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A Note on Non-cut Points, VH-sets, Almost VH-sets and COTS

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Abstract

Two concepts of VH-sets and almost VH-sets are introduced to study connected ordered topological spaces (COTS). We show that every $R(i)$ subset of a connected space is an almost VH-set. Two characterizations of COTS with endpoints using the concept of almost VH-set have been obtained. It is also proved that if X is a connected non-cut point inclined space and the removal of any two-point disconnected set of it leaves the space disconnected, then each one of $H \cup \{a, b\}$ and $K \cup \{a, b\}$ has exactly two non-cut points, and is homeomorphic to a finite connected subspace of the Khalimsky line where H and K are separating sets of $X \setminus \{a, b\}$, $a, b \in X$.

Keywords: connected space; COTS; VH-set; almost VH-set; non-cut point.
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1. Introduction

Topological spaces having at least three points are assumed to be connected for any consideration of cut points. The concept of COTS (= connected ordered topological space), is defined by Khalimsky, Kopperman and Meyer in [8]. Some properties of COTS are studied in [8] and [7]. Several characterizations of COTS with endpoints are obtained in [2–6]. In notation and terminology, we will follow [6] and [9]. In this paper, we introduce two concepts namely VH-set and almost VH-set to study connected ordered topological spaces (COTS). The main results of the paper appear in Sections 2, 3 and 4. In Section 2, we prove that if H is a VH-set of a COTS, then $H \subset S[a, b]$ for some a and b of H. Using this result, it is shown that if H is a VH-set of COTS X, then there exist a and b in H such that H is contained in a COTS with endpoints a and b. In Section 3, we show that every $R(i)$ subset of a connected space is an almost VH-set. Also, we prove that if a connected space X has a non-degenerate almost VH-set H such that there is no proper regular closed connected subset of X the interior of which contains H, then there is no proper non-empty cut point convex subset of X containing all the non-cut points of X. Thus Theorem 4.2 of [6] is strengthened since $R(i)$ subsets of

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connected spaces are almost VH-sets. A connected space X is a COTS with endpoints iff X has at most two non-cut points and an almost VH-set H such that there is no proper regular closed, connected subset of X the interior of which contains H. In Section 4, it is proved that if X is a connected non-cut point inclined space and the removal of any two-point disconnected set of it leaves the space disconnected, then each one of $H \cup \{a, b\}$ and $K \cup \{a, b\}$ has exactly two non-cut points, and is homeomorphic to a finite connected subspace of the Khalimsky line where H and K are separating sets of $X \setminus \{a, b\}$, $a, b \in X$.

2. VH-sets and COTS

We say that a non-degenerate subset H of a topological space X is a VH-set if for every non-empty cut point convex subset Y of X such that $\emptyset \neq (H \setminus Y) \subset \mathrm{ct}X$, there exists some $q \in H \setminus Y$ such that $H \subset \mathrm{cl}_X((A_q(Y))^{+q})$.

In view of Lemma 3.13 of [6], if X is connected, then every non-degenerate H-set is VH-set. In fact, every non-degenerate cut point H-set [6] of a connected space is VH-set. Therefore, following Theorems 2.1, 2.2, 2.3, 2.4 and Corollary 2.5 strengthens Lemma 3.13, Theorems 3.14, 3.15, 3.20 and Corollary 3.21 of [6] respectively. We also note that the proofs of following theorems 2.1, 2.2, 2.3, 2.4 and Corollary 2.5 follow on the lines of proofs of Lemma 3.13, Theorems 3.14, 3.15, 3.20 and Corollary 3.21 of [6] respectively. For completeness, we have included the proofs.

