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Notion of Non-archimedean Pseudo-differential Operators Associated with Fractional Fourier Transform

Abhisekh Shekhar[1,*], Annu Kumari[2]

[1] *Department of Mathematics, C. M. Science College, Darbhanga, Bihar, India*
[2] *Research Scholar, University Department of Mathematics, L.N.M.U. Darbhanga, Bihar, India*

Abstract

In this manuscript, we define the first type of non-archimedean pseudo-differential operator associated with the fractional Fourier transform and Bessel potentials, denoted by $\mathcal{J}^{\omega},\ \omega > 1$ and second type of non-archimedean pseudo-differential operator $\mathcal{A}^{\omega}$ on $\mathcal{D}(\mathbb{Q}_p)$. We show that these operators holds the positive maximum principle and a strongly continuous, positive, contraction semigroup on $\mathbb{C}_0(\mathbb{Q}_p)$. Also, we solve Cauchy problem (the inhomogeneous initial value problem) related to fractional Fourier transform and these operators.

Keywords: Non-archimedean analysis; Pseudo-differential operators; Fractional Fourier transform; M-dissipative operators; The positive maximum principle.
2020 Mathematics Subject Classification: 26E30, 35S05, 46E35, 47G30, 46F12.

1. Introduction

The connections of non-archimedean pseudo-differential operators with certain p-adic pseudo-differential equations that describe certain physical models [1–9]. Therefore, non-archimedean pseudo-differential operators have received a lot of attention in two decades.Non-archimedean pseudo-differential operators have gained popularity in recent years due to their utility in studying certain equations associated with new physical models/Models in physical form [10–16]. The interest in pseudo-differential operators in the p-adic context has grown significantly in recent years as a result of their utility in modelling various types of physical phenomena. For example, modelling geological processes (such as the formation of petroleum micro-scale reservoirs and fluid flows in porous media such as rock); the dynamics of complex systems such as macromolecules, glasses, and proteins; the study of Coulomb gases, etc. [17–22]. Nonlocal diffusion problems arise in a wide range of applications in the archimedean setting, including biology, image processing, particle systems, and coagulation models. This work is motivated/ inspired by the works of Ismael Gutiérrez García and

*Corresponding author (abhi08.iitkgp@gmail.com)

Anselmo Torresblanca-Badillo [14,21,23]. In 2020, Ismael Gutiérrez García and Anselmo Torresblanca-Badillo studied a class of non-archimedean pseudo-differential operators associated via Fourier transform to the Bessel potentials [14].

In the present manuscript, we introduce notion of non-archimedean pseudo-differential operators associated with Fractional Fourier transform and fractional Bessel potential. Let $\mathbb{Q}_p$ be the set of p-adic numbers and $\mathcal{D}'(\mathbb{Q}_p)$ be the space of distributions in $\mathbb{Q}_p$. If $\psi \in \mathcal{D}'(\mathbb{Q}_p)$ and $\omega \in \mathbb{C}$, we introduce one dimensional p-adic fractional Bessel potential of ψ of order ω as follows:

$$\widehat{(\mathcal{J}^{\omega}\psi)}_{\vartheta}(\zeta) = (\max\{1, |\zeta|\})^{-\omega}\widehat{\psi}_{\theta}(\zeta) = (\max\{1, |\zeta|\})^{-\omega}(\mathcal{F}_{\vartheta}\psi)(\zeta), \quad \forall \zeta \in \mathbb{Q}_p, \tag{1}$$

where $\widehat{\psi}_{\theta}$ is the fractional Fourier transform of ψ in [24–28]. The first type of non-archimedean pseudo-differential operator associated with the fractional Fourier transform and Bessel potentials will be denoted by $\mathcal{J}^{\omega}$, $\omega > 1$. Let $\mathcal{D}(\mathbb{Q}_p)$ be the space of locally constant functions on $\mathbb{Q}_p$ with compact support. Taking inverse fractional Fourier transform on both sides of (1), we get that

$$(\mathcal{J}^{\omega}\psi)_{\vartheta}(\eta) = \mathcal{F}_{\vartheta}^{-1}\left[(\max\{1, |\zeta|\})^{-\omega}\widehat{\psi}_{\vartheta}(\zeta)\right](\eta) = \mathcal{F}_{\vartheta}^{-1}\left[(\max\{1, |\zeta|\})^{-\omega}(\mathcal{F}_{\vartheta}\psi)(\zeta)\right](\eta), \tag{2}$$

