International Journal of
Mathematics And its Applications
ISSN: 2347-1557

Int. J. Math. And Appl., **13(3)**(2025), 137–162
Available Online: http://ijmaa.in

Multifactor Balanced Asymmetrical Factorial Designs of Type II

N. J. Wanyoike[1,*], Guangzhou Chen[2,3]

[1]*School of Technology, KCA University, Ruaraka, Nairobi, Kenya*

[2]*School of Mathematics and Information Science, Henan Normal University, Xinxiang, People's Republic of China*

[3]*Henan Engineering Laboratory for Big Data Statistical Analysis and Optimal Control, Henan Normal University, Xinxiang, People's Republic of China*

Abstract

This manuscript gives methods of constructing multifactor BAFDs of type II. Multi-factor BAFDS of type II are constructed from two factor BAFDs. Two methods of construction are given. The first method is the product of balanced arrays which is similar to the product of orthogonal arrays defined by [4]. The second methods was given by [28] which generates a BAFD from two given BAFD's. These methods can provide efficient BAFD's if efficient two factor BAFD's are used. The designs constructed are balanced with orthogonal factorial structure.

Keywords: Balanced Arrays; Orthogonal Arrays; effiecient BAFD's; Designs; Balance; Orthogonal Factorial Structure; BAFD's.

1. Introduction

In many situations there arise scenarios when an experimenter has to use factors at different levels. The problem of obtaining confounded plans for such cases has received a good deal of attention. To this extent, [37], by trial and hit methods obtained confounded plans of the type $3^m \times 2^n$, where m and n are any positive integers. Using orthogonal arrays of strength 2 [25] gave methods for constructing Extended Group Divisible Designs $\{EGD\}$ for $s_1 \times s_2$ experiments in blocks of size $s_1 < s_2$. [33] starting from a basic $s_1 \times s_2$ design in blocks of size s_2 ($s_2 < s_1$, s_1 being a prime number or power of prime) obtained three factor designs. [27] constructed some series of designs from orthogonal latin squares for $s_1 \times s_2$ experiments in block of size s_1 and $s_2 - 1$ replications. [32] gave a class of balanced designs with OFS. [23] considered the use of balanced incomplete block designs for the construction of $s_1 \times s_2$ balanced factorial designs with OFS when $s_1 > s_2$. Informative accounts and subsequent developments have been done by [14,19,31]. [29] proposed a general method of obtaining block designs for asymmetrical confounded factorial experiments using block designs for

*Corresponding author (wajongaii@yahoo.com)

symmetrical factorial experiments. [15] describes a general method of construction of supersaturated designs for asymmetric factorials obtained by exploiting the concept of resolvable orthogonal arrays and Hadamard matrices. [12] considered three forms of a general null hypothesis Ho on the factorial parameters of a general asymmetrical factorial paired comparison experiment in order to determine optimal or efficient designs. [18] constructed designs by using confounding through equation methods. Construction of confounded asymmetrical factorial experiments in row-column settings and efficiency factor of confounded effects was worked out. [1] attempted to construct asymmetrical factorial type switch over designs having strip type arrangement of combination of the levels. To start with, two factors at different levels were considered. [11] described a method of constructing cost-efficient response surface designs (RSDs) as compared to the replicated central composite designs (RCCDs), that are useful for modelling and optimization of the asymmetric experiments. [34] identified a Kronecker product structure for a particular class of asymmetric factorial designs in blocks, including the classes of designs generated by several of the classical methods in literature. [8] focuses on the construction and analysis of an extra ordinary type of asymmetrical factorial experiment which corresponds to a fraction of symmetrical factorial experiment as indicated by [9]. [6] establishes a lower bound to measure optimality with respect to a main effects model in a general asymmetric factorial experiment. [21] conceptualized the fundamental aspects of the Complete, Fractional, Central Composite Rotational and Asymmetrical factorial designs. Recent applications of these powerful tools were described. [3] developed a method for the construction of $p \times 3 \times 2$ asymmetrical factorial experiments with $(p-1)$ replications. [30] proposed a general method of obtaining block designs for asymmetrical confounded factorial experiments using the block designs for symmetrical factorial experiments. [38] Constructed asymmetrical factorial designs containing clear effects. [22] explained how to choose an optimal $(s^2)s^n$ design for the practical need, where s is any prime or prime power and accordingly considered the clear effects criterion for selecting good designs. [24] dealt with situations where there was a need for designing an asymmetrical factorial experiment involving interactions. Failing to get a satisfactory answer to this problem from the literature, the authors developed an adhoc method of constructing the design. It is transparent that the design provides efficient estimates for all the required main effects and interactions. The later part of this paper deals with the issues of how this method is extended to more general situations and how this adhoc method is translated into a systematic approach. [26] developed The R package DoE.base which can be used for creating full factorial designs and general factorial experiments based on orthogonal arrays. Besides design creation, some analysis functionality is also available, particularly (augmented) half-normal effects plots. [17] Published a monograph that is an outcome of the research works on the construction of factorial experiments (symmetrical and asymmetrical). In this booklet, construction frameworks have been described for factorial experiments. The construction frameworks include general construction method of p^n factorial experiments, construction methods with confounded effects and detection methods of confounded effects in a confounded plan.

The concepts of combinatorial, matrix operations and linear equation technique have been deployed to develop the methods. [8] discussed an Alternative Method of Construction and Analysis of Asymmetrical Factorial Experiment of the type 6×22 in Blocks of Size 12. [8] focuses on the construction and analysis of an extra ordinary type of asymmetrical factorial design which corresponds to fraction of a symmetrical factorial design as indicated by [9]. For constructing this design, they used 3 choices and for each choice they used 5 different cases. Finding the block contents for each case shows that there are mainly two different cases for each choice. In case of analysis of variance, is seen that, for the case where the highest order interaction effect is confounded in 4 replications, the loss of information is same for all the choices. [16] in his book chapter discusses different methods of constructing systems of confounding for asymmetrical factorial designs, including: Combining symmetrical systems of confounding via the Kronecker product method, use of pseudo-factors, the method of generalized cyclic designs, method of finite rings (this method is also used to extend the Kempthorne parameterization from symmetrical to asymmetrical factorials), and the method of balanced factorial designs. He showed the equivalence of balanced factorial designs and extended group divisible partially balanced incomplete block designs, establishing again a close link between incomplete block designs and confounding in factorial designs. [10] in her book chapter discusses confounding in single replicate experiments in which at least one factor has more than two levels. First, the case of three-levelled factors is considered and the techniques are then adapted to handle m-levelled factors, where m is a prime number.

Next, pseudofactors are introduced to facilitate confounding for factors with non-prime numbers of levels. Asymmetrical experiments involving factors or pseudofactors at both two and three levels are also considered, as well as more complicated situations where the treatment factors have a mixture of $2, 3, 4$, and 6 levels. Analysis of an experiment with partial confounding is illustrated using the SAS and R software packages. [13] shows that Asymmetrical single replicate factorial designs in blocks are constructed using the deletion technique. Results are given that are useful in simplifying expressions for calculating loss of information on main effects and interactions, due to confounding with blocks. Designs for estimating main effects and low order interactions are also given. [20] in his work presents the results of a systematic literature review (SLR) and a taxonomical classification of studies about run orders for factorial designs published between 1952 and 2021.

The objective here is to describe the findings, main and future research directions in this field. The main components considered in each study and the methodologies they used to obtain run sequences are also highlighted, allowing professionals to select an appropriate ordering for their problem. This review shows that obtaining orderings with good properties for an experimental design with any number of factors and levels is still an unresolved issue. [36] in his present book gives, for the first time, a comprehensive and up-to-date account of the modern theory of factorial designs.