Theorem 2.1. *Let X be a connected space. If H is a non-degenerate VH-set of X, then for every non-empty cut point convex subset Y of X such that $\emptyset \neq (H \setminus Y) \subset ctX$, there exists some $z \in H \setminus Y$ such that $H \subset (A_z(Y))^{+z}$.*

Proof. Let Y be a non-empty cut point convex subset of X such that $\emptyset \neq (H \setminus Y) \subset \mathrm{ct}X$. Since H is VH-set, there exists some $q \in H \setminus Y$ such that $H \subset \mathrm{cl}_X((A_q(Y))^{+q})$. If $H \not\subset (A_q(Y))^{+q}$, then there exists some $p \in H \cap B_q$. Since $A_q(Y) \cap B_q = \emptyset$, $p \in H \setminus Y$ and so p is a cut point of X. Since $(A_q(Y))^{+q}$ is connected by Lemma 2.1 of [6] and $(A_q(Y))^{+q} \subset X \setminus \{p\}$, we have $(A_q(Y))^{+q} \subset A_p(Y)$, where $A_p(Y)$ is the separating set of $X \setminus \{p\}$ containing Y. Therefore $H \subset \mathrm{cl}_X((A_q(Y))^{+q}) \subset \mathrm{cl}_X(A_p(Y)) \subset (A_p(Y))^{+p}$, the last containment is in view of Lemma 2.1 (a) of [4]. The proof is complete. □

Theorem 2.2. *Let X be a connected space. Let $H \subset X$ be a non-degenerate VH-set such that $H \subset ctX$.*

(i) Let $z \in H$. There is some $q_z \in H$ such that $H \subset (A_{q_z})^{+q_z}$.

(ii) If H is not connected, then for each connected component K of H, there exists $q_K \in H \setminus K$ such that $H \subset (A_{q_K}(K))^{+q_K}$. Therefore, $H \subset \bigcap\{(A_{q_K}(K))^{+q_K} : K$ is a connected component of $H\}$.

Proof. (i) If $H \not\subset (A_z)^{+z}$, then $H \setminus (A_z)^{+z} \neq \emptyset$. Now $(A_z)^{+z}$ is cut point convex by Proposition 3.1(i) of [5]. Since H is VH-set, so using Theorem 2.1, there is some $q_z \in H \setminus (A_z)^{+z}$ such that $H \subset (A_{q_z}((A_z)^{+z}))^{+q_z}$.

(ii) K is cut point convex by Proposition 3.1 (i) of [5], and $H \setminus K \neq \emptyset$ as H is not connected. Therefore using Theorem 2.1, there is some $q_K \in H \setminus K$ such that $H \subset (A_{q_K}(K))^{+q_K}$ as H is VH-set. Thus $H \subset \bigcap\{(A_{q_K}(K))^{+q_K} : K \text{ is a connected component of } H\}$. □

Theorem 2.3. *Let X be a connected space and $H \subset X$ is a non-degenerate VH-set.*

(i) If $z \in H$ is such that $H \setminus \{z\} \subset ctX$, then there exist distinct points q and p of H such that either $H \subset S[p,q] \subset (A_q(p))^{+q}$, or $H \subset (A_q(z))^{+q} \cap (A_p(S[z,q]))^{+p}$.

(ii) If $H \subset ctX$, then for each $z \in H$, there exist distinct points q and p of H such that $H \subset S[p,q] \subset (A_q(p))^{+q} \cap (A_p(q))^{+p}$, or $H \subset (A_q(z))^{+q} \cap (A_p(S[z,q]))^{+p}$.

Proof. (i) By Theorem 2.1, there is some $q \in H \setminus \{z\}$ such that $H \subset (A_q(z))^{+q}$. If $H \subset S[z,q]$, then the result hold by taking $z = p$ and by Proposition 3.1(i) of [5]. Otherwise $H \setminus S[z,q] \neq \emptyset$. $S[z,q]$ is a cut point convex set using Lemma 2.1(II) of [2], so, by Theorem 2.1, there is some $p \in H \setminus S[z,q]$ such that $H \subset (A_p(S[z,q]))^{+p}$.
(ii) Let $z \in H$. Since $H \subset$ ct$X$, $H \setminus \{z\} \subset$ ct$X$. Therefore, by (i), there exist distinct points q and p of H such that either $H \subset S[p,q] \subset (A_q(p))^{+q}$, or $H \subset (A_p(S[z,q]))^{+p} \cap (A_q(z))^{+q}$. Since $H \subset$ ct$X$, $p \in$ ct$X$. $A_p(q)^{+p}$ being connected by Lemma 2.1 of [6], is cut point convex by Proposition 3.1(i) of [5]. So $S[p,q] \subset A_p(q)^{+p}$. This completes the proof of (ii). □