$\forall \psi \in \mathcal{D}(\mathbb{Q}_p), \ \forall \eta \in \mathbb{Q}_p.$

It implies that $\mathcal{J}^{\omega}$ is an non-archimedean pseudo-differential operator related to the symbol $(\max\{1, |\zeta|\})^{-\omega}$, $\zeta \in \mathbb{Q}_p$. The interaction of non-archimedean pseudo-differential operators and stochastic processes on p-adics has received a lot of attention in recent decades because of the connection of the p-adic pseudo-differential equations associated with certain physical models, see [29–31]. This fact sparked a great deal of interest in the possibility of obtaining second type of non-archimedean pseudo-differential operators associated with certain stochastic processes on p-adics, resulting in second type of non-archimedean pseudo-differential operators related to Bessel potentials and fractional Fourier transform. This also motivates to define second type of pseudo-differential operators related to Bessel potentials and fractional Fourier transform, denoted by $\mathcal{A}^{\omega}$, $\omega > 0$. This operator $\mathcal{A}^{\omega}$ is defined as follows:

$$\begin{aligned}(\mathcal{A}^{\omega}\psi)(\zeta) &= (-\mathcal{J}^{\omega}\psi)(\zeta) + \psi(\zeta) \\ &= \mathcal{F}_{\vartheta}^{-1}\left[\{1 - (\max\{1, |\eta|\})^{-\omega}\}\widehat{\psi}_{\vartheta}(\eta)\right](\zeta), \ \forall \psi \in \mathcal{D}(\mathbb{Q}_p), \ \forall \zeta \in \mathbb{Q}_p.\end{aligned} \tag{3}$$

The symbol $1 - (\max\{1, |\eta|\})^{-\omega}$, $\eta \in \mathbb{Q}_p$ defines second type of non-archimedean pseudo-differential operator $\mathcal{A}^{\omega}$ on $\mathcal{D}(\mathbb{Q}_p)$.

2. Mathematical Background of Fractional Fourier Analysis on $\mathbb{Q}_p$

Definition 2.1 (The field of p-adic numbers). *Let p be a prime number. Through out this manuscript p will denote a prime number. Firstly we define p-adic norm* $|.|_p$ *on* $\mathbb{Q}$ *as follows*

$$|\eta|_p = \begin{cases} 0, & \text{if } \eta = 0, \\ p^{-\tau}, & \text{if } \eta = p^{\tau}\frac{\rho}{\sigma}, \end{cases}$$

where ρ *and* σ *are integers coprime with p. The integer* $\tau := ord(\eta)$, *with ord(0):=+∞, is called the p-adic order of* η. *The unique expansion of any p-adic number* $\eta \neq 0$ *is of the form*

$$\eta = p^{ord(\eta)} \sum_{i=0}^{\infty} \eta_i p^i, \tag{4}$$

where $\eta_i \in \{0, 1, 2, \ldots, p-1\}$ *and* $\eta_0 \neq 0$. *Using (4), we define the fractional part of* $\eta \in \mathbb{Q}_p$, *denoted by* $\{\eta\}_p$, *as the rational number*

$$\{\eta\}_p = \begin{cases} 0, & \text{if } \eta = 0 \text{ or } ord(\eta) \geq 0, \\ p^{ord(\eta)} \sum\limits_{i=0}^{-ord_p(\eta)-1} \eta_i p^i, & \text{if } ord(\eta) < 0. \end{cases}$$

Extention of the p-adic norm on $\mathbb{Q}_p$ *is given by*

$$||\eta||_p = |\eta|, \quad \forall \eta \in \mathbb{Q}_p.$$

Let $r_0 \in \mathbb{Z}$ *and* $a_0 \in \mathbb{Q}_p$. *We consider* $I_{r_0}(\eta_0) = \{\eta \in \mathbb{Q}_p : ||\eta - \eta_0||_p \leq p^{r_0}\}$. *The empty set and the points are the only connected subsets of* $\mathbb{Q}_p$. *Therefore, the topological space* $(\mathbb{Q}_p, \ ||.||_p)$ *is totally disconnected. The necesssary and sufficient condition for the compactness of a subset of* $\mathbb{Q}_p$ *is that bounded and closed subdet of* $\mathbb{Q}_p$, *see e.g [8].*