Many major classes of designs are covered in the book. While maintaining a high level of mathematical rigor, it also provides extensive design tables for research and practical purposes. [2] in

his work discusses the construction of 'inter-class orthogonal' main effect plans (MEPs) for asymmetrical experiments. In such a plan, the factors are partitioned into classes so that any two factors from different classes are orthogonal. The researcher also defined the concept of "partial orthogonality" between a pair of factors. In many of his plans, partial orthogonality has been achieved when (total) orthogonality is not possible due to divisibility or any other restriction. He presented a method of obtaining inter-class orthogonal MEPs. Using this method and also a method of 'cut and paste' he obtained several series of inter-class orthogonal MEPs. One of them happens to be a series of orthogonal MEP (OMEPs), which includes an OMEP for a 330 experiment on 64 runs. [7] in his book, provides a rigorous, systematic, and up-to-date treatment of the theoretical aspects of factorial design. To prepare readers for a general theory, the author first presents a unified treatment of several simple designs, including completely randomized designs, block designs, and row-column designs. As such, the book is accessible to readers with minimal exposure to experimental design.
In [5], Lee discrepancy has wide applications in design of experiments, which can be used to measure the uniformity of fractional factorials. An improved lower bound of Lee discrepancy for asymmetrical factorials with mixed two-, three- and four-level is presented. The new lower bound is more accurate for a lot of designs than other existing lower bounds, which is a useful complement to the lower bounds of Lee discrepancy and can be served as a benchmark to search uniform designs with mixed levels.
In this manuscript, we going to use already known methods and some known balanced factorial designs to construct multifactor balanced asymmetrical factorial designs of Type II. We are especially interested in designs in which main effects and lower order interactions can be estimated with higher efficiencies.

Definition 1.1. *An experiment involving* $m \geq 2$ *factors* $F_1, F_2, \ldots, F_m$ *that appear at* $s_1, \ldots, s_m (\geq 2)$ *levels is called an* $s_1 \times \cdots \times s_m$ *factorial experiment (or an* $s_1 \times \cdots \times s_m$ *factorial for brevity).*

The purpose of this paper is to give simplified methods of constructing multifactor asymmetrical factorial designs of type II that are characterized by balance with orthogonal factorial structure.
In this paper we shall involve well known arrangements of arrays such as Difference Schemes, Orthogonal Arrays, Balanced Arrays, Transitive Arrays, and Hadarmard Matrices whose definitions are given below.

Definition 1.2. *An* $r \times c$ *array* D *with entries from* $\mathcal{A}$ *is called a difference scheme based on* $(\mathcal{A}, +)$ *if it has the property that for all* i *and* j *with* $1 \leq i, j \leq c$*, the vector difference between the* i^{th} *and* j^{th} *columns contains every element of* $\mathcal{A}$ *equally often if* $i \neq j$

0	0	0	0	0	0	0	0	0
0	1	2	0	1	2	0	1	2
0	2	1	0	2	1	0	2	1
0	0	0	1	1	1	2	2	2
0	1	2	1	2	0	2	0	1
0	2	1	1	0	2	2	1	0
0	0	0	2	2	2	1	1	1
0	1	2	2	0	1	1	2	0
0	2	1	2	1	0	1	0	2

Table 1: This difference scheme is derived from $(GF(3), +)$

Where $GF(3)$ is a galois Field with 3 elements.

Definition 1.3. *A $k \times b$ array A with entries from a set of v symbols is called an orthogonal array of strength t if each $t \times b$ subarray of A contains all possible v^t column vectors with the same frequency $\lambda = \frac{b}{v^t}$. It is denoted $OA(b, k, v, t; \lambda)$; the number λ is called the index of the array. The numbers b and k are known as the number of assemblies and constraints of the orthogonal array respectively.*

Example 1.4.

0	1	1	1	1	0	0	0
1	0	1	1	0	1	0	0
1	1	0	1	0	0	1	0
1	1	1	0	0	0	0	1

$OA(8, 4, 2, 3; 1)$

Definition 1.5. *Let A be a $k \times b$ array with entries from a set of v symbols. Consider the v^t ordered t-tuples $(x_1, \ldots, x_t)$ that can be formed from a t-rowed subarray of A, and let there be associated a non-negative integer $\lambda(x_1, \ldots, x_t)$ that is invariant under permutations of $x_1, \ldots, x_t$. If for any t-rowed subarray of A the v^t ordered t-tuples $(x_1, \ldots, x_t)$, each occur $\lambda(x_1, \ldots, x_t)$ times as a column, then A is said to ba a balanced array of strength t. It is denoted by $BA(b, k, v, t)$ and the numbers $\lambda(x_1, \ldots, x_t)$ are called the index parameters of the array.*

Clearly a $BA(b, k, v, t)$ with $\lambda(x_1, \ldots, x_t) = \lambda$ for all t-tuples $(x_1, \ldots, x_t)$ is simply an orthogonal array $OA(b, k, v, t; \lambda)$.

Example 1.6.

0	1	0	1	0	1	0	1	0	1
1	1	1	0	1	1	0	0	0	0
0	0	1	1	1	0	0	0	1	1
1	1	0	0	0	0	1	0	1	1
0	0	0	0	1	1	1	1	1	0

$BA(10, 5, 2, 2)$

In this manuscript we are going to construct multi-factor BAFDs using balanced arrays of strength $t = 2$ with parameters $\lambda(x, y) = \lambda_1$ or λ_2 according as $x = y$ or not. In particular we are interested in the $BA[(ks-1)s\lambda, ks, s, 2]$ with parameters $\lambda(x,y) = (k-1)\lambda$ or $k\lambda$ according as $x = y$ or not. For brevity we shall call it the balanced array of type T with index λ and denote it by $BA[T][k,s,\lambda]$. It is clear that a

$$\begin{aligned} BA[T][1,s,\lambda] &= BA[\lambda s(s-1), s, s, 2] \\ &= TA[\lambda s(s-1), s, s, 2]. \end{aligned}$$

In constructing a $BA[T][k,s,\lambda]$ for any given k and s we would like λ to be as small as possible so that the size of the balanced array is not too large. However if there is no restriction on λ, we can always construct a $BA[T][k,s,\lambda]$ for any k and s.

Definition 1.7. *A transitive array $TA(b,k,v,t;\lambda)$ is a $k \times b$ array of v symbols such that for any choice of t rows, the $\frac{v!}{(v-t)!}$ ordered t-tuples of distinct symbols each occur λ times as a column.*

Example 1.8.

$$\begin{matrix} 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 & 2 & 3 & 0 & 1 & 3 & 2 & 1 & 0 \\ 2 & 3 & 0 & 1 & 3 & 2 & 1 & 0 & 1 & 0 & 3 & 2 \\ 3 & 2 & 1 & 0 & 1 & 0 & 3 & 2 & 2 & 3 & 0 & 1 \end{matrix}$$

$TA(12,4,4,2;1)$

Definition 1.9. *A Hadamard matrix of order n is an $n \times n$ matrix H_n of $+1's$ and $-1's$ whose rows are orthogonal, that is, which satisfies*

$$H_n H_n^T = nI_n \tag{1}$$

For example, here are Hadamard matrices of order $1, 2$ and 4.

$$H_1 = [1], H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \tag{2}$$

Theorem 1.10. *For all k and s,there always exists a $BA[T][k,s,\lambda]$ for some λ. Where $BA[T][k,s,\lambda]$ is a balanced array with parameters $\lambda(x,y) = (k-1)$ or $k\lambda$ accordingly as $x = y$ or not.*

Proof. For all k and s, there exists a $TA[(ks-1)ksn, ks, ks, 2]$ for some n. Let the symbols of the transitive array be denoted by $[0, 1, ..., ks-1]$. If we replace each symbol in the transitive array by $x(mod k)$. Then the transitive array becomes a $BA[(ks-1)ksn, ks, s, 2]$ with parameters $\lambda(x,y) = (k-1)kn$ or k^2n according as $x = y$ or not, which is a $BA[T][ks, s, kn]$. The method of construction in Theorem 1.10 does not usually provide balanced arrays with a small number of assemblies as we desire. □

Definition 1.11. *An orthogonal array $OA[N,k,s,2]$ is said to be a-resolvable if it is statistically equivalent to the juxtaposition of $\frac{N}{as}$ arrays such that each factor occurs in each of these arrays a times at each level. A* 1*-resolvable orthogonal array is also called completely resolvable, otherwise it is called Partly resolvable.*

Definition 1.12. *A $v \times v$ matrix G where $v = \Pi s_i$ will be said to have property A if it is of the form $G = \sum_{y\in\Omega^*} h(y)Z^y$, where $h(y), y \in \Omega^*$, are real numbers, Ω^* is a set of all m component binary vectors.*