Theorem 2.4. *Let X be a COTS. If H is an VH-set of X, then $H \subset S[a,b]$ for some $a,b \in H$.*

Proof. By Proposition 2.5 of [8], X has at most two non-cut points. So we can find $a,b \in H$, with $a < b$ in the given total order $\leq$ of X, such that $H \setminus \{a,b\} \subset$ ct$X$. If $H \subset S[a,b]$, we are done. Otherwise $H \setminus S[a,b] \neq \emptyset$. In view of Lemma 2.1(II) of [2], $S[a,b]$ is a cut point convex set. Therefore, $H \subset (A_p(S[a,b]))^{+p}$ for some $p \in H \setminus S[a,b]$ using Theorem 2.1 as H is VH-set. Since $p \in$ ct$X$, $A_p(S[a,b])$ equals $L(p)$ or $R(p)$ in view of Theorem 2.7 of [8].
Case 1: $A_p(S[a,b]) = L(p)$. For every $h \in H$, $h < p$, and $a < b < p$. If $H \subset S[a,p]$, we are done. Otherwise $H \setminus S[a,p] \neq \emptyset$. In view of Lemma 3.19 of [6], $b \in S(a,p)$, so $H \setminus S[a,p] \subset$ ct$X$. $S[a,p]$ is cut point convex, so $H \subset (A_t(S[a,p]))^{+t}$ for some $t \in H \setminus S[a,p]$ using Theorem 2.1 since H is VH-set. Since $t \in$ ct$X$, in view of Theorem 2.7 of [8], $(A_t(S[a,p]))^{+t}$ equals $L(t)$ or $R(t)$. But $t < p$, so $(A_t(S[a,p]))^{+t} = R(t)$. This implies that for every $h \in H$, $t < h$. Therefore $H \subset S[t,p]$ using Lemma 3.19 of [6].
Case 2: $A_p(S[a,b]) = R(p)$. In this case, $p < a < b$, and, for every $h \in H$, $p < h$. If $H \subset S[p,b]$, we are done. Otherwise $H \setminus S[p,b] \neq \emptyset$. In view of Lemma 3.19 of [6], $a \in S(p,b)$, so $H \setminus S[p,b] \subset$ ct$X$. $S[p,b]$ is cut point convex, so $H \subset (A_t(S[p,b]))^{+t}$ for some $t \in H \setminus S[p,b]$ using Theorem 2.1 as H is VH-set. Since $t \in$ ct$X$, So $(A_t(S[p,b]))^{+t}$ equals $L(t)$ or $R(t)$. But $p < t$. So $(A_t(S[p,b]))^{+t} = L(t)$. Therefore $H \subset S[p,t]$ using Lemma 3.19 of [6]. The proof is complete. □

Corollary 2.5. *Let X be a COTS. If H is a VH-set of X, then there exist a and b in H such that H is contained in a COTS with endpoints a and b.*

Proof. By Theorem 2.4, $H \subset S[a,b]$ for some $a, b \in H$. Since by Theorem 3.2(b) of [8], $S[a,b]$ is a COTS, the result follows. □

3. Almost VH-sets and COTS

Call a non-degenerate subset H of a topological space X an almost VH-set if for every non-empty cut point convex subset Y of X such that $\emptyset \neq (X \setminus Y) \subset \text{ct}X$, there exists some $q \in X \setminus Y$ such that $H \subset \text{cl}_X((A_q(Y))^{+q})$. In view of Lemma 3.1 of [6], we note that every non-degenerate subset of a connected space with endpoints is an almost VH-set.