3. Few Functional Spaces

A function $f : \mathbb{Q}_p \to \mathbb{C}$ is called locally constant if for any $\eta \in \mathbb{Q}_p$ there exists an integer $r(\eta) \in \mathbb{Z}$ such that $f(\eta + \eta') = f(\eta)$ for all $\eta' \in I_{r(\eta)}$. A function $f : \mathbb{Q}_p \to \mathbb{C}$ is called a test function (or a Bruhat-Schwartz function) if it is a compact support with locally constant. The set of all complex valued test functions on $\mathbb{Q}_p$ is denoted by $\mathcal{D}(\mathbb{Q}_p)$ or simply $\mathcal{D}$. The set of all distributions (all continous functionals) on $\mathcal{D}$ is denoted by $\mathcal{D}'(\mathbb{Q}_p)$ or simply $\mathcal{D}'$. The mapping $\langle U, \psi \rangle : \mathcal{D}'(\mathbb{Q}_p) \times \mathcal{D}(\mathbb{Q}_p) \to \mathbb{C}$ for $U \in \mathcal{D}'(\mathbb{Q}_p)$ and $\psi \in \mathcal{D}(\mathbb{Q}_p)$ is defined as follows:

$$\langle U, \psi \rangle = \int_{\mathbb{Q}_p} U(\zeta)\psi(\zeta) d\zeta.$$

Definition 3.1 (Regular Distribution). *Let M be an arbitrary compact subset of* $\mathbb{Q}_p$. *i.e.* $M \subset \mathbb{Q}_p$. *Then*

$L^1_{loc}(\mathbb{Q}_p) = \{\phi | \phi : \mathbb{Q}_p \to \mathbb{C}$ such that $\phi \in L^1(M)\}$. A distribution $\phi \in \mathcal{D}(\mathbb{Q}_p)$ is defined by every function $\phi \in L^1_{loc}(\mathbb{Q}_p)$ according to the formula

$$\langle \phi, \psi \rangle = \int_{\mathbb{Q}_p} \phi(\zeta)\psi(\zeta)d\zeta.$$

This type of distributions is known as regular distributions.

Let $\sigma \in [0, \infty)$. Then the set $L^\sigma\big(\mathbb{Q}_p,\ dx\big) = \big\{h : \mathbb{Q}_p \to \mathbb{C}$ such that $\int_{\mathbb{Q}_p} |h(x)|^\sigma dx < \infty\big\}$, the set $L^\infty\big(\mathbb{Q}_p,\ dx\big) = \big\{h : \mathbb{Q}_p \to \mathbb{C}$ such that essential supremum of $|h| < \infty\big\}$, the set $C\big(\mathbb{Q}_p, \mathbb{C}\big) = \big\{h : \mathbb{Q}_p \to \mathbb{C}$ and h is a continous function$\big\}$, and the set

$$C_0\big(\mathbb{Q}_p, \mathbb{C}\big) = \big\{h : \mathbb{Q}_p \to \mathbb{C} \text{ and } h \text{ is a continous function and } \lim_{||\zeta||_p \to \infty} h(\zeta) = 0\big\}$$

are complex vector space under the binary operation vector addition (+) and scalar multiplication (.). It also implies that $\big(C_0(\mathbb{Q}_p, \mathbb{C}),\ ||.||_{L^\infty}\big)$ is a Banach space.

4. Fractional Fourier Transform on $\mathbb{Q}_p$

In this chapter, we introduce the definition of fractional Fourier transform on the field of p-adic numbers $\mathbb{Q}_p$. Firstly, the map $\chi_p^\vartheta(.\ ,\ .)$ is defined on $\mathbb{Q}_p$ as follows:
$\forall \zeta,\ \eta \in \mathbb{Q}_p$,

$$\chi_p^\vartheta(\zeta, \eta) = \begin{cases} C^\vartheta e^{\frac{i(\zeta^2+\eta^2)\cot\vartheta}{2} - i\zeta\eta\csc\vartheta}, & \vartheta \neq n\pi, n \in \mathbb{Z} \\ \frac{1}{\sqrt{2\pi}} e^{-i\zeta\eta}, & \vartheta = \frac{\pi}{2}, \end{cases}$$
$$C^\vartheta = \sqrt{\frac{1 - i\cot\vartheta}{2\pi}}.$$

