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Algebraic Description of Monadic Second Order Logic

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Abstract

A direct algebraic description of monadic second order logic in terms of Boolean algebras of unary and binary relations are given by extending the method of [1].

Keywords: Boolean algebras; Monadic logics; Dyadic logics; Second order logics.

1. Introduction

Propositional logic is represented algebraically by a Boolean algebra. The standard approach is due to Lindenbaum [5]. There are mainly two approaches to algebraize first order logic. They are due to Halmos [3] and Tarski [5]. Our approach is nearer to Halmos but it is less abstract. It retains the form and content of logic. We confine ourselves to monadic and dyadic logics and monadic second order logics. The algebraic descriptions are given in terms of Boolean algebras of unary and binary relations. The quantifiers are described by operators in terms of infimum and supremum.

2. Monadic Logic

Consider the complete Boolean algebra $\mathbb{B} = (\{0,1\}, \wedge, \vee, ', 0, 1)$ with order $0 \leq 1$. Assume that X is a nonempty set. $R_1(X) = \{\alpha : \alpha \subseteq X\}$, the set of all unary relations on X. A unary relation α on X can be identified with a function $\alpha : X \to \mathbb{B}$. Thus $R_1(X) = \mathbb{B}^X = \{\alpha \mid \alpha : X \to \mathbb{B}\}$. Define the operations $\Lambda, \text{V}, ', 0, 1$ on $R_1(X)$ pointwise as follows:
$(\alpha \wedge \beta)(x) = \alpha(x) \wedge \beta(x), (\alpha \vee \beta)(x) = \alpha(x) \vee \beta(x), \alpha'(x) = (\alpha(x))'\alpha^t(x)$, $\vee 0(x) = 0$, and $\vee 1(x) = 1$. The order $\vee\alpha \leq \beta$, if $\vee\alpha(x) \leq \beta(x)$ for any $\vee x \in X$ where $\vee\alpha, \beta \in R_1(X)$. Thus

Theorem 2.1. $(R_1(X)\wedge, \text{V}, ', 0, 1)$ *is a complete Boolean algebra.*

The quantifier operators $\forall$ and $\exists$ on $R_1(X)$ are defined in terms of infimum and supremum as follows: $(\forall x)\alpha(x) = \inf_x\{\alpha(x) : x \in X\}$ and $(\exists x)\alpha(x) = \sup_x\{\alpha(x) : x \in X\}$. Briefly, $(\forall x)\alpha(x) = \inf_x \alpha(x)$ and

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$(\exists x)\alpha(x) = \sup_x \alpha(x)$. By completeness (Theorem 2.1) $(\forall x)\alpha(x)$ and $(\exists x)\alpha(x)$ exist. By properties of order of our Boolean algebra we get the following.

Lemma 2.2.

i). $inf_x\alpha(x)$ *and* $sup_x\alpha(x)$ *are unique.*

ii). $inf_x\alpha(a) = \sup_x \alpha(a) = \alpha(a)$, *where a is a constant.*

iii). $\inf_x \alpha(x) \leq \sup_x \alpha(x)$ *for any* $x \in X$.

iv). *if* $\alpha(x) \leq c$ *for any* $x \in X$, *then* $\sup_x \alpha(x) \leq c$, *where c is a constant.*

v). *if* $c \leq \alpha(x)$ *for any* $x \in X$, *then* $c \leq inf_x\alpha(x)$, *where c is a constant.*

vi). *if* $\alpha(x) \leq \beta(x)$ *for any* $x \in X$, *then*

  1). $\sup_x \alpha(x) \leq \sup_x \beta(x)$ *and*

  2). $\inf_x \beta(x) \leq \sup_x \alpha(x)$.

By using Lemma 2.2 and the properties of Boolean algebra we prove the following form of DeMorgan theorem.

Theorem 2.3.

i). $(inf_x\alpha(x))' = \sup_x \alpha'(x)$

ii). $(\sup_x \alpha(x))' = \inf_x \alpha'(x)$

Proof.

i). $inf_x\alpha(x) \leq \alpha(x)$. Then $(\alpha(x))' \leq (inf_x\alpha(x))'$. Therefore $\alpha(x)' \leq (inf_x\alpha(x))'$. Thus $(\sup_x \alpha'(x))' \leq (\inf f_x\alpha(x))'$. On the other hand $\alpha'(x) \leq \sup_x \alpha'(x)$. Then $(\sup_x \alpha'(x))' \leq \alpha(x)$. Therefore $(\sup_x \alpha'(x))' \leq \inf_x \alpha(x)$. Thus $(inf_x\alpha(x))' = \sup_x \alpha'(x)$.

ii). Apply part (i) to $\alpha'(x)$. $(\inf_x \alpha(x))' = \sup_x \alpha''(x) = \sup_x \alpha(x)$. Therefore $(\sup_x \alpha(x))' = \inf_x \alpha'(x)$.