Theorem 3.1. *Let X be a connected space. If H is an $R(i)$ subset of X, then H is an almost VH-set.*

Proof. Let Y be a proper non-empty cut point convex set such that $\emptyset \neq (X \setminus Y) \subset \text{ct}X$. As $X \setminus Y \subset \text{ct}X$, by Theorem 3.7 of [6], there exists an infinite chain α of proper regular closed connected sets of the form $(A_x(Y))^{+x}$ where $x \in X \setminus Y$, x a closed point of X, covering X. Since H is $R(i)$, $H \cap B_x = \emptyset$ for some $(A_x)^{+x} \in \alpha$ using Lemma 4.1 of [6]. Therefore $H \subset A_x(Y)^{+x}$. Thus H is an almost VH-set. □

In view of Theorem 3.1, the following result strengthens Theorem 4.2 of [6].

Theorem 3.2. *If a connected space has a non-degenerate almost VH-set H such that there is no proper regular closed connected subset of X the interior of which contains H, then there is no proper non-empty cut point convex subset of X containing all the non-cut points of X.*

Proof. We prove the result by contradiction. Let Y be a proper non-empty cut point convex set of X with $X \setminus Y \subset \text{ct}X$. Then there exists some $q \in X \setminus Y$ such that $H \subset \text{cl}_X((A_q(Y))^{+q})$ as H is almost VH-set. So in view of Remark 2.3 of [4] and Lemma 3.5 (i) of [6], we have $H \subset A_y(Y)^{+y}$ for some closed point $y \in X \setminus Y$. Take $z \in B_y$. Then $z \in \text{ct}X$. So $A_y(Y)^{+y} \subset A_z(Y)$. Therefore $H \subset A_z(Y)$. Now by remark 2.3 of [4], $H \subset \text{int}_X(A_z(Y)^{+z})$. So $H \subset \text{int}_X(\text{cl}_X(A_z(Y)^{+z}))$. By Lemma 3.5(ii) of [6], $\text{cl}_X(A_z(Y)^{+z})$ is a proper connected regular closed subset of X. This gives a contradiction to the given condition. The proof is complete. □

Theorem 3.3. *If a connected space has a non-degenerate almost VH-set H such that there is no proper regular closed connected subset of X the interior of which contains H, then X has at least two non-cut point.*

Theorem 3.4. *A connected space X has at most two non-cut points and an almost VH-set H such that there is no proper regular closed, connected subset of X the interior of which contains H iff X is a COTS with endpoints.*

Proof. If X has an almost VH-set H such that there is no proper regular closed, connected subset of X containing H, then, by Theorem 3.3, X has at least two non-cut points. Therefore, by the given condition, X has exactly two non-cut points, say, a and b. Let $x \in X \setminus \{a, b\}$. Then $x \in \text{ct}X$. Since each of $(A_x)^{+x}$ and $(B_x)^{+x}$ is connected, so using Theorem 3.2, there is no proper connected subset of X containing $X \setminus \mathrm{ct}X$. But $X \setminus \mathrm{ct}X = \{a, b\}$. So $a \in A_x$ and $b \in B_x$ or conversely. This implies that $x \in S(a, b)$. Hence $X = S[a, b]$. Now by Theorem 3.2 of [2], X is a COTS with endpoints a and b. Conversely suppose that X is a COTS with endpoints. Then X is a connected space with endpoints, say, a and b. Therefore, by Lemma 3.1 of [6], there is no proper cut point convex set of X containing $\{a, b\}$ since using theorem 3.1 of [2], a and b are the only non-cut points of X. This implies that there is no proper regular closed, connected subset of X the interior of which contains $\{a, b\} = H$. Also, H is almost VH-set. The proof is complete. □