If $\psi \in L^1(\mathbb{Q}_p)$, its fractional Fourier transform of one dimension is defined as follows:

$$(\mathcal{F}_\vartheta \psi)(\eta) = \widehat{\psi}_\vartheta(\eta) = \int_{\mathbb{Q}_p} \chi_p^\vartheta(\zeta, \eta)\psi(\zeta)d\zeta, \quad for\ \eta \in \mathbb{Q}_p. \tag{5}$$

The inverse fractional Fourier transform of a map $\phi \in L^1(\mathbb{Q}_p)$ is

$$(\mathcal{F}_\vartheta^{-1}\phi)(\zeta) = \int_{\mathbb{Q}_p} \chi_p^{-\vartheta}(\zeta, \eta)\phi(\eta)d\eta, \quad for\ \zeta \in \mathbb{Q}_p. \tag{6}$$

The fractional Fourier transform is an isomorphism, continuous and linear map of $\mathcal{D}(\mathbb{Q}_p)$ onto itself holding

$$(\mathcal{F}_\vartheta(\mathcal{F}_\vartheta^{-1}\phi))(\zeta) = (\mathcal{F}_\vartheta^{-1}(\mathcal{F}_\vartheta\phi))(\zeta) = \phi(\zeta), \tag{7}$$

for every $\phi \in \mathcal{D}(\mathbb{Q}_p)$.

5. The Fractional Fourier Transform Related to Non-archimedean Pseudo-differential Operator $\mathcal{J}^{\omega}$

We define first type of non-archimedean pseudo-differential operator $\mathcal{J}^{\omega}$, $\omega > n$ associated with fractional Fourier transform to the Bessel potentials in this chapter.

Definition 5.1. *If $\psi \in \mathcal{D}(\mathbb{Q}_p)$, $\omega \in \mathbb{C}$, we discuss the one dimensional p-adic Bessel potential related to fractional Fourier transform of order ω of ψ as follows:*

$$\widehat{(\mathcal{J}^{\omega}\psi)}_{\vartheta}(\zeta) = (max\ \{1, |\zeta|\})^{-\omega}\widehat{\psi}_{\vartheta}(\zeta) = (max\ \{1, |\zeta|\})^{-\omega}(\mathcal{F}_{\vartheta}\psi)(\zeta), \quad \forall\zeta \in \mathbb{Q}_p. \tag{8}$$

The one dimensional p-adic gamma function Γ_p is defined as

$$\Gamma_p(\omega) = \frac{1 - p^{\omega-1}}{1 - p^{-\omega}}\ \ for\ \omega \neq 0 \in \mathbb{C}.$$

Definition 5.2. *First type of non-archimedean pseudo-differential operator $\mathcal{J}^{\omega} : \mathcal{D}(\mathbb{Q}_p) \to \mathcal{D}(\mathbb{Q}_p)$, $\omega > n$ associated with fractional Fourier transform to the Bessel potentials is defined as follows:*

$$(\mathcal{J}^{\omega}\psi)_{\vartheta}(\eta) = \mathcal{F}_{\vartheta}^{-1}\big[(max\ \{1, |\zeta|\})^{-\omega}\widehat{\psi}_{\vartheta}(\zeta)\big](\eta) = \mathcal{F}_{\vartheta}^{-1}\big[(max\ \{1, |\zeta|\})^{-\omega}(\mathcal{F}_{\vartheta}\psi)(\zeta)\big](\eta), \tag{9}$$

$\forall\psi \in \mathcal{D}(\mathbb{Q}_p)$, $\forall\eta \in \mathbb{Q}_p$, with the symbol $(max\ \{1, |\zeta|\})^{-\omega}$.