Let Ω^* be the set of all $m-$ component binary vectors, that is $\Omega^* = \Omega \cup \{(0,0,\dots,0)\}$ where Ω is the set of all none null binary component vectors for $y = (y_1, y_2, \dots, y_m) \in \Omega$ then

$$Z^y = \otimes_{i=1}^m Z_i^{y_i} \tag{3}$$

where for $1 \le i \le m$, and

$$Z_i^{y_i} = I_i \quad \text{if} \quad y_i = 1 \tag{4}$$

$$= J_i \quad \text{if} \quad y_i = 0 \tag{5}$$

where I is an identity matrix and J is matrix of $1's$ both of order $m \times m$. From (3) and (4) we can ultimately obtain

$$C = r(\otimes_{i=1}^m I_i) - k^{-1}NN' \tag{6}$$

where C is the design matrix and N is the incidence matrix of a BAFD. (6) shows that the design has property A see [35]. For connected equireplicate designs with property A and a common replication number r the interaction efficiencies are given by

$$E(y) = 1 - \frac{1}{rk}g(y) \text{ and } E(y) = 1 \text{ if and only if } g(y) = 0 \tag{7}$$

2. Type II Designs

Let $s_m = n_1 s_1 = n_2 s_2 = \dots = n_{m-1} s_{m-1}$ and there exists $BA(T)[n_i, s_i, 1]$ for $i = 1, 2, \dots, m-1$. By Theorem A.5 there exists an $s_1 \times s_2 \times \dots \times s_m$ BAFD with $k = s_m$, $b = (s_m - 1)^{m-1} s_1 s_2 \dots s_{m-1}$, $r = [s_m - 1]^{m-1}$, $\lambda(y_1, y_2, \dots y_{m-1}, 0) = 0$ and $\lambda(y_1, y_2, \dots y_{m-1}, 1) = \prod_{i=1}^{m-1} n_i^{x_i}(n_i - 1)^{1-x_i}$. By Theorem A.6 the eigenvalues of the NN^T of the BAFD are given by

$$g[y_1, y_2, \dots, y_m] = r + \sum_{x\in\Omega} \lambda(x) \left\{ \prod_{i=1}^m [(1-y_i)s_i - 1]^{x_i} \right\} \tag{8}$$

$$= r + \sum_{x\in\Omega} \prod_{i-1}^{m-1} n_i^{x_i}(n_i - 1)^{1-x_i} \left\{ \prod_{i=1}^m [(1-y_i)s_i - 1]^{x_i} \right\} \tag{9}$$

Let $y_m = 1$, $y_i = 0$ for $i = 1, 2, \ldots, m-1$ in equation (9)

$$g(0,0,\ldots,0,1) = r + \left\{ \begin{array}{c} \lambda^1(x_1,x_2,\ldots,x_{m-1},1) \left[\begin{array}{c} \left\{(1-0)s_1-1\right\}^{x_1} \left\{(1-0)s_2-1\right\}^{x_2} \\ \ldots \left\{(1-1)s_m-1\right\}^{1} \end{array} \right] + \\ \lambda^2(x_1,x_2,\ldots x_{m-1},1) \left[\begin{array}{c} \left\{(1-0)s_1-1\right\}^{x_1} \left\{(1-0)s_2-1\right\}^{x_2} \\ \ldots \left\{(1-1)s_m-1\right\}^{1} \end{array} \right] + \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \\ \lambda^{m-1}(x_1,x_2,\ldots,x_{m-1},1) \left[\begin{array}{c} \left\{(1-0)s_1-1\right\}^{x_1} \left\{(1-0)s_2-1\right\}^{x_2} \\ \ldots \left\{(1-1)s_m-1\right\}^{1} \end{array} \right] \end{array} \right\} \tag{10}$$

where $\lambda^h(x_1,x_2,\ldots x_{m-1},1)$, $h = 1, 2, \ldots, m-1$ is the h^{th} distinct term of $\lambda(x_1,x_2,\ldots,x_{m-1},1)$ and hence we have $2^m - 1$ distinct terms of $\lambda(x_1,x_2,\ldots,x_{m-1},1)$. After expanding equation (10) we have

$$\begin{aligned} g(0,0,\ldots,0,1) &= r + \left\{ -s_m^{m-1} + (m-1)s_m^{m-2} - \tfrac{(m-1)(m-2)}{2} s_m^{m-3} + \ldots - s_m^0(-1)^{m-1} \right\} \\ &= r - \left\{ s_m^{m-1} - (m-1)s_m^{m-2} + \tfrac{(m-1)(m-2)}{2} s_m^{m-3} - \ldots + s_m^0(-1)^{m-1} \right\} \\ &= r - \left\{ \begin{array}{c} {}^{m-1}c_0 s_m^{m-1}(-1)^0 + {}^{m-1}c_1 s_m^{m-2}(-1)^1 + {}^{m-1}c_2 s_m^{m-3}(-1)^2 + \ldots + \\ {}^{m-1}c_{m-1} s_m^0(-1)^{m-1} \end{array} \right\} \\ &= r - (s_m - 1)^{m-1} = r - r = 0 \end{aligned}$$

similarly we can show that $g(1,0,0,\ldots,0) = g(0,1,0,\ldots,0) = \ldots = g(0,0,0,\ldots,1,0) = 0$. Thus all main effects are estimated with full efficiency. Let $y_{m-1} = y_m = 1$ and $y_i = 0$ for $i = 1,2,\ldots m-2$ then (9) becomes

$$g(0,0,\ldots,0,1,1) = r + \left\{ \begin{array}{c} \lambda^1(x_1,x_2,\ldots,x_{m-1},1) \left[\begin{array}{c} \left\{(1-0)s_1-1\right\}^{x_1} \left\{(1-0)s_2-1\right\}^{x_2} \\ \ldots (a) \left\{(1-1)s_m-1\right\}^{1} \end{array} \right] \\ +\lambda^2(x_1,x_2,\ldots x_{m-1},1) \left[\begin{array}{c} \left\{ \begin{array}{c} (1-0) \\ s_1-1 \end{array} \right\}^{x_1} \left\{(1-0)s_2-1\right\}^{x_2} \\ \ldots (a) \left\{(1-1)s_m-1\right\}^{1} \end{array} \right] + \ldots \\ +\lambda^{2m-1}(x_1,x_2,\ldots,x_{m-1},1) \left\{(1-0)s_1-1\right\}^{x_1} \left\{(1-0)s_2-1\right\}^{x_2} \\ \ldots \left\{(1-1)s_{m-1}-1\right\}^{x_{m-1}} \left\{(1-1)s_m-1\right\}^{1} \end{array} \right\} \tag{11}$$

(In the above equation (11), let $\left\{(1-1)s_{m-1}-1\right\}^{x_{m-1}}$ be a value represented by (a)), where $\lambda^h(x_1,x_2,\ldots,x_{m-1},1), h = 1,2,\ldots,2^m-1$ is the h^{th} distinct term of $\lambda(x_1,x_2,\ldots,x_{m-1},1)$ and hence we have $2^m - 1$ distinct terms of $\lambda(x_1, x_2, \ldots, x_{m-1}, 1)$. After expanding equation (11) we have

$$\begin{aligned} g(0,0,\ldots,0,1,1) &= r + \left\{ s_m^{m-2} - (m-2)s_m^{m-3} + \tfrac{(m-2)(m-3)}{2} s_m^{m-4} - \ldots + s_m^0(-1)^{m-2} \right\} \\ &= r + \left\{ \begin{matrix} {}^{m-2}c_0 s_m^{m-2}(-1)^0 + {}^{m-2}c_1 s_m^{m-3}(-1)^1 + {}^{m-2}c_2 s_m^{m-4}(-1)^2 + \ldots + \\ {}^{m-2}c_{m-2} s_m^0 (-1)^{m-2} \end{matrix} \right\} \\ &= r + (s_m - 1)^{m-2} \\ &= (s_m - 1)^{m-1} + (s_m - 1)^{m-2} \end{aligned}$$

In general, when $\sum_{i=1}^m y_i = q$ equation (9) is

$$g(y_1, y_2, \ldots, y_m) = (s_m - 1)^{m-1} + (-1)^q (s_m - 1)^{m-q} \tag{12}$$

hence by Corollary A.7

$$\begin{aligned} E[y_1, y_2, \ldots, y_m] &= 1 - \frac{g[y_1, y_2, \ldots, y_m]}{rk} \\ &= 1 - \frac{(s_m - 1)^{m-1} + (-1)^q (s_m - 1)^{m-q}}{(s_m - 1)^{m-1} s_m} \\ &= 1 - \frac{1}{s_m} - \frac{(-1)^q}{s_m (s_m - 1)^{q-1}} \end{aligned} \tag{13}$$

where $\sum_{i=1}^m y_i = q$. It can be seen that equation (13) is the same as the equation (23) with $s_m = s$, hence the efficiencies are equal to those of s_m^m symmetrical balanced factorial design in Lemma A.8 with $j = m$.