□

By using Lemma 2.2 and Theorem 2.3 we prove the following.

Theorem 2.4. *For any* $\alpha, \beta \in R_1(X)$ *the following hold*

i). $\sup_x(\alpha(x) \vee \beta(x)) = \sup_x \alpha(x) \vee \sup_x \beta(x)$.

ii). $inf_x(\alpha(x) \wedge \beta(x)) = inf_x\alpha(x) \wedge inf_x\beta(x)$.

iii). $\sup_x(\alpha(x) \wedge \beta(x)) \leq \sup_x \alpha(x) \wedge \sup_x \beta(x)$.

iv). $\inf_x(\alpha(x) \vee \beta(x)) \leq \inf_x \alpha(x) \vee \inf_x \beta(x)$.

Proof.

i). $\alpha(x) \leq \alpha(x) \vee \beta(x)$ and $\beta(x) \leq \alpha(x) \vee \beta(x)$. Therefore $\sup_x \alpha(x) \leq \sup_x(\alpha(x) \vee \beta(x))$ and $\sup_x \beta(x) \leq \sup_x(\alpha(x) \vee \beta(x))$. Then $\sup_x \alpha(x) \vee \sup_x \beta(x) \leq \sup_x(\alpha(x) \vee \beta(x))$. On the other hand $\alpha(x) \leq \sup_x \alpha(x)$ and $\beta(x) \leq \sup_x \beta(x)$. Therefore $\alpha(x) \vee \beta(x) \leq \sup_x \alpha(x) \vee \sup_x \beta(x)$. Therefore $\sup_x(\alpha(x) \vee \beta(x)) \leq \sup_x \alpha(x) \vee \sup_x \beta(x)$. Thus the equality holds.

ii). Apply (i) to $\alpha'(x)$ and $\beta'(x)$. $\sup_x \left(\alpha'^{(x)} \vee \beta'(x)\right) = \sup_x \alpha'(x) \vee \sup_x \beta'(x)$. By taking the complement of both sides and using DeMorgan rules we have, $\inf_x(\alpha(x) \wedge \beta(x)) = \inf_x \alpha(x) \wedge \inf_x \beta(x)$.

iii). $\alpha(x) \leq \sup_x \alpha(x)$ and $\beta(x) \leq \sup_x \beta(x)$. Then $\alpha(x) \wedge \beta(x) \leq \sup_x \alpha(x) \wedge \sup_x \beta(x)$. Therefore $\sup_x(\alpha(x) \wedge \beta(x)) \leq \sup_x \alpha(x) \wedge \sup_x \beta(x)$.

iv). Apply (iii) to $\alpha'(x)$ and $\beta'(x)$. as in (ii).

□

Thus we have the following:

Corollary 2.5. *For any $\alpha, \beta \in R_1(X)$ the following hold.*

i). $(\exists x)(\alpha(x) \vee \beta(x)) = (\exists x)\alpha(x) \vee (\exists x)\beta(x)$

ii). $(\forall x)(\alpha(x) \wedge \beta(x)) = (\forall x)\alpha(x) \wedge (\forall x)\beta(x)$

iii). $(\exists x)(\alpha(x) \wedge \beta(x)) \leq (\exists x)\alpha(x) \wedge (\exists x)\beta(x)$

iv). $(\forall x)\alpha(x) \vee (\forall x)\beta(x) \leq (\forall x)(\alpha(x) \vee \beta(x))$.

The following example shows that the equalities in (iii) and (iv) do not hold.
Assume that $X = \{a, b\}$ and $\alpha, \beta \in R_1(X)$ are defined by $\alpha : a \to 1, b \to 0$ and $\beta : a \to 0, b \to 1$. Then $\alpha \vee \beta : a, b \to 1$ and $\alpha \wedge \beta : a, b \to 0$. Then the converse of (iii) becomes $(\exists x)\alpha(x) \wedge (\exists x)\beta(x) \leq (\exists x)(\alpha(x) \wedge \beta(x))$ that is $1 \wedge 1 \leq 0$. It is impossible. Similarly, for the converse of part (iv).