Theorem 3.5. *If a connected space has a non-degenerate almost VH-set H such that there is no proper regular closed connected subset of X the interior of which contains H, then for each $x \in ctX$, A_x contains a non-cut point of X.*

Proof. If we suppose to the contrary, then $A_x \subset \mathrm{ct}X$. This implies that $X \setminus \mathrm{ct}X \subset (B_x)^{+x}$, where B_x is the other separating set of $X \setminus \{x\}$. But this contradicts Theorem 3.2 as $(B_x)^{+x}$ is connected. □

Theorem 3.6. *If a connected and locally connected space X has at most two non-cut points and a non-degenerate almost VH-set such that there is no proper regular closed connected subset of X the interior of which contains H, then X is a compact COTS with endpoints.*

Proof. By Theorem 3.4, X is a COTS with endpoints. Now the theorem follows by Theorem 4.4 of [3]. □

Theorem 3.7. *If a T_1 separable, connected and locally connected space X has at most two non-cut points and a non-degenerate almost VH-set such that there is no proper regular closed connected subset of X the interior of which contains H, then X is homeomorphic to the closed unit interval.*

Proof. By Theorem 3.4, X is a COTS with endpoints. Now the theorem follows by Corollary 6.2 (i) of [3]. □

The following result is an analogous to Theorem 2.1.

Theorem 3.8. *Let X be a connected space. If H is a non-degenerate almost VH-set of X, then for every non-empty cut point convex subset Y of X such that $\emptyset \neq (X \setminus Y) \subset ctX$, there exists some $z \in X \setminus Y$ such that $H \subset (A_z(Y))^{+z}$.*

The following results are analogous to Theorems 3.2 and 3.3 respectively.

Theorem 3.9. *Let P be a cut point hereditary property. If a connected space having the property P has a non-degenerate almost VH-set H such that there is no proper regular closed connected subset having the property P, of X the interior of which contains H, then there is no proper non-empty cut point convex subset of X containing all non-cut points of X.*

Proof. Let Y be a proper non-empty cut point convex set of X with $X \setminus Y \subset \text{ct}X$. Then there exists some $q \in X \setminus Y$ such that $H \subset (A_q(Y))^{+q}$ using Theorem 3.8 as H is almost VH-set. By Lemma 3.5 of [1], B_q contains a closed point, say z of X. Then $z \in \text{ct}X$. Since P is cut point hereditary, it follows that $A_q(Y)^{+q} \subset A_z(Y)$ and $A_z(Y)^{+z}$ has Property P. Therefore $H \subset A_z(Y)$. Now by Remark 2.3 of [4], $H \subset \text{int}_X(A_z(Y)^{+z})$ which is a contradiction to given condition as $A_z(Y)^{+z}$ is a proper connected regular closed subset having property P, of X. The proof is complete. □

Theorem 3.10. *Let P be a cut point hereditary property. If a connected space having the property P has a non-degenerate almost VH-set H such that there is no proper regular closed connected subset having the property P, of X the interior of which contains H, then X has at least two non-cut point.*

The following theorems are analogous to Theorems 3.4, 3.6 and 3.7 respectively.

Theorem 3.11. *Let P be a cut point hereditary property. A connected space X having the property P has at most two non-cut points and an almost VH-set H such that there is no proper regular closed, connected subset having property P, of X the interior of which contains H iff X is a COTS with endpoints.*

Theorem 3.12. *Let P be cut point hereditary. If a connected and locally connected space X has at most two non-cut points and a non-degenerate almost VH-set such that there is no proper regular closed connected subset having property P, of X the interior of which contains H, then X is a compact COTS with endpoints.*

Theorem 3.13. *Let P be a cut point hereditary property. If a T_1 separable, connected and locally connected space X has at most two non-cut points and a non-degenerate almost VH-set such that there is no proper regular closed connected subset having property P of X the interior of which contains H, then X is homeomorphic to the closed unit interval.*

4. Non-cut Points and Finite Connected Subspaces of the Khalimsky Line

In Theorem 3.5 of [3], it is assumed that $\text{cd}(X)$ is finite, which is too stringent a condition. Assuming a space to be non-cut point inclined [6] is comparatively weaker. To prove a result like Theorem 3.5 of [3] by assuming the space to be non-cut point inclined we need the help of following lemma.