Lemma 5.3. *Let t be a possitive real number. Then, the symbol $e^{-t(max\ \{1,|\zeta|\})^{-\omega}} \in L^1(\mathbb{Q}_p)$.*

Proof. For $r \geq 1$, we obtain $e^{-tp^{-r\omega}} \leq 1$. Now, we get that

$$\begin{aligned}\int_{\mathbb{Q}_p} e^{-t(\max\ \{1,|\zeta|\})^{-\omega}} d\zeta &= e^{-t} + (1 - p^{-1})\sum_{r=1}^{\infty} e^{-tp^{-r\omega}} p^r \\ &\leq e^{-t} + \sum_{r=1}^{\infty}(p^r - p^{r-1}) \\ &= e^{-t} - 1 < \infty.\end{aligned}$$

Therefore, the symbol $e^{-t(\max\ \{1,|\zeta|\})^{-\omega}} \in L^1(\mathbb{Q}_p)$. □

Lemma 5.4. *The Cauchy problem involving fractional Fourier transform*

$$\begin{cases} \frac{\partial\varphi(\zeta,\ t)}{\partial t} = \mathcal{J}^{\omega}\varphi(\zeta,\ t), \quad 0 \leq t < \infty, \quad \zeta \in \mathbb{Q}_p \\ \varphi(\zeta,\ 0) = \varphi_0(\zeta) \in \mathcal{D}(\mathbb{Q}_p). \end{cases}$$

Then

$$\varphi(\zeta,\ t) = \int_{\mathbb{Q}_p} \chi_p^{-\vartheta}(\zeta, \eta)\frac{e^{-t(max\{1,\ |\eta|\})^{-\omega}}}{C^{-\vartheta}}(\mathcal{F}_{\vartheta}\varphi_0)(\eta)d\eta$$

is a classical solution of the above Cauchy problem.

Proof. Since, we have

$$\begin{aligned}\frac{\partial \varphi(\zeta,\ t)}{\partial t} &= \mathcal{J}^{\omega}\varphi(\zeta,\ t)\\ &= \mathcal{F}_{\vartheta}^{-1}\big[(\max\{1,|\zeta|\})^{-\omega}\frac{e^{-t(\max\{1,\ |\eta|\})^{-\omega}}}{C^{-\vartheta}}(\mathcal{F}_{\vartheta}\varphi_0)(\eta)\big](\zeta)\\ &= \int_{\mathbb{Q}_p}\chi_p^{-\vartheta}(\zeta,\eta)(\max\{1,|\zeta|\})^{-\omega}\frac{e^{-t(\max\{1,\ |\eta|\})^{-\omega}}}{C^{-\vartheta}}(\mathcal{F}_{\vartheta}\varphi_0)(\eta)d\eta,\end{aligned}$$

for $0\le t<\infty,\ \zeta\in\mathbb{Q}_p$. The formula is based on the fact that

$$\left|\chi_p^{-\vartheta}(\zeta,\eta)\frac{e^{-t(\max\{1,\ |\eta|\})^{-\omega}}}{C^{-\vartheta}}(\mathcal{F}_{\vartheta}\varphi_0)(\eta)\right|\le\big|(\mathcal{F}_{\vartheta}\varphi_0)(\eta)\big|\in L^1(\mathbb{Q}_p)$$

and that

$$\left|\chi_p^{-\vartheta}(\zeta,\eta)(\max\{1,|\zeta|\})^{-\omega}\frac{e^{-t(\max\{1,\ |\eta|\})^{-\omega}}}{C^{-\vartheta}}(\mathcal{F}_{\vartheta}\varphi_0)(\eta)\right|\le\big|(\mathcal{F}_{\vartheta}\varphi_0)(\eta)\big|\in L^1(\mathbb{Q}_p).$$

Now, using the Dominated Convergence Theorem and [16], Lemma 1-(i), we have

$$\varphi(\zeta,.)\in C^1([0.\ \infty)).$$

Since

$$\mathcal{J}^{\omega}\varphi(\zeta,\ t)=\int_{\mathbb{Q}_p}\chi_p^{-\vartheta}(\zeta,\eta)(\max\{1,|\zeta|\})^{-\omega}\frac{e^{-t(\max\{1,\ |\eta|\})^{-\omega}}}{C^{-\vartheta}}(\mathcal{F}_{\vartheta}\varphi_0)(\eta)d\eta$$

for $0\le t<\infty,\ \zeta\in\mathbb{Q}_p$. The formula is based on the fact that