3. Dyadic Logic

Again assume that X is a non-empty set. The set of all binary relations on X is $R_2(X) = \{\alpha : \alpha \subseteq X \times X\} \cdot R_2(X) \equiv B^{X \times X} = \{\alpha \mid \alpha : X \times X \to B\}$. Define the operations $\Lambda, V, ', 0, 1$ on $R_2(X)$ pointwise. The order $\leq$ on $R_2(X)$ is also defined pointwise by $\alpha \leq \beta$ if $\alpha(x, y) \leq \beta(x, y)$ for any $(x, y) \in X \times X$, where $\alpha, \beta \in R_2(X)$. Then we have

Theorem 3.1. $(R_2(X), \wedge, \vee, ', 0, 1)$ *is a complete Boolean algebra.*

The quantifying operators $\forall$ and $\exists$ on $R_2(X)$ are defined by infimum and supremum respectively. The double (nested) operators on $R_2(X)$ are defined by iteration on two variables x and y. So $(\forall x)(\exists y)\alpha(x,y)$ is given as follows:
$(\forall x)(\exists y)\alpha(x,y) \equiv (\forall x)[(\exists y)\alpha(x,y)] \equiv \inf_x \left[\sup_y \alpha(x,y)\right] \equiv \inf_x \sup_y \alpha(x,y)$.

Theorem 3.2.

i). $\sup_x \sup_y \alpha(x,y) = \sup_y \sup_x \alpha(x,y)$.

ii). $\sup_y \inf_x \alpha(x,y) \leq \inf_x \sup_y \alpha(x,y)$.

Proof.

i). $\alpha(x,y) \leq \sup_x \alpha(x,y) \leq \sup_y \sup_x \alpha(x,y)$. Then $\sup_y \alpha(x,y) \leq \sup_y \sup_x \alpha(x,y)$. Therefore $\sup_x \sup_y \alpha(x,y) \leq \sup_y \sup_x \alpha(x,y)$. The converse is similar. Thus $\sup_x \sup_y \alpha(x,y) = \sup_y \sup_x \alpha(x,y)$.

ii). $\alpha(x,y) \leq \sup_y \alpha(x,y)$. Then $\inf_x \alpha(x,y) \leq \inf_x \sup_y \alpha(x,y)$. Therefore $\sup_y \inf_x \alpha(x,y) \leq \inf_x \sup_y \alpha(x,y)$.

□

Theorem 3.3.

i). $(\exists x)(\exists y)\alpha(x,y) = (\exists y)(\exists x)\alpha(x,y)$

ii). $(\forall x)(\forall y)\alpha(x,y) = (\forall y)(\forall x)\alpha(x,y)$

iii). $(\exists y)(\forall x)\alpha(x,y) \leq (\forall x)(\exists y)\alpha(x,y)$

iv). $(\exists x)(\forall y)\alpha(x,y) \leq (\forall y)(\exists x)\alpha(x,y)$

Proof. (i) and (iii) are induced by Theorem 3.2. For (ii) apply part (i) to $\alpha'(x,y)$. Then $(\exists x)(\exists y)\alpha'(x,y) = (\exists y)(\exists x)\alpha'(x,y)$. By using DeMorgan we get $(\forall x)(\forall y)\alpha(x,y) = (\forall y)(\forall x)\alpha(x,y)$. Part (iv) is obtained by applying part (iii) to $\alpha'(x,y)$.
The following example shows that the equalities in (iii) and (iv) do not hold. Let $X = \{a,b\}$ and $\alpha \in R_2(X)$ be defined by
If $x \neq y$, then $\alpha(x,y) = 1$, otherwise $\alpha(x,y) = 0$. For part (iii), $(\forall x)(\exists y)\alpha(x,y) = (\exists y)\alpha(a,y) \wedge (\exists y)\alpha(b,y) = [\alpha(a,a) \vee \alpha(a,b)] \wedge [\alpha(b,a) \vee \alpha(b,b)] = 1$ and $(\exists y)(\forall x)\alpha(x,y) = (\forall x)\alpha(x,a) \vee (\forall x)\alpha(x,b) = [\alpha(a,a) \wedge \alpha(b,a)] \vee [\alpha(a,b) \wedge \alpha(b,b)] = 0$. Thus $(\exists y)(\forall x)\alpha(x,y) \neq (\forall x)(\exists y)\alpha(x,y)$. Part (iv) is similar. □