Lemma 4.1. *For a connected space X, let $a, b \in X$ with $a \neq b$ be non-cut points of X. Let H be a separating set of $X \setminus \{a, b\}$. Then*

(1) $cd(H \cup \{a, b\}) \subset cd(X)$.

(2) If B is a regular closed connected subset of $H \cup \{a, b\}$, then either B is regular closed in X or $cl_X(B)$ is regular closed in X.

(3) If X has non-cut point inclined property, then $H \cup \{a, b\}$ has non-cut point inclined property.

Proof. Let K be the other separating set of $X \setminus \{a,b\}$ corresponding to H. Following the proof of Lemma 3.11(I) of [3], we have either $\text{cl}_X(H) = H$ and $\text{cl}_X(K) = K$ or $\text{cl}_X(H) = H \cup \{a,b\}$ and $\text{cl}_X(K) = K \cup \{a,b\}$ or $\text{cl}_X(H) = H \cup \{a\}$ and $\text{cl}_X(K) = K \cup \{a\}$ or $\text{cl}_X(H) = H \cup \{b\}$ and $\text{cl}_X(K) = K \cup \{b\}$.

(1) To prove (1), it suffices to show that $a,b \in \text{cd}(X)$ whenever $a,b \in \text{cd}(H \cup \{a,b\})$. Suppose that $a \in \text{cd}(H \cup \{a,b\})$. Then $a \in \text{cd}(H \cup \{a\})$. If $\text{cl}_X(H) = H \cup \{a,b\}$ or $\text{cl}_X(H) = H \cup \{a\}$, then we are done. If $\text{cl}_X(H) = H$, then $H \cup \{a\}$ becomes disconnected which is not possible because using Lemma 2.1 of [6], $H \cup \{a\}$ is connected as $b \in X \setminus \text{ct}X$. Finally consider the case $\text{cl}_X(H) = H \cup \{b\}$ and $\text{cl}_X(K) = K \cup \{b\}$. This implies that $\{a\}$ is open in X, and so $\{a\}$ is open in $H \cup \{a\}$ which gives a contradiction as $a \in \text{cd}(H \cup \{a\})$ and $H \cup \{a\}$ is connected. Thus $a \in \text{cd}(X)$. Similarly, $b \in \text{cd}(X)$ if $b \in \text{cd}(H \cup \{a,b\})$. This completes proof of (1).

(2) Using given conditions, $a \in \text{ct}(X \setminus \{b\})$ and $b \in \text{ct}(X \setminus \{a\})$. So, using Lemma 2.2 (e) of [4], the following four cases arise.

Case (i): $\{a\}$ is open in $X \setminus \{b\}$ and $\{b\}$ is open in $X \setminus \{a\}$. This implies that $a,b \notin \text{cl}_X(K)$. So, K is closed in X as $\text{cl}_X(K) \subset K \cup \{a,b\}$. Thus $H \cup \{a,b\}$ is open in X. Therefore $\text{cl}_X(B)$ is regular closed in X by Remark 2.3(b)(i) of [6].

Case (ii): $\{a\}$ is closed in $X \setminus \{b\}$ and $\{b\}$ is closed in $X \setminus \{a\}$. Then $a,b \in \text{cl}_X(H) \cap \text{cl}_X(K)$. Therefore $\text{cl}_X(H) = H \cup \{a,b\}$ and $\text{cl}_X(K) = K \cup \{a,b\}$. This implies that $H \cup \{a,b\}$ is regular closed in X. Therefore, B is regular closed in X by Lemma 2.4 of [6].