Example 2.1. *A $BA(T)(3,2,1)$ given in Example A.17 can be used to construct a $2^2 \times 6$ BAFD with $k = 6$, $b = 100$, $r = 25$, $\lambda(1,0) = \lambda(2,0) = 0$, $\lambda(0,1) = 4$, $\lambda(1,1) = 6$, $\lambda(2,1) = 9$. The efficiencies are;*

$$\begin{aligned} E[0,1] &= E[1,0] = 1.0 \\ E[1,1] &= E[2,0] = \frac{4}{5} \\ E[2,1] &= \frac{21}{25} \end{aligned}$$

Example 2.2. *A $BA(T)(2,3,1)$ given in Example A.14 can be used to construct a $3^3 \times 6$ BAFD with $k = 6, b = 3{,}375, r = 125, \lambda(1,0) = \lambda(2,0) = \lambda(3,0) = 0$, $\lambda(0,1) = 1, \lambda(1,1) = 2, \lambda(2,1) = 4, \lambda(3,1) = 8$. The efficiencies are;*

$$\begin{aligned} E[0,1] &= E[1,0] = 1.0 \\ E[1,1] &= E[2,1] = \frac{4}{5} \\ E[2,1] &= \frac{21}{25} \end{aligned}$$

$$E[3,1] = \frac{104}{125}$$

Example 2.3. *Example A.18 is also of this type. Other examples include* $2 \times 2 \times 4$, $2 \times 4 \times 4$, $3 \times 3 \times 6$, $2 \times 5 \times 10, \ldots$ *and so on.*

The following example is also a $2^2 \times 6$ BAFD with only 5 replications;the main effects are estimated with full efficiencies and some interactions are not estimable.

Example 2.4. *A* $2^2 \times 6$ *BAFD with* $k = 6, b = 20, r = 5, \lambda(1,1) = \lambda(1,0) = \lambda(2,0) = 0$ *and* $\lambda(0,1) = 2$, $\lambda(2,1) = 3$. *The efficiencies are;* $E[0,1] = E[1,0] = E[2,1] = 1.0$; $E[1,1] = \frac{4}{5}, E[2,0] = 0$ *can be constructed using Theorem A.19 and by letting N be the incidence matrix of the* 2×6 *BAFD that was corresponding to a* $BA(T)[3,2,1]$ *which was given in Example A.17.*

In this case, we shall let N^* be the incidence matrix of the following 2^2 design with block size 1.

00	11	01	10

Table 2: resolvable 2^2 Symmetrical design

and if we let N^*_{22} be the following 2^2 balanced factorial design with interaction confounded

00	01
11	10

Table 3: 2^2 balanced factorial design with interactions confounded

Applying Theorem A.19 we get the following $2^2 \times 6$ BAFD.

Blocks	1	2	3	4	5	6	7	8	9	10
Levels of F_3	Levels	of	F_1	and	F_2					
0	00	00	00	00	00	11	11	11	11	11
1	00	00	11	11	11	00	00	00	11	11
2	11	00	00	11	11	11	11	00	00	00
3	00	11	11	00	11	00	11	11	00	00
4	11	11	11	00	00	11	00	00	00	11
5	11	11	00	11	00	00	00	11	11	00

Blocks	11	12	13	14	15	16	17	18	19	20
Levels of F_3	Levels	of	F_1	and	F_2					
0	01	01	01	01	01	10	10	10	10	10
1	01	01	10	10	10	01	01	01	10	10
2	10	01	01	10	10	10	10	01	01	01
3	01	10	10	01	10	01	10	10	01	01
4	10	10	10	01	01	10	01	01	01	10
5	10	10	01	10	01	01	01	10	10	01

Table 4: $2^2 \times 6$ BAFD

Example 2.5. *A* $2^2 \times 4$ *BAFD with* $k = 4$, $b = 24$, $r = 6$, $\lambda(1,1) = \lambda(1,0) = \lambda(2,0) = 0$ *and* $\lambda(0,1) = 2$,$\lambda(2,1) = 4$ *and efficiencies are;*

$$E[0,1] = E[1,0] = E[2,1] = 1.0$$
$$E[1,1] = \frac{2}{3}, E[2,0] = 0$$

can be constructed by letting N be the incidence matrix of the 2×4 *BAFD that corresponds to the* $BA[12,4,2,2]$ *in Example A.21.*

In this case, we let N^* be the incidence matrix of the following 2^2 design with block size 1

00	11	01	10

Table 5: 2^2 design

If we let N^*_{22} to be the following 2^2 balanced factorial design with interaction confounded

00	01
11	10

Table 6: 2^2 BFD with interactions confounded

Then applying Theorem A.19 we get the following $2^2 \times 4$ BAFD.

Blocks	1	2	3	4	5	6	7	8	9	10	11	12
Levels of F_3	Levels	of	F_1	and	F_2							
0	00	00	00	00	00	00	11	11	11	11	11	11
1	11	11	00	11	00	11	00	00	11	00	11	00
2	00	00	11	11	11	11	11	11	00	00	00	00
3	11	11	11	00	11	00	00	00	00	11	00	11

Blocks	13	14	15	16	17	18	19	20	21	22	23	24
Levels of F_3	Levels	of	F_1	and	F_2							
0	01	01	01	01	01	01	10	10	10	10	10	10
1	10	10	01	10	01	10	01	01	10	01	10	01
2	01	01	10	10	10	10	10	10	01	01	01	01
3	10	10	10	01	10	01	01	01	01	10	01	10

Table 7: $2^2 \times 4$ BAFD

Example 2.6. *A* $3^2 \times 9$ *BAFD with* $k = 9$, $b = 144$, $r = 16$, $\lambda(1,1) = \lambda(1,0) = \lambda(2,0) = 0$ *and* $\lambda(0,1) = 4, \lambda(2,1) = 3$ *and efficiencies;*

$$E[0,1] = E[1,0] = 1.0$$
$$E[1,1] = \frac{7}{8}, E[2,0] = \frac{1}{2}$$
$$E[2,1] = \frac{15}{16}$$

can be constructed by letting N be the incidence matrix of the 3×9 *BAFD that corresponds to the* $BA(T)[3,3,1]$ *in Example A.16.*

In this case, we let N^* be the incidence matrix of the following 3^2 design with block size 1.

00	12	21	01	10	22	02	11	20	00	11	22	02
10	21	01	12	20								

Table 8: 3^2 design with block size 1

if we also let N^*_{22} be the incidence matrix of the following 3^2 balanced factorial design with interaction confounded.

00	01	02	00	02	01
12	10	11	11	10	12
21	22	20	22	21	20

Table 9: 3^2 BFD with interactions confounded

Then applying Theorem A.19 we get the following $3^2 \times 9$ BAFD.