4. Deduction

Deduction in our algebraic systems is defined as follows 0 $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \vdash \beta$, if $\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_n \leq \beta$. The rules used are the algebraic substitution and simplification. For example $\{(\forall x)(\alpha(x) \to \alpha(s(x)), \alpha(a)\} \vdash \alpha(s(a))$ where a is a constant and $s : X \to X$ is a function (term). $(\forall x)(\alpha(x) \to \alpha(s(x)) \leq \alpha(a) \to \alpha(s(a)) = \alpha(a)' \vee \alpha(s(a)).(\forall x)(\alpha(x) \to \alpha(s(x)) \wedge \alpha(a) = (\alpha(a) \to \alpha(s(a))) \wedge \alpha(a) \leq (\alpha(a)' \vee \alpha(s(a))) \wedge \alpha(a) = \alpha(s(a)) \wedge \alpha(a) \leq \alpha(s(a))$.

5. Monadic Second Order Logic [2]

Assume that X is a nonempty set and $\mathbb{B}$ is a Boolean algebra, $R_1(X) = \{\alpha : \alpha \subseteq X\}$, the set of unary relations on X. A unary relation α on X can be identified with a function $\alpha : X \to \mathbb{B}$. Thus $R_1(X) \equiv \mathbb{B}^X = \{\alpha \mid \alpha : X \to \mathbb{B}\}$.

Quantifications

1. On Variables

$$(\forall x)\alpha(x) = \inf_x \alpha(x)$$
$$\exists(x)\alpha(x) = \sup_x \alpha(x)$$

2. On Unary Relations

$$\forall\alpha = \inf\alpha$$
$$\exists\alpha = \sup\alpha$$

Lemma 5.1.

i). $\forall\alpha$ *and* $\exists\alpha$ *are unique*

ii). $\forall\alpha = \exists\alpha = \alpha$

iii). $\forall\alpha \leq \alpha \leq \exists\alpha$

iv). *if* $\alpha \leq \mathrm{c}$ *then* $\exists\alpha \leq \mathrm{c}$

v). *if* $\mathrm{c} \leq \alpha$ *then* $\mathrm{c} \leq \forall\alpha$

vi). *if* $\alpha \leq \beta$ *then*

1). $\exists\alpha \leq \exists\beta$ *and*

2). $\forall\beta \leq \forall\alpha$.

Theorem 5.2.

i). $(\forall\alpha)' = \exists\alpha'$

ii). $(\exists\alpha)' = \forall\alpha'$

Proof.

i). $\forall\alpha \leq \alpha$ then $\alpha' \leq (\forall\alpha)' \Rightarrow \exists\alpha' \leq (\forall\alpha)'$, on the other hand $\alpha' \leq \exists\alpha' \Rightarrow (\exists\alpha')' \leq \alpha \Rightarrow (\exists\alpha')' \leq \forall\alpha \Rightarrow$ $\therefore (\forall\alpha)' \leq \exists\alpha'$.

ii). Apply (i) on α' we get $(\exists\alpha)' = \forall\alpha'$.

□

Theorem 5.3. *For $\alpha, \beta \in R_1(X)$ the following hold:*

i). $\sup(\alpha \vee \beta) = \sup\alpha \vee \sup\beta$

ii). $\inf(\alpha \wedge \beta) = \inf\alpha \wedge \inf\beta$

iii). $\sup(\alpha \wedge \beta) \leq \sup\alpha \wedge \sup\beta$

iv). $\inf\alpha \vee \inf\beta \leq \inf(\alpha \vee \beta)$

Proof.

i). $\alpha \leq \alpha \vee \beta$ and $\beta \leq \alpha \vee \beta \Rightarrow \sup\alpha \leq \sup(\alpha \vee \beta)$ and $\sup\beta \leq \sup(\alpha \vee \beta)$ $\therefore \sup\alpha \vee \sup\beta \leq \sup(\alpha \vee \beta)$, $\alpha \leq \sup\alpha$ and $\beta \leq \sup\beta \Rightarrow \alpha \vee \beta \leq \sup\alpha \vee \sup\beta \Rightarrow \sup(\alpha \vee \beta) \leq \sup\alpha \vee \sup\beta$, $\therefore \sup(\alpha \vee \beta) = \sup\alpha \vee \sup\beta$.

ii). $\sup(\alpha' \vee \beta') = \sup\alpha' \vee \sup\beta'$. By DeMorgan $(\sup(\alpha' \vee \beta'))' = (\sup\alpha' \vee \sup\beta)'$, $\therefore \inf(\alpha \wedge \beta) = \inf\alpha \wedge \inf\beta$.

iii). $\alpha \leq \sup\alpha$ and $\beta \leq \sup\beta \Rightarrow \alpha \wedge \beta \leq \sup\alpha \wedge \sup\beta \Rightarrow \sup(\alpha \wedge \beta) \leq \sup\alpha \wedge \sup\beta$.

iv). Apply DeMorgan on iii we get $\inf\alpha \vee \inf\beta \leq \inf(\alpha \vee \beta)$.

