomorphism ζ by:*

$$\mathcal{J}(x_1,x_2,x_3,x_4,x_5) = (\wp x_1, \wp x_2, \wp x_3, (p-\wp)x_4, (p-\wp)x_5),$$

and

$$\zeta(x_1,x_2,x_3,x_4,x_5) = (\chi x_1, \chi x_2, \chi x_3, -\tilde{\chi}x_4, -\tilde{\chi}x_5),$$

with $\tilde{\chi} = \frac{\wp+\sqrt{\wp^2+4p}}{2}$ satisfying the relation $\tilde{\chi}^2 = \wp\tilde{\chi} + pI$ and $(p-\wp)\tilde{\chi}^2 = -q\tilde{\chi} + p(p-\wp)$. The pair $(\mathcal{J}, \mathrm{g}^*)$ is indeed a metallic structure.

$$\begin{aligned}
\mathcal{J}\zeta^2(x_1,x_2,x_3,x_4,x_5) &= \mathcal{J}(\chi^2x_1, \chi^2x_2, \chi^2x_3, \tilde{\chi}^2x_4, \tilde{\chi}^2x_5)\\
&= (\wp\chi^2x_1, \wp\chi^2x_2, \wp\chi^2x_3, (p-\wp)\tilde{\chi}^2x_4, (p-\wp)\tilde{\chi}^2x_5).
\end{aligned}$$

If χ is a meta-metallic number then one has the relation $\wp\chi^2 = q\chi + p\wp$ and :

$$\begin{aligned}
\mathcal{J}\zeta^2\mathbb{X} &= ((q\chi + p\wp)x_1, (q\chi + p\wp)x_2, (q\chi + p\wp)x_3, (-q\tilde{\chi} + p(p-\wp))x_4, (-q\tilde{\chi} + p(p-\wp))x_5) \\
&= q(\chi x_1, \chi x_2, \chi x_3, -\tilde{\chi}x_4, -\tilde{\chi}x_5) + p(\wp x_1, \wp x_2, \wp x_3, (p-\wp)x_4, (p-\wp)x_5) \\
&= q\zeta\mathbb{X} + p\mathcal{J}\mathbb{X},
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{g}^*(\zeta\mathbb{X}, \zeta\mathbb{Y}) &= \mathrm{g}^*(\mathbb{X}, \zeta^2\mathbb{Y}) = \mathrm{g}^*(\mathbb{X}, (\mathcal{J}\zeta - p\zeta + pI)\mathbb{Y}) \\
&= \mathrm{g}^*(\mathbb{X}, \mathcal{J}\zeta\mathbb{Y}) - p\mathrm{g}^*(\mathbb{X}, \zeta\mathbb{Y}) + p\mathrm{g}^*(\mathbb{X}, \mathbb{Y}).
\end{aligned}$$

The triplet $(\mathcal{J}, \mathrm{g}^*, \zeta)$ is a nearly metametallic pesudo-Riemannian structure and $(\mathcal{M}^*, \mathrm{g}^*, \mathcal{J}, \zeta)$ is a nearly metametallic pesudo-Riemannian manifold. Consider the hypersurface $\mathcal{H}$ defined by

$$\begin{aligned}
x_1 &= u_1, \\
x_2 &= -u_1 \sin\alpha + u_3 \cos\alpha, \\
x_3 &= u_1 \cos\alpha + u_3 \sin\alpha, \\
x_4 &= u_2, \\
x_5 &= u_4.
\end{aligned}$$

The tangent bundle $\mathcal{TH}$ is generated by the vectors:

$$\begin{aligned}
\mathbb{T}_1 &= (0, -\sin\alpha, \cos\alpha, 0, 0), \quad \mathbb{T}_2 = (0,0,0,1,0), \\
\mathbb{T}_3 &= (1, \cos\alpha, \sin\alpha, 0, 0), \quad \mathbb{T}_4 = (0,0,0,0,1).
\end{aligned}$$

Thus, the hypersurface $\mathcal{H}$ is of light type. The radical distribution $Rad(\mathcal{TH})$ is generated by the vector:

$$\mathbb{T}_3 = (1, \cos\alpha, \sin\alpha, 0, 0)$$

and, the screen distribution $S(\mathcal{TH})$ is generated by the vectors:

$$\mathbb{T}_1 = (0, -\sin\alpha, \cos\alpha, 0, 0), \quad \mathbb{T}_2 = (0,0,0,1,0), \quad \mathbb{T}_4 = (0,0,0,0,1).$$