Case (iii): $\{a\}$ is closed in $X \setminus \{b\}$ and $\{b\}$ is open in $X \setminus \{a\}$. Then $a \in \text{cl}_X(H) \cap \text{cl}_X(K)$ and $b \notin \text{cl}_X(H) \cup \text{cl}_X(K)$. This implies that $\{a\}$ is closed in X and $H \cup \{b\}$ is open in X. Let $B = \text{cl}_{H \cup \{a,b\}}(G)$ for some G open in $H \cup \{a,b\}$. Since $H \cup \{a,b\}$ is connected, $G \setminus \{a\} \neq \emptyset$. Also, $G \setminus \{a\}$ is open in $H \cup \{a,b\}$ as $G \setminus \{a\} \subset H \cup \{b\}$ and $H \cup \{b\}$ is open in X. If $a \notin G$, then $B = \text{cl}_{H \cup \{a,b\}}(G \setminus \{a\})$. If $a \in G$, then, since $B = (\text{cl}_{H \cup \{a,b\}}(G \setminus \{a\})) \cup \{a\}$ and B is connected, $a \in \text{cl}_{H \cup \{a,b\}}(G \setminus \{a\})$. So, $B = \text{cl}_{H \cup \{a,b\}}(G \setminus \{a\})$. Use of Remark 2.3(a) of [6] now implies $\text{cl}_X(B)$ is regular closed in X.

Case (iv): $\{a\}$ is open in $X \setminus \{b\}$ and $\{b\}$ is closed in $X \setminus \{a\}$. In this case, $\text{cl}_X(B)$ is regular closed in X follows on the lines of case (iii). The proof of (2) is complete.

(3) Let B be a proper non-degenerate regular closed connected subset of $H \cup \{a,b\}$. Then by part (2), either B or $\text{cl}_X(B)$ is regular closed in X. Also, $\text{cl}_X(B) \neq X$ as B is a proper subset of $H \cup \{a,b\}$. Therefore, by given condition, $\text{cd}(X) \cap B$ is finite. Using part (I), $\text{cd}(H \cup \{a,b\}) \cap B$ is finite.

□

In the outcome of the above lemma, $H \cup \{a,b\}$ and $K \cup \{a,b\}$ behave alike. Also, we note below that if a connected space has non-cut point inclined property, then $H \cup \{a,b\}$ and $K \cup \{a,b\}$ behave alike regarding non-cut points and being COTS. The following is a result like Theorem 3.5 of [3] proved under weaker assumption.

Theorem 4.2. *Let X be a connected non-cut point inclined space and the removal of any two-point disconnected set of it leaves the space disconnected. If H and K are separating sets of $X \setminus \{a, b\}$, $a, b \in X$, then each one of $H \cup \{a, b\}$ and $K \cup \{a, b\}$ has exactly two non-cut points, and is homeomorphic to a finite connected subspace of the Khalimsky line.*

Proof. By Corollary 5.9 of [6], $\text{ct}X = \emptyset$. So each one of $H \cup \{a, b\}$ and $K \cup \{a, b\}$ is connected by Lemma 3.3 of [3]. Using Lemma 4.1 (3), each one of $H \cup \{a, b\}$ and $K \cup \{a, b\}$ has non-cut point inclined property. By part (i) of Theorem 3.4 of [3], one of $H \cup \{a, b\}$ and $K \cup \{a, b\}$ has exactly two non-cut points. If $H \cup \{a, b\}$ has exactly two non-cut points, then using Theorem 5.6 of [6], $H \cup \{a, b\}$ is homeomorphic to a finite connected subspace of the Khalimsky line. So, $H \cup \{a, b\}$ is a COTS by definition of the Khalimsky line ([1]). Thus, in view of Theorem 3.4 (ii) of [3], each one of $H \cup \{a, b\}$ and $K \cup \{a, b\}$ has exactly two non-cut points. Now using Theorem 5.6 of [6], each one of $H \cup \{a, b\}$ and $K \cup \{a, b\}$ is homeomorphic to a finite connected subspace of the Khalimsky line. □

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