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Some Remarks on Weakly Tripotent Rings

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Abstract

In the mathematical literature there are three approaches to study a weakly tripotent ring. We provide a comparative study of these approaches and exhibit that a weakly tripotent ring defined as per one of these approaches is in harmony with a tripotent ring in various aspects as it is a commutative ring, a reduced ring and satisfies the identity $a^5 = a$ for each $a \in R$ like tripotent rings however a weakly tripotent ring defined as per rest of two approaches fail to be in harmony with a tripotent ring with respect to these aspects.

Keywords: idempotent; tripotent; tripotent ring; weakly tripotent ring.
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1. Introduction

Let R be a ring. An element $a \in R$ is called tripotent if $a^3 = a[1-3]$. A ring R is called tripotent if every element of R is tripotent. It may be noted that several authors have shown their interests in tripotent and weakly tripotent rings. A weakly tripotent ring is not necessarily a tripotent ring however a tripotent ring is a weakly tripotent ring. In this note we consider the following three approaches which are used to study a weakly tripotent ring.

(AP 1): A ring R is called a weakly tripotent ring if $a^3 = a$ or $(1+a)^3 = 1+a$ for each $a \in R$ [4-5].

(AP 2): A ring R is called a weakly tripotent ring if $a^3 = a$ or $(a)^3 = -a$ for each $a \in R$ [6].

(AP 3): A ring R is called a weakly tripotent ring if $a^3 = a$ or $(1-a)^3 = 1-a$ for each $a \in R$ [7-8].

It may be noted that (AP3) used in [7] has already been introduced and seen equivalent to (AP1) in [4-5, Proposition 3] and therefore we consider mainly (AP1) and (AP2) in this note. It has been seen that if R is a weakly tripotent ring as per (AP1), then it is not necessarily a weakly tripotent ring as per (AP2) and vice versa. In this article we note that every weakly tripotent ring defined as per (AP2) is a commutative ring however a weakly tripotent ring defined as per (AP1) is not necessarily a commutative ring. We also note that weakly tripotent ring defined as per (AP 2) is a reduced ring

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however a weakly tripotent ring defined as per (AP1) is not necessarily a reduced ring. It is seen that if R is a weakly tripotent ring of characterictic two as per (AP1), then R is not necessarily a tripotent ring however if R is a weakly tripotent ring of characterictic two as per (AP2), then R is a tripotent ring. In addition to above it is noted that each tripotent ring as well as each weakly tripotent as per (AP2) satisfy the identity $a^5 = a$ for each $a \in R$ however a weakly tripotent ring as per (AP1) does not necessarily satisfy this identity. In this note R is a unital and associative ring. In the next section we provide some results.

2. Some Results

Proposition 2.1. *Every weakly tripotent ring as per (AP2) is a commutative ring however a weakly tripotent ring as per (AP1) is not necessarily a commutative ring.*

Proof. Let R is a weakly tripotent ring as per (AP2). Then R is commutative weakly tripotent ring [6]. For a noncommutative weakly tripotent ring as per (AP1), we provide the following example. Let

$$R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

One may verify that R is a unital ring of characteristic two under matrix addition and multiplication modulo two. It is easy to note that $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is tripotent. Similarly we have $B^3 = B$ or $(I + B)^3 = I + B$ for each $B \in R$. Therefore R is weakly tripotent. Clearly R is a noncommutative ring. □

Proposition 2.2. *Let R is a weakly tripotent ring as per (AP2). Then R is a reduced ring. However, if R is a weakly tripotent ring as per (AP1), then it is not necessarily a reduced ring.*

Proof. Let R is a weakly tripotent ring as per (AP2). If $a \in R$ then we have $a^3 = a$ or $a^3 = -a$. Both the identities gives $a^5 = a$ for each $a \in R$. Therefore there does not exist a positive integer n such that $a^n = 0$. Hence R is a reduced ring.

Let R is a weakly tripotent ring as per (AP1), then it is not necessarily a reduced ring. For such a ring one may refer above example given in the proof of Proposition 2.2. Clearly $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in R$ is a non-zero nilpotent element of R. Therefore it is not a reduced ring. □

Proposition 2.3. *Let R is a weakly tripotent ring of characteristic two as per (AP2). Then R is a tripotent ring. However, if R is a weakly tripotent ring of characteristic two as per (AP1), then it is not necessarily a tripotent ring.*

Proof. Let R is a weakly tripotent ring as per (AP2). If $a \in R$ is any element of R, then we have $a^3 = a$ or $a^3 = -a$. This gives $a^3 = a$ for each $a \in R$ (since $1 = -1$). Therefore R is tripotent.

Further let $R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$. Then R is a weakly tripotent ring of characteristic two as per (AP1) under addition and multiplication of matrices modulo two. Clearly R is not a tripotent ring. □

Proposition 2.4. *A weakly tripotent ring as per (AP1) does not necessarily satisfy the identity $a^5 = a$ for each $a \in R$ however each tripotent ring as well as a weakly tripotent ring as per (AP2) satisfy the identity $a^5 = a$ for $a \in R$.*

Proof. It easily follows that each weakly tripotent ring as per (AP2) and each tripotent ring satisfy $a^5 = a$ for $a \in R$. For the rest part please refer above example. □

3. Conclusion

(AP1) includes noncommutative and non-reduced rings as well. However (AP2) does not include non-reduced rings and noncommutative rings. From the above results it follows that a weakly tripotent ring as per (AP2) is in harmony with a tripotent ring in various aspects as it is a commutative ring, a reduced ring and satisfies the identity $a^5 = a$ for each $a \in R$ like tripotent rings however a weakly tripotent ring as per (AP1) fails to be in harmony with a tripotent ring with respect to these aspects.

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