□

Corollary 5.4. *For any $\alpha, \beta \in R_1(X)$ the following hold:*

i). $\exists(\alpha \vee \beta) = \exists\alpha \vee \exists\beta$

ii). $\forall(\alpha \wedge \beta) = \forall\alpha \wedge \forall\beta$

iii). $\exists(\alpha \wedge \beta) \leq \exists\alpha \wedge \exists\beta$

iv). $\forall\alpha \vee \forall\beta \leq \forall(\alpha \vee \beta)$

Theorem 5.5.

i). $\forall\alpha(\forall x)\alpha(x) = (\forall x)\forall\alpha\alpha(x)$

ii). $\exists\alpha(\exists x)\alpha(x) = (\exists x)\exists\alpha\alpha(x)$

iii). $(\exists x)\forall\alpha\alpha(x) \leq \forall\alpha(\exists x)\alpha(x)$

iv). $\exists\alpha(\forall x)\alpha(x) \leq (\forall x)\exists\alpha\alpha(x)$

Proof.

i). $\forall\alpha(\forall x)\alpha(x) \leq \alpha(x) \Rightarrow \alpha'(x) \leq (\forall\alpha(\forall x)\alpha(x))'$. Therefore $\exists\alpha'\alpha'(x) \leq (\forall\alpha(\forall x)\alpha(x))'$ $\Rightarrow (\exists x)\exists\alpha'\alpha'(x) \leq (\forall\alpha(\forall x)\alpha(x))'$. Therefore $(\forall x)\forall\alpha\alpha(x) \leq ((\exists x)\exists\alpha'\alpha(x))' \leq \forall\alpha(\forall x)\alpha(x)$. On the other hand $\alpha'(x) \leq (\exists x)\alpha'(x) \Rightarrow ((\exists x)\alpha'(x))' \leq \alpha(x)$. Therefore $((\exists x)\alpha'(x))' \leq \alpha(x)$ $\Rightarrow ((\exists x)\alpha'(x))' \leq (\forall x)\alpha(x) \leq (\forall x)\forall\alpha\alpha(x)$. Therefore $\forall\alpha(\forall x)\alpha(x) \leq (\forall x)\forall\alpha\alpha(x)$. Therefore $\forall\alpha(\forall x)\alpha(x) = (\forall x)\forall\alpha\alpha(x)$.

ii). Apply (i) to α'

iii). $\forall\alpha\alpha(x) \leq \alpha(x) \leq (\exists x)\alpha(x) \leq \forall\alpha(\exists x)\alpha(x)$. Therefore $(\exists x)\forall\alpha\alpha(x) \leq \forall\alpha(\exists x)\alpha(x)$.

iv). $(\forall x)\exists\alpha\alpha(x) \leq \alpha(x) \Rightarrow \exists\alpha(\forall x)\alpha(x) \leq (\forall x)\exists\alpha\alpha(x)$.

□

6. A finite Case

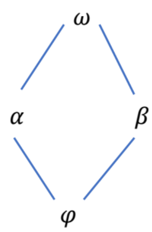
Take $X = \{a, b\}$. $R_1(X) = \{\alpha : \alpha \subseteq X\} = \{\varphi, \{a\}, \{b\}, X\}$ as unry relations. Also as functions $R_1(X) = \{\varphi, \alpha, \beta, \omega\}$, where

$\varphi : a \mapsto 0, b \mapsto 0$

$\alpha : a \mapsto 1, b \mapsto 0$

$\beta : a \mapsto 0, b \mapsto 1$

$\omega : a \mapsto 1, b \mapsto 1$



Now, $\forall\alpha \equiv \inf\alpha = \varphi, \exists\alpha \equiv \sup\alpha = \omega; \forall\beta \equiv \inf\beta = \varphi, \exists\beta \equiv \sup\beta = \omega$.

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