Blocks	1	2	3	4	5	6	7	8	9	10	11	12
Levels of F_3	Levels	of	F_1	and	F_2							
0	00	00	00	00	00	00	00	00	12	12	12	12
1	12	21	00	12	21	00	12	21	21	00	12	21
2	21	12	00	21	12	00	21	12	00	21	12	00
3	00	00	12	12	12	21	21	21	12	12	21	21
4	12	21	12	21	00	21	00	12	21	00	21	00
5	21	12	12	00	21	21	12	00	00	21	21	12
6	00	00	21	21	21	12	12	12	12	12	00	00
7	12	21	21	00	12	12	21	00	21	00	00	12
8	21	12	21	12	00	12	00	21	00	21	00	21

Blocks	13	14	15	16	17	18	19	20	21	22	23	24
Levels of F_3	Levels	of	F_1	and	F_2							
0	12	12	12	12	21	21	21	21	21	21	21	21
1	00	12	21	00	00	12	21	00	12	21	00	12
2	21	12	00	21	12	00	21	12	00	21	12	00
3	21	00	00	00	21	21	00	00	00	12	12	12
4	12	00	12	21	00	12	00	12	21	12	21	00
5	00	00	21	12	12	00	00	21	12	12	00	21
6	00	21	21	21	21	21	12	12	12	00	00	00
7	21	21	00	12	00	12	12	21	00	00	12	21
8	12	21	12	00	12	00	12	00	21	00	21	12

Blocks	25	26	27	28	29	30	31	32	33	34	35	36
Levels of F_3	Levels	of	F_1	and	F_2							
0	01	01	01	01	01	01	01	01	10	10	10	10
1	10	22	01	10	22	01	10	22	22	01	10	22
2	22	10	01	22	10	01	22	10	01	22	10	01
3	01	01	10	10	10	22	22	22	10	10	22	22
4	10	22	10	22	01	22	01	10	22	01	22	01
5	22	10	10	01	22	22	10	01	01	22	22	10
6	01	01	22	22	22	10	10	10	10	10	01	01
7	10	22	22	01	10	10	22	01	22	01	01	10
8	22	10	22	10	01	10	01	22	01	22	01	22

Blocks	37	38	39	40	41	42	43	44	45	46	47	48
Levels of F_3	Levels	of	F_1	and	F_2							
0	10	10	10	10	22	22	22	22	22	22	22	22
1	01	10	22	01	01	10	22	01	10	22	01	10
2	22	10	01	22	10	01	22	10	01	22	10	01
3	22	01	01	01	22	22	01	01	01	10	10	10
4	10	01	10	22	01	10	01	10	22	10	22	01
5	01	01	22	10	10	01	01	22	10	10	01	22
6	01	22	22	22	22	22	10	10	10	01	01	01
7	22	22	01	10	01	10	10	22	01	01	10	22
8	10	22	10	01	10	01	10	01	22	01	22	10

Blocks	49	50	51	52	53	54	55	56	57	58	59	60
Levels of F_3	Levels	of	F_1	and	F_2							
0	02	02	02	02	02	02	02	02	11	11	11	11
1	11	20	02	11	20	02	11	20	20	02	11	20
2	20	11	02	20	11	02	20	11	02	20	11	02
3	02	02	11	11	11	20	20	20	11	11	20	20
4	11	20	11	20	02	20	02	11	20	02	20	02
5	20	11	11	02	20	20	11	02	02	20	20	11
6	02	02	20	20	20	11	11	11	11	11	02	02
7	11	20	20	02	11	11	20	02	20	02	02	11
8	20	11	20	11	02	11	02	20	02	20	02	20

Blocks	61	62	63	64	65	66	67	68	69	70	71	72
Levels of F_3	Levels	of	F_1	and	F_2							
0	11	11	11	11	20	20	20	20	20	20	20	20
1	02	11	20	02	02	11	20	02	11	20	02	11
2	20	11	02	20	11	02	20	11	02	20	11	02
3	20	02	02	02	20	20	02	02	02	11	11	11
4	11	02	11	20	02	11	02	11	20	11	20	02
5	02	02	20	11	11	02	02	20	11	11	02	20
6	02	20	20	20	20	20	11	11	11	02	02	02
7	20	20	02	11	02	11	11	20	02	02	11	20
8	11	20	11	02	11	02	11	02	20	02	20	11

Blocks	73	74	75	76	77	78	79	80	81	82	83	84
Levels of F_3	Levels	of	F_1	and	F_2							
0	00	00	00	00	00	00	00	00	11	11	11	11
1	11	22	00	11	22	00	11	22	22	00	11	22
2	22	11	00	22	11	00	22	11	00	22	11	00
3	00	00	11	11	11	22	22	22	11	11	11*	22
4	11	22	11	22	00	22	00	11	22	00	22	00
5	22	11	11	00	22	22	11	00	00	22	22	11
6	00	00	22	22	22	11	11	11	11	11	22	00
7	11	22	22	00	11	11	22	00	22	00	00	11
8	22	11	22	11	00	11	00	22	00	22	00	22

Blocks	85	86	87	88	89	90	91	92	93	94	95	96
Levels of F_3	Levels	of	F_1	and	F_2							
0	11	11	11	11	22	22	22	22	22	22	22	22
1	00	11	22	00	00	11	22	00	11	22	00	11
2	22	11	00	22	11	00	22	11	00	22	11	00
3	22	00	00	00	22	22	00	00	00	11	11	11
4	11	00	11	22	00	11	00	11	22	11	22	00
5	00	00	22	11	11	00	00	22	11	11	00	22
6	00	22	22	22	22	22	11	11	11	00	00	00
7	22	22	00	11	00	11	11	22	00	00	11	22
8	11	22	11	00	11	00	11	00	22	00	22	11

Blocks	97	98	99	100	101	102	103	104	105	106	107	108
Levels of F_3	Levels	of	F_1	and	F_2							
0	02	02	02	02	02	02	02	02	10	10	10	10
1	10	21	02	10	21	02	10	21	21	02	10	21
2	21	10	02	21	10	02	21	10	02	21	10	02
3	02	02	10	10	10	21	21	21	10	10	21	21
4	10	21	10	21	02	21	02	10	21	02	21	02
5	21	10	10	02	21	21	10	02	02	21	21	10
6	02	02	21	21	21	10	10	10	10	10	02	02
7	10	21	21	02	10	10	21	02	21	02	02	10
8	21	10	21	10	02	10	02	21	02	21	02	21

Blocks	109	110	111	112	113	114	115	116	117	118	119	120
Levels of F_3	Levels	of	F_1	and	F_2							
0	10	10	10	10	21	21	21	21	21	21	21	21
1	02	10	21	02	02	10	21	02	10	21	02	10
2	21	10	02	21	10	02	21	10	02	21	10	02
3	21	02	02	02	21	21	02	02	02	10	10	10
4	10	02	10	21	02	10	02	10	21	10	21	02
5	02	02	21	10	10	02	02	21	10	10	02	21
6	02	21	21	21	21	21	10	10	10	02	02	02
7	21	21	02	10	02	10	10	21	02	02	10	21
8	10	21	10	02	10	02	10	02	21	02	21	10

Blocks	121	122	123	124	125	126	127	128	129	130	131	132
Levels of F_3	Levels	of	F_1	and	F_2							
0	01	01	01	01	01	01	01	01	12	12	12	12
1	12	20	01	12	20	01	12	20	20	01	12	20
2	20	12	01	20	12	01	20	12	01	20	12	01
3	01	01	12	12	12	20	20	20	12	12	20	20
4	12	20	12	20	01	20	01	12	20	01	20	01
5	20	12	12	01	20	20	12	01	01	20	20	12
6	01	01	20	20	20	12	12	12	12	12	01	01
7	12	20	20	01	12	12	20	01	20	01	01	12
8	20	12	20	12	01	12	01	20	01	20	01	20

Blocks	133	134	135	136	137	138	139	140	141	142	143	144
Levels of F_3	Levels	of	F_1	and	F_2							
0	12	12	12	12	20	20	20	20	20	20	20	20
1	01	12	20	01	01	12	20	01	12	20	01	12
2	20	12	01	20	12	01	20	12	01	20	12	01
3	20	01	01	01	20	20	01	01	01	12	12	12
4	12	01	12	20	01	12	01	12	20	12	20	01
5	01	01	20	12	12	01	01	20	12	12	01	20
6	01	20	20	20	20	20	12	12	12	01	01	01
7	20	20	01	12	01	12	12	20	01	01	12	20
8	12	20	12	01	12	01	12	01	20	01	20	12

Table 10: $3^2 \times 9$ BAFD

A. Appendix

Theorem A.1. *For all k and s,there always exists a $BA[T][k,s,\lambda]$ for some λ.*

Proof. For all k and s, there exists a $TA[(ks-1)ksn, ks, ks, 2]$ for some n. Let the symbols of the transitive array be denoted by $[0, 1, ..., ks-1]$. If we replace each symbol in the transitive array by $x(modk)$. Then the transitive array becomes a $BA[(ks-1)ksn, ks, s, 2]$ with parameters $\lambda(x,y) = (k-1)kn$ or k^2n according as $x = y$ or not, which is a $BA[T][ks, s, kn]$. The method of construction in Theorem A.1 does not usually provide balanced arrays with a small number of assemblies as we desire. □