$$\varphi(\zeta,\ t)=\mathcal{F}_{\vartheta}^{-1}\big[(\max\{1,|\zeta|\})^{-\omega}\frac{e^{-t(\max\{1,\ |\eta|\})^{-\omega}}}{C^{-\vartheta}}(\mathcal{F}_{\vartheta}\varphi_0)(\eta)\big](\zeta)\in L^2(\mathbb{Q}_p),$$

for any $t\in\mathbb{R}_+$ and that

$$(\max\{1,|\zeta|\})^{-\omega}\frac{e^{-t(\max\{1,\ |\eta|\})^{-\omega}}}{C^{-\vartheta}}(\mathcal{F}_{\vartheta}\varphi_0)(\eta)\in L^2(\mathbb{Q}_p)$$

for any $t\in\mathbb{R}_+$, by [16], Lemma 1 (i). It implies that

$$\varphi(\zeta,\ t)=\int_{\mathbb{Q}_p}\chi_p^{-\vartheta}(\zeta,\eta)\frac{e^{-t(\max\{1,\ |\eta|\})^{-\omega}}}{C^{-\vartheta}}(\mathcal{F}_{\vartheta}\varphi_0)(\eta)d\eta$$

is a classical solution of the above Cauchy problem. □

6. On $\mathbb{C}_0(\mathbb{Q}_p)$, the Positive Maximum Principle and a Strongly Continuous, Positive, Contraction Semigroup

In this section, we will demonstrate that the operator $-\mathcal{J}^{\omega}$ holds the positive maximum principle on $\mathbb{C}_0(\mathbb{Q}_p)$, as well as that the closure $-\overline{\mathcal{J}^{\omega}}$ of the operator $-\mathcal{J}^{\omega}$ on $\mathbb{C}_0(\mathbb{Q}_p)$ is single-valued and generates a strongly continuous, positive, contraction semigroup $\{U(\zeta)\}$ on $\mathbb{C}_0(\mathbb{Q}_p)$.

Definition 6.1 ([14]). *An operator $\Phi : \mathbb{C}_0(\mathbb{Q}_p) \to \mathbb{C}_0(\mathbb{Q}_p)$ is said to satisfy the positive maximum principle if whenever $\phi \in$ domain of $\Phi \subseteq \mathbb{C}_0(\mathbb{Q}_p)$, $\zeta_0 \in \mathbb{Q}_p$, and $\sup_{\zeta\in\mathbb{Q}_p} \phi(\zeta) = \phi(\zeta_0) \geq 0$ we have $\Phi(\phi(\zeta_0)) \leq 0$.*

Theorem 6.2. *On $\mathbb{C}_0(\mathbb{Q}_p)$,*

(i) the operator

$$(-\mathcal{J}^{\omega}\psi)(\eta) = -\mathcal{F}_{\vartheta}^{-1}\left[(max\ \{1,|\zeta|\})^{-\omega}\widehat{\psi}_{\vartheta}(\zeta)\right](\eta) = -\mathcal{F}_{\vartheta}^{-1}\left[(max\ \{1,|\zeta|\})^{-\omega}(\mathcal{F}_{\vartheta}\psi)(\zeta)\right](\eta),$$

$\eta,\ \zeta \in \mathbb{Q}_p$ and $\psi \in \mathcal{D}(\mathbb{Q}_p)$ holds the positive maximum principle.

(ii) The closure $-\overline{\mathcal{J}^{\omega}}$ of the operator $-\mathcal{J}^{\omega}$ on $\mathbb{C}_0(\mathbb{Q}_p)$ is single-valued and generates a strongly continuous, positive, contraction semigroup $\{U(\zeta)\}$ on $\mathbb{C}_0(\mathbb{Q}_p)$.

Proof. The proof of the Theorem 6.1 is similar to the one given in [14,32,33]. □

Definition 6.3. *The graph of $-\mathcal{J}^{\omega}$ is denoted by $\mathbb{G}(-\mathcal{J}^{\omega})$, defined as follows:*

$$\mathbb{G}(-\mathcal{J}^{\omega}) = \Big\{(\phi,\Psi) \in L^2(\mathbb{Q}_p) \times L^2(\mathbb{Q}_p); \phi \in \mathcal{D}(-\mathcal{J}^{\omega}) \text{ and } \Psi = -\mathcal{J}^{\omega}\phi\Big\}.$$

It is also closed in $L^2(\mathbb{Q}_p)$.