Let's calculate:

$$\begin{aligned}
\mathcal{J}\mathbb{T}_1 &= \mathcal{J}(0, -\sin\alpha, \cos\alpha, 0, 0) = (0, -\wp\sin\alpha, \wp\cos\alpha, 0, 0) = p\wp\mathbb{T}_1 \in Rad(\mathcal{TH}), \\
\mathcal{J}\mathbb{T}_2 &= \mathcal{J}(0,0,0,1,0) = (0,0,0,(p-\wp),0) = (p-\wp)(0,0,0,1,0) = \wp\mathbb{T}_2 \in S(\mathcal{TH}), \\
\mathcal{J}\mathbb{T}_3 &= \mathcal{J}(1, \cos\alpha, \sin\alpha, 0, 0) = (\wp, \wp\cos\alpha, \wp\sin\alpha, 0, 0) = \wp\mathbb{T}_3 \in Rad(\mathcal{TH}),
\end{aligned}$$

$$\mathcal{J}\mathbb{T}_4 = \mathcal{J}(0,0,0,0,1) = (0,0,0,0,(p-\wp)) = (p-\wp)\mathbb{T}_4 \in Rad(\mathcal{TH}).$$

The distributions $Rad(\mathcal{TH})$, $S(\mathcal{TH})$ are $\mathcal{J}$-invariant. Thus the transverse tangent bundle is generated by the vector:

$$\mathbb{N} = \frac{1}{2}(-\cos\alpha, \sin\alpha, 1, 0, 0).$$

Moreover,

$$\mathcal{J}\mathbb{N} = \frac{1}{2}\mathcal{J}(-\cos\alpha, \sin\alpha, 1, 0, 0) = \frac{1}{2}(-\wp\cos\alpha, \wp\sin\alpha, \wp, 0, 0) = \wp\mathbb{N} \in (ltr(\mathcal{TH})).$$

The 1*-formes $u(X) = g(X, \mathcal{J})$ and $\flat(X) = g(X, \zeta)$ are given as :*

$$\begin{aligned} \flat(X) &= \chi(-\flat_1 + \flat_2\cos\alpha + \flat_3\sin\alpha), \qquad \flat(\mathbb{N}) = \frac{\chi}{4}\sin^2\alpha. \\ u(X) &= \wp(-u_1 + u_2\cos\alpha + u_3\sin\alpha), \qquad u(\mathbb{N}) = \frac{\wp}{4}\sin^2\alpha. \end{aligned}$$

Let's calculate:

$$\begin{aligned} \zeta\mathbb{T}_1 &= (0, -\sin\alpha, \cos\alpha, 0, 0) = (0, -\chi\sin\alpha, \chi\cos\alpha, 0, 0) = \chi\mathbb{T}_1 \in S(\mathcal{TH}), \\ \zeta\mathbb{T}_2 &= (0,0,0,1,0) = (0,0,0,\tilde{\chi},0) = \tilde{\chi}\mathbb{T}_2 \in S(\mathcal{TH}), \\ \zeta\mathbb{T}_3 &= (1, \cos\alpha, \sin\alpha, 0, 0) = (\chi, \chi\cos\alpha, \chi\sin\alpha, 0, 0) = \chi(1, \cos\alpha, \sin\alpha, 0, 0) = \chi\mathbb{T}_3 \in S(\mathcal{TH}), \\ \zeta\mathbb{T}_4 &= \zeta(0,0,0,0,1) = (0,0,0,0,-\tilde{\chi}) = -\tilde{\chi}\mathbb{T}_4 \in Rad(\mathcal{TH}), \end{aligned}$$

and,

$$\zeta\mathbb{N} = \frac{1}{2}\zeta(-\cos\alpha, \sin\alpha, 1, 0, 0) = \frac{1}{2}(-\chi\cos\alpha, \chi\sin\alpha, \chi, 0, 0) = \chi\mathbb{N} \in ltr(\mathcal{TH}).$$

Then, $\mathcal{H}$ is an invariant lightlike hypersurface of a meta-metallic pseudo-Riemannian manifold.

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