Definition A.2. *Suppose $(\mathcal{F}, \mathcal{A})$ is a (v,k,λ)-BIBD, a parallel class in $(\mathcal{F}, \mathcal{A})$ is a subset of disjoint blocks from $\mathcal{A}$ whose union is $\mathcal{F}$. A partition of $\mathcal{A}$ into r parallel classes is called a resolution; and $(\mathcal{F}, \mathcal{A})$ is said to be a resolvable BIBD if $\mathcal{A}$ has at least one resolution. We say that $\mathcal{F}$ is a finite set of points called treatments, where*

$$\mathcal{F} = \left\{0, 1, 2, \cdots, v-1\right\}$$

An example of a BIBD $[4, 6, 2]$ has been captured in table 16

Theorem A.3. *If there exists a resolvable BIBD with qs treatments and block size q, then there exists a $ps \times qs$ BAFD with block size pqs such that all main effects are estimated with full efficiency.*

Proof. Construct a $BA(T)(p,s,n)$ for some integer n by Theorem A.1. In the resolvable BIBD, there being s blocks in each replication , we can number the block in each replication by $0,1,\cdots,s-1$. Replacing each symbol in the balanced array by a group of symbols which represents blocks in the BIBD for each replication, we obtain a $pqs \times [ps-1]snr^{'}$ matrix, where $r^{'}$ is the number of replications in the BIBD. Assign i^{th} level of F_1 to the rows from the $(i_q+1)^{th}$ to the $(i+1)^{th}$, where $i=0,1,\cdots,ps-1$.Identifying columns and symbols with blocks and the levels of F_2, we get a $ps \times qs$ design with block size pqs.

We shall show that all the main effects of the design constructed above are estimated with full efficiency. Let $\lambda^{'}$ be the number of blocks in which two treatments occur together in the BIBD, then $(qs-1)\lambda^{'} = (q-1)r^{'}$. Assume that $r^{'} = (qs-1)m$ and $\lambda^{'} = (q-1)m$, where m need not be an integer. Let $\lambda_{01}, \lambda_{10}, \lambda_{11}$ denote the parameters and r denote the number of replications in the $ps \times qs$ design, then through inspection we have

$$\lambda(x,y) = (ps-1)^{x+1}(qs-1)^{y+1}(p-1)^{x}(q-1)^{y}mn + (xy)(pq)(s-1)^{xy}mn \tag{14}$$

$x, y = 0$ or $1 \pmod 2$ so

$$\left\{\begin{array}{c} \lambda_{01} = (ps-1)(q-1)mn \\ \lambda_{10} = (qs-1)(p-1)mn \\ \lambda_{11} = (p-1)(q-1)mn + pq(s-1)mn \\ \lambda_{00} = r = (ps-1)(qs-1)mn \end{array}\right\} \tag{15}$$

if we substitute the parameters of the equations (17), (18) and (19) in equations (15) and Corollary A.7 we get

$$E[0,1] = E[1,0] = 1 \quad \text{and} \quad E[1,1] = -\frac{s-1}{(ps-1)(qs-1)} + 1$$

Given any q and s, there always exists a resolvable BIBD with qs treatments and block size q if the number of replications is allowed to be large.

Example A.4. *The irreducible BIBD of qs treatments with block size q in which each of the* $\binom{qs}{q}$ *possible q− element combinations form a block is resolvable with parameters*

$$v = qs, b = \binom{qs}{q}, r = \binom{qs-1}{q-1}, k = q, \lambda = \binom{qs-2}{q-2} \tag{16}$$

The eigenvalues of $N^{T}N$ of a BAFD are given by

$$g(1,0) = r + (s_2-1)\lambda_{01} - \lambda_{10} - (s_2-1)\lambda_{11} \tag{17}$$

$$g(0,1) = r - \lambda_{01} + (s_1-1)\lambda_{10} - (s_1-1)\lambda_{11} \tag{18}$$

$$g(1,1) = r - \lambda_{01} - \lambda_{10} + \lambda_{11} \tag{19}$$

where N is the incidence matrix of the $BAFD$. □

Theorem A.5. *If there exists a $BA[N_i, s_m, s_i, 2](i = 1, \ldots, m-1)$ with parameters $\lambda_i(x, y) = \mu_0^i$ or μ_1^i according as $x = y$ or not then there exists an $s_1 \times s_2 \times \ldots \times s_m$ BAFD with $k = s_m$, $b = N_1 \ldots N_{m-1}, \lambda_{\alpha_1\alpha_2\ldots\alpha_{m-1}0=0}$,*

$$\lambda_{\alpha_1\alpha_2\ldots\alpha_{m-1}\ \ 1} = \mu_{\alpha_1}^1 \mu_{\alpha_2}^2 \ldots \mu_{\alpha_{m-1}}^{m-1}$$

where $\alpha_i = 0$ or 1.

Proof. Multiply the $m-1$ balanced arrays to obtain a $BA[N_1N_2 \ldots N_{m-1},\ s_m,\ s_1s_2 \ldots s_{m-1}, 2]$ with parameters $\lambda[(x_1, x_2, \ldots, x_{m-1}),\ (y_1,\ y_2, \ldots,\ y_{m-1})] = \mu_{\alpha_1}^1 \mu^2\alpha_2 \ldots \mu_{\alpha_{m-1}}^{m-1}$ where $\alpha_i = 0$ or 1 according as $x = y$ or not. Identifying the symbols with the levels of F_1, $F_2, \ldots, F_{m-1}$, rows with the levels of F_m and columns with blocks, we obtain an $s_1 \times s_2 \times \ldots \times s_m$ BAFD with the specified parameters. The method used in Theorem A.5 can usually produce efficient BAFDS if we use balanced arrays corresponding to efficient two factor BAFDS. While applying this method, the block size remains the same but the number of blocks increases very rapidly. Hence this method is used when the number of assemblies in the balanced arrays are not too large. □

Theorem A.6. *The eigenvalues of NN' of a BAFD are $g(y_1, y_2, \ldots, y_m)'s$ with corresponding eigenvectors given by the columns of $p^{y'}$, where $y = (y_1, y_2, \ldots, y_m) \in \Omega$ and N is the incidence matrix of a $BAFD$. y is an interaction effect and C is the design matrix.*

It should be noted that the multiplicity of $g(y_1, y_2, \ldots, y_m)$ is $\prod_{i=1}^{m} (s_i - 1)^{y_i}$. Since $C = r(\otimes_{i=1}^{m} I_i) - k^{-1}NN'$. The columns of $P^{y'}$ $y \in \Omega$ are also the eigenvectors of C with corresponding eigenvalues

$$\rho(y) = r - \frac{1}{k} g(y_1, y_2, \ldots, y_m) \tag{20}$$

$$= r - \frac{1}{k} g(y), \qquad y \in \Omega \tag{21}$$

Let $E(y)$ denote the interaction efficiencies, then

Corollary A.7. *$E(y) = 1 - \frac{1}{rk} g(y)$ and $E(y) = 1$ if and only if $g(y) = 0$.*

Lemma A.8. *If s is a prime power, then given j $(1 \leq j \leq m)$ there exists an s^m symmetrical balanced factorial design with block size s and parameters $\lambda_j = 1$, $\lambda_i = 0$ for all $i \neq j$.*

The efficiencies of the symmetrical balanced factorial design constructed in Lemma A.8 can be calculated by equation (22)

$$E_i = 1 - \frac{1}{s} - \frac{P_j(i; m, s)}{\binom{m}{j}(s-1)^{j-1}s}; \quad i = 1, 2, \ldots, m \tag{22}$$

In particular when $j = m$,

$$P_m(i;m,s) = (-1)^i(s-1)^{m-i}$$

and equation (22) becomes (23)

$$E_i = 1 - \frac{1}{s} - \frac{(-1)^i}{(s-1)^{i-1}s} \quad i = 1,2,\ldots,m. \tag{23}$$

The main effects of this balanced design are estimated with full efficiency since $E_1 = 1$ in equation (23)