Definition 6.4. *An operator Ψ with domain $D(\Psi)$ in $L^2(\mathbb{Q}_p)$ is said to be dissipative if $re\langle\Psi\phi,\phi\rangle \leq 0$, where $L^2(\mathbb{Q}_p)$ is the Hilbert space with the inner product*

$$\langle\phi,\varphi\rangle = \int_{\mathbb{Q}_p} \phi(\zeta)\overline{\varphi(\zeta)}d\zeta, \quad \phi,\ \varphi \in L^2(\mathbb{Q}_p).$$

Theorem 6.5. *The operator $-\mathcal{J}^{\omega}$ is dissipative in $L^2(\mathbb{Q}_p)$.*

Proof.

$$\begin{aligned}\langle -\mathcal{J}^{\omega}\phi,\phi\rangle &= \Big\langle -\mathcal{F}_{\vartheta}^{-1}\left[(\max\ \{1,|\zeta|\})^{-\omega}\widehat{\psi}_{\vartheta}(\zeta)\right](\eta),\psi\Big\rangle \\ &= \langle -(\max\ \{1,|\zeta|\})^{-\omega}\widehat{\psi}_{\vartheta}(\zeta),\widehat{\psi}_{\vartheta}(\zeta)\rangle \\ &= -\int_{\mathbb{Q}_p}(\max\ \{1,|\zeta|\})^{-\omega}|\widehat{\psi}_{\vartheta}(\zeta)|^2 d\zeta \leq 0.\end{aligned}$$

The proof of the Theorem 6.5 is achieved. □

Definition 6.6 ([34]). *An operator Ψ in $L^2(\mathbb{Q}_p)$ is m-dissipative if*

(i) Ψ *is dissipative;*

(ii) for all $\rho > 0$ and all $\phi \in L^2(\mathbb{Q}_p)$, there exists φ belongs to domain of Ψ such that $\varphi - \rho\Psi\varphi = \phi$.

Theorem 6.7. *The operator $-\mathcal{J}^\omega$ is self-adjoint. It implies that*

$$\langle -\mathcal{J}^\omega \phi, \psi \rangle = \langle \phi, -\mathcal{J}^\omega \psi \rangle, \qquad \forall \phi, \psi \in L^2(\mathbb{Q}_p).$$

Proof. Let $\phi, \psi \in\in L^2(\mathbb{Q}_p)$. Using the Parseval-Steklov equality, we obtain that

$$\begin{aligned}
\langle -\mathcal{J}^\omega \phi, \psi \rangle &= \left\langle -\mathcal{F}_\vartheta^{-1}[(\max\{1, |\zeta|\})^{-\omega} \widehat{\phi}_\vartheta(\zeta)], \psi \right\rangle \\
&= -\int_{\mathbb{Q}_p} (\max\{1, |\zeta|\})^{-\omega} \widehat{\phi}_\vartheta(\zeta) \overline{\widehat{\psi}_\vartheta(\zeta)} d\zeta \\
&= -\int_{\mathbb{Q}_p} \overline{(\max\{1, |\zeta|\})^{-\omega}} \widehat{\phi}_\vartheta(\zeta) \left[\int_{\mathbb{Q}_p} \chi_p^\vartheta(\eta, \zeta) \overline{\psi(\eta)} d\eta \right] d\zeta \\
&= \int_{\mathbb{Q}_p} \widehat{\phi}_\vartheta(\zeta) \overline{(\max\{1, |\zeta|\})^{-\omega} \widehat{\psi}_\vartheta(\zeta)} d\zeta \\
&= \left\langle \phi, -\mathcal{F}_\vartheta^{-1}[(\max\{1, |\zeta|\})^{-\omega} \widehat{\psi}_\vartheta(\zeta)] \right\rangle \\
&= \langle \phi, -\mathcal{J}^\omega \psi \rangle, \qquad \forall \phi, \psi \in L^2(\mathbb{Q}_p).
\end{aligned}$$

The proof of the Theorem 6.7 is achieved. □

Theorem 6.8. *M-dissipative property is satisfied by the operator $-\mathcal{J}^\omega : L^2(\mathbb{Q}_p) \to L^2(\mathbb{Q}_p)$.*