Theorem A.9. *In an $s_1 \times s_2$ BAFD with block size $s_1(s_1 \leq s_2)$, if the main effects of F_1 are estimated with full efficiency and the main effects F_2 are estimated with maximum efficiency $\frac{(s_1-1)s_2}{s_1(s_2-1)}$ then the BAFD has parameters $\lambda_{10} = \lambda_{01} = 0$ and $\lambda_{11} \neq 0$. This design is equivalent to a $TA[\lambda_{11}s_2(s_2-1), s_1, s_2, 2]$. Since $\lambda_{10} = 0$ means that two treatments at the same level of F_2 do not occur together in the same block, which implies $s_2 \geq k = s_1$ we do not need $s_1 \leq s_2$ in the construction of the designs in Theorem A.9.*

The construction of $TA[s_2(s_2-1)\lambda_{11}, s_1, s_2, 2]$ has been discussed in [31]. Deleting any $(s_2 - s_1)$ constraints from a $TA[s_2(s_2-1)\lambda_{11}, s_2, s_2, 2]$ we obtain a $TA[s_2(s_2-1)\lambda_{11}, s_1, s_2, 2]$. If we restrict $\lambda_{11} = 1$ then the existence of a $TA[s_2(s_2-1), s_1, s_2, 2]$ is equivalent to the existence of $s_1 - 1$ mutually orthogonal latin squares of order s_2 or $s_1 - 2$ mutually orthogonal latin squares of order s_2 with different elements in the diagonal.

Example A.10. *A 3×6 BAFD with $b = 30, k = 3, r = 5$ $\lambda_{01} = \lambda_{10} = 0$ and $\lambda_{11} = 1$ can be constructed from a $TA[30,3,6,2]$. The efficiencies are $E[1,0] = 1.0$,$E[0,1] = \frac{4}{5}$ and $E[1,1] = \frac{3}{5}$.*

Theorem A.11. *If s is a power of an odd prime then there exists a difference scheme $D(2s, 2s, s)$ and an orthogonal array $OA(2s^2, 2s+1, s, 2)$.*

Proof. We construct four $s \times s$ matrices $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$, $F = (f_{ij})$, $0 \leq i, j \leq s-1$ whose entries are given by

$$\left\{ \begin{array}{c} a_{ij} = k_i k_j \\ b_{ij} = k_i k_j + h k_j^2 \\ c_{ij} = k_i k_j + m k_i^2 \\ f_{ij} = n k_i k_j + g k_j^2 + e k_i^2 \end{array} \right\} \tag{24}$$

Where h, m, n, g, e are elements of $GF(s)$ that satisfy the conditions

$$n = 1 + 4he = e/m = n^2 - 4ge, \tag{25}$$

In particular we may take

$$n = k, h = \frac{1}{2}, m = \frac{k-1}{2k}, g = \frac{k}{2} \text{ and } e = \frac{k-1}{2}, \tag{26}$$

Then

$$D = \begin{bmatrix} A & C \\ B & F \end{bmatrix} \tag{27}$$

is a difference scheme $D(2s, 2s, s)$ based on the additive group $GF(s)$ □

Example A.12. *Table 11 shows a difference scheme $D(6, 6, 3)$ constructed in a similar way from $GF(3)$*

0	0	0	0	0	0
0	1	2	1	2	0
0	2	1	1	0	2
0	2	2	0	1	1
0	0	1	2	2	1
0	1	0	2	1	2

Table 11: A difference Scheme $D(6, 6, 3)$

Corollary A.13. *If $s = p^n, k = 2s^l$ where p is an odd prime, $n \geq 1$ and $l \geq 0$, then a $BA(T)[k, s, 1]$ can always be constructed.*

Proof. By using Theorem A.11, we can construct $OA[ks^2, ks, s, 2]$ by developing a difference scheme $D(2s, 2s, s)$. We then apply Theorem A.23 to construct a $BA(T)(k, s, \lambda)$ □

Example A.14. *For $s = 3$ and $k = 2$ implies $3 = 3^1, k = 2.3^0 \mapsto n = 1$ and $l = 0$ We can therefore construct*

$$OA[2.3^2, 2.3, 3, 2] = OA[18, 6, 3, 2]$$

by developing a difference scheme $D(2s, 2s, s) = D(6, 6, 3)$ which is exhibited in table 11

0	0	0	0	0	0	1	1	1	1	1	1	2	2	2	2	2	2
0	1	2	1	2	0	1	2	0	2	0	1	2	0	1	0	1	2
0	2	1	1	0	2	1	0	2	2	1	0	2	1	0	0	2	1
0	2	2	0	1	1	1	0	0	1	2	2	2	1	1	2	0	0
0	0	1	2	2	1	1	1	2	0	0	2	2	2	0	1	1	0
0	1	0	2	1	2	1	2	1	0	2	0	2	0	2	1	0	1

Table 12: Table $OA[18, 6, 3, 2]$

Applying Theorem A.23 to this orthogonal array we obtain $BA(T)[2, 3, 1]$

0	0	0	0	0	1	1	1	1	1	2	2	2	2	2
1	2	1	2	0	2	0	2	0	1	0	1	0	1	2
2	1	1	0	2	0	2	2	1	0	1	0	0	2	1
2	2	0	1	1	0	0	1	2	2	1	1	2	0	0
0	1	2	2	1	1	2	0	0	2	2	0	1	1	0
1	0	2	1	2	2	1	0	2	0	0	2	1	0	1

Table 13: Table $BA(T)[2, 3, 1] = BA[15, 6, 3, 2]$

Parameters of $BA(T)[2, 3, 1]$

- $\lambda(0,0) = \lambda(1,1) = \lambda(2,2) = 1$
- $\lambda(0,1) = \lambda(1,0) = \lambda(0,2) = \lambda(2,0) = \lambda(1,2) = \lambda(2,1) = 2$

Corollary A.15. *If k and s are both powers of the same prime p a $BA(T)[k,s,1]$ can always be constructed.*

Proof. We can always construct a completely resolvable orthogonal array $OA[\lambda s^2, \lambda(s+1)+1, s, 2]$ by deleting any $\lambda + 1$ constraints(factors) we obtain $OA[\lambda s^2, \lambda s, s, 2]$. Then Theorem A.23 is applied. □

Example A.16. *For $k = 3$ and $s = 3$ we can construct a $BA(T)[3,3,1]$ by first constructing a completely resolvable $OA[27,9,3,2]$ which is exhibited in table 14. Applying Theorem A.23, we obtain $BA(T)[3,3,1]$*

0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2
0	1	2	0	1	2	0	1	2	1	2	0	1	2	0	1	2	0	2	0	1	2	0	1	2	0	1
0	2	1	0	2	1	0	2	1	1	0	2	1	0	2	1	0	2	2	1	0	2	1	0	2	1	0
0	0	0	1	1	1	2	2	2	1	1	1	2	2	2	0	0	0	2	2	2	0	0	0	1	1	1
0	1	2	1	2	0	2	0	1	1	2	0	2	0	1	0	1	2	2	0	1	0	1	2	1	2	0
0	2	1	1	0	2	2	1	0	1	0	2	2	1	0	0	2	1	2	1	0	0	2	1	1	0	2
0	0	0	2	2	2	1	1	1	1	1	1	0	0	0	2	2	2	2	2	2	1	1	1	0	0	0
0	1	2	2	0	1	1	2	0	1	2	0	0	1	2	2	0	1	2	0	1	1	2	0	0	1	2
0	2	1	2	1	0	1	0	2	1	0	2	0	2	1	2	1	0	2	1	0	1	0	2	0	2	1

Table 14: An $OA(27,9,3,2)\lambda = 3$

0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2
1	2	0	1	2	0	1	2	2	0	1	2	0	1	2	0	0	1	2	0	1	2	0	1
2	1	0	2	1	0	2	1	0	2	1	0	2	1	0	2	1	0	2	1	0	2	1	0
0	0	1	1	1	2	2	2	1	1	2	2	2	0	0	0	2	2	0	0	0	1	1	1
1	2	1	2	0	2	0	1	2	0	2	0	1	0	1	2	0	1	0	1	2	1	2	0
2	1	1	0	2	2	1	0	0	2	2	1	0	0	2	1	1	0	0	2	1	1	0	2
0	0	2	2	2	1	1	1	1	1	0	0	0	2	2	2	2	2	1	1	1	0	0	0
1	2	2	0	1	1	2	0	2	0	0	1	2	2	0	1	0	1	1	2	0	0	1	2
2	1	2	1	0	1	0	2	0	2	0	2	1	2	1	0	1	0	1	0	2	0	2	1

Table 15: Table $BA(T)[3,3,1] = BA[24,9,3,2]$

Parameters are

- $\lambda(0,0) = \lambda(1,1) = \lambda(2,2) = 2$
- $\lambda(0,1) = \lambda(1,0) = \lambda(0,2) = \lambda(2,0) = \lambda(1,2) = \lambda(2,1) = 3$

Example A.17. *Using Example A.4 a 4×6 BAFD with block size 12 can be constructed using $BA[10,6,2,2]$ and a resolvable BIBD with 4 treatments and block size 2 as shown below.*

Consider the following BIBD with 4 treatments and block size 2 where $X_0, X_1, Y_0, Y_1, Z_0, Z_1$ represents the blocks.