Proof. The proof of the Theorem 6.8 is followed by the Theorem 6.5 and Theorem 6.7, some well-understood results in the theory of dissipative operator [34]. □

7. Second Type of Non-archimedean Pseudo-differential Operator $\mathcal{A}^\omega$ on $\mathcal{D}(\mathbb{Q}_p)$

We will define second type of non-archimedean pseudo-differential operator $\mathcal{A}^\omega$, $\omega > 0$ on $\mathcal{D}(\mathbb{Q}_p)$ with the help of fractional Fourier transform and Bessel potentials in this chapter. This operator is combined by a linear combination of the Bessel potentials and the Identity operator.

Definition 7.1. *Let $\psi \in \mathcal{D}_p(\mathbb{Q}_p)$. We will define and denote the non-archimedean pseudo-differential operator $\mathcal{A}^\omega$, $\omega > 0$ as follows:*

$$(\mathcal{A}^\omega \psi)(\zeta) = (-\mathcal{J}^\omega + \mathcal{I})\psi(\zeta), \quad \zeta \in \mathbb{Q}_p,$$

where $\mathcal{I}$ is the identity operator. Now, we get that

$$\begin{aligned}
(\mathcal{A}^\omega \psi)(\zeta) &= (-\mathcal{J}^\omega \psi(\zeta) + \psi(\zeta) \\
&= -\mathcal{F}_\vartheta^{-1}\left[(max\{1, |\zeta|\})^{-\omega} \widehat{\psi}_\vartheta(\zeta) \right] + \psi(\zeta)
\end{aligned}$$

$$= -\mathcal{F}_{\vartheta}^{-1}\Big[(max\{1,|\zeta|\})^{-\omega}\widehat{\psi}_{\vartheta}(\zeta) - \widehat{\psi}_{\vartheta}(\zeta)\Big]$$
$$= \mathcal{F}_{\vartheta}^{-1}\Big[\Big\{1-(max\{1,|\zeta|\})^{-\omega}\Big\}\widehat{\psi}_{\vartheta}(\zeta)\Big],$$

i.e, using the symbol $1-(max\{1,|\zeta|\})^{-\omega}$, $\omega > 0$, $\zeta \in \mathbb{Q}_p$, *second type of non-archimedean pseudo-differential operator* $\mathcal{A}^{\omega}$ *on* $\mathcal{D}(\mathbb{Q}_p)$ *is defined.*

Lemma 7.2. *The Cauchy problem involving the inverse of the fractional Fourier transform*

$$\begin{cases} \frac{\partial\psi(\zeta,\, t)}{\partial t} = -(\mathcal{A}^{\omega}\psi)(\zeta,\, t), \quad 0 \le t < \infty, \quad \zeta \in \mathbb{Q}_p \\ \psi(\zeta,\, 0) = \psi_0(\zeta) \in \mathcal{D}(\mathbb{Q}_p). \end{cases}$$

Then

$$\psi(\zeta,\, t) = \int_{\mathbb{Q}_p} \chi_p^{-\vartheta}(\zeta,\eta)\frac{e^{-t(max\{1,\, |\eta|\})^{-\omega}}}{C^{-\vartheta}}(\mathcal{F}_{\vartheta}\psi_0)(\eta)d\eta$$

is a classical solution of the above Cauchy problem.

Proof. Proof of the Lemma 7.2 is similar to the Lemma 5.4. □

8. Conclusion

We will show that the heat equation associated to the operator $\mathcal{J}^{\omega}$, $\omega > 1$, describes the cooling (or loss of heat) in a given region over time, since the fundamental solution $\mathcal{Z}(\zeta, t)$ (explicitly represented), of real positive time variable and p-adic spatial variables satisfies either $\mathcal{Z}(\zeta, t) \ge 0$ or $\mathcal{Z}(\zeta, t) \le 0$. We will also study Feller semigroups and stochastic processes related to the second type of non-archimedean pseudo-differential operator $\mathcal{A}^{\omega}$ on $\mathcal{D}(\mathbb{Q}_p)$.

Discussion for Further Research

There are potential directions for future research of my manuscript by using many types of integral transformations.

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