Also consider the $BA(T)(3,2,1)$ given below

X_0	X_1	Y_0	Y_1	Z_0	Z_1
0	2	0	1	0	1
1	3	2	3	3	2

Table 16: Table of BIBD[4,6,2]

0	0	0	0	0	1	1	1	1	1
0	0	1	1	1	0	0	0	1	1
1	0	0	1	1	1	1	0	0	0
0	1	1	0	1	0	1	1	0	0
1	1	1	0	0	1	0	0	0	1
1	1	0	1	0	0	0	1	1	0

Table 17: Table of BA(T)[3,2,1]

By Theorem A.3 we can construct a 4×6 BAFD with $k = 12, r = \lambda_{00} = 15, b = 30, \lambda_{10} = 5, \lambda_{01} = 6, \lambda_{11} = 8$ with $E[1,0] = 1, E[0,1] = 1, E[1,1] = \frac{14}{15}$

Blocks	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
levels of F_2	Levels	of	F_1												
0	X_0	X_0	X_0	X_0	X_0	X_1	X_1	X_1	X_1	X_1	Y_0	Y_0	Y_0	Y_0	Y_0
1	X_0	X_0	X_1	X_1	X_1	X_0	X_0	X_0	X_1	X_1	Y_0	Y_0	Y_1	Y_1	Y_1
2	X_1	X_0	X_0	X_1	X_1	X_1	X_1	X_0	X_0	X_0	Y_1	Y_0	Y_0	Y_1	Y_1
3	X_0	X_1	X_1	X_0	X_1	X_0	X_1	X_1	X_0	X_0	Y_0	Y_1	Y_1	Y_0	Y_1
4	X_1	X_1	X_1	X_0	X_0	X_1	X_0	X_0	X_0	X_1	Y_1	Y_1	Y_1	Y_0	Y_0
5	X_1	X_1	X_0	X_1	X_0	X_0	X_0	X_1	X_1	X_0	Y_1	Y_1	Y_0	Y_1	Y_0

Blocks	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
levels of F_2															
0	Y_1	Y_1	Y_1	Y_1	Y_1	Z_0	Z_0	Z_0	Z_0	Z_0	Z_1	Z_1	Z_1	Z_1	Z_1
1	Y_0	Y_0	Y_0	Y_1	Y_1	Z_0	Z_0	Z_1	Z_1	Z_1	Z_0	Z_0	Z_0	Z_1	Z_1
2	Y_1	Y_1	Y_0	Y_0	Y_0	Z_1	Z_0	Z_0	Z_1	Z_1	Z_1	Z_1	Z_0	Z_0	Z_0
3	Y_0	Y_1	Y_1	Y_0	Y_0	Z_0	Z_1	Z_1	Z_0	Z_1	Z_0	Z_1	Z_1	Z_0	Z_0
4	Y_1	Y_0	Y_0	Y_0	Y_1	Z_1	Z_1	Z_1	Z_0	Z_0	Z_1	Z_0	Z_0	Z_0	Z_1
5	Y_0	Y_0	Y_1	Y_1	Y_0	Z_1	Z_1	Z_0	Z_1	Z_0	Z_0	Z_0	Z_1	Z_1	Z_0

Table 18: Table of a 4×6 BAFD

Example A.18. *The product of $BA(T)(3,2,1)$ in Example A.17 and a $BAT(2,3,1)$ in Example A.14 generates a $2 \times 3 \times 6$ BAFD with $r = 25, b = 150, k = 6, \lambda(0,1,0) = \lambda(1,0,0) = \lambda(1,1,0) = 0, \lambda(0,0,1) = 2, \lambda(0,1,1) = 4, \lambda(1,0,1) = 3$ and $\lambda(1,1,1) = 6$. The efficiencies are*

$$E(0,0,1) = E(0,1,0) = E(1,0,0) = 1.0$$
$$E(0,1,1) = E(1,0,1) = E(1,1,0) = \frac{4}{5}$$
$$E(1,1,1) = \frac{21}{25}.$$

We can also obtain an efficient $2 \times 3 \times 6$ BAFD by collapsing the first factor of the 6^2 symmetrical balanced factorial design in Example A.10 into two factors one at 2 levels and the other at 3 levels. The BAFD has parameters $r = 10, b = 60, k = 6, \lambda(0,0,1) = \lambda(0,1,0) = \lambda(1,0,0) = \lambda(0,1,0) = 0$ and $\lambda(0,1,1) = \lambda(1,0,1) = \lambda(1,1,1) = 2$. The efficiencies are $E(0,0,1) = E(0,1,0) = E(1,0,0) = E(1,1,0) = 1.0$ and $E(0,1,1) = E(1,0,1) = E(1,1,1) = \frac{4}{5}$ all the main effects are also estimated with full efficiency like in Example A.18 but we only need 10 replications in this design.

Assume that there exists a BAFD with m factors $F_1, F_2, \ldots, F_m$ at $s_1, s_2, \ldots, s_m$ levels respectively,each of the $v* (= s_1 s_2 \ldots s_m)$ treatments replicated $r*$ times in b^* blocks of k^* plots each,with the incidence matrix.

$$N^* = [A_1^* | A_2^* | \ldots | A_b^* *] \tag{28}$$

Further assume that $b^* = pq$, and the pq blocks can be divided into p groups of q blocks each, such that the design consisting of p blocks formed by adding together all the blocks of a group is a BAFD.The incidence matrix is :

$$N^*_{pq} = \left[\sum_{j=1}^{q} A^*{}_j | \sum_{j=1}^{q} A^*{}_{j+q} | \ldots | \sum_{j=1}^{q} A^*{}_{pq-q+j}\right] \tag{29}$$

for a resolvable design N^*, the corresponding $N^*{}_{pq}$ exists with $p = r^*$. The following theorem was proven by [28].

Theorem A.19. *Let there be a BAFD with the incidence matrix N in $n+1$ factors $F_0, F_{m+1}, \ldots, F_{m+n}$ at $q, s_{m+1}, \ldots, s_{m+n}$ levels respectively in b blocks of k plots each.Also let there be two BAFDs with incidence matrices N^* and $N^*{}_{pq}$ as given by equations* (28) *and* (29) *respectively.If the level $j-1$ of the factor F_0 is replaced by the block $A_{iq+j}(j = 1, 2, \ldots, q)$ in each of the treatments of N ,then the design obtained by adjoining the p designs so formed (for $i = 0, 1, 2, \ldots, p-1$) is a BAFD in $m+n$ factors in bp blocks of kk^* plots each.*

This method generates an $m+n$ factor BAFD from an $n+1$ factor BAFD and an m factor BAFD.Thus from the two-factor BAFD's we can generate a three-factor BAFD. If the two-factor BAFD's are efficient, then three-factor BAFD is also efficient. We can therefore construct efficient multi-factor BAFD's step by step from efficient two-factor BAFD's. While applying this method, the number of blocks does not increase so quickly as in the first method, but the block size does increase. The method of Difference Schemes used in the construction of orthogonal arrays can also be used to construct some type of balanced arrays discussed in this manuscript.

Theorem A.20. *Let M be a module of s elements. It is possible to choose k rows and N columns ($N = \lambda_1 + \lambda_2(s-1), \lambda_1$ and λ_2 integers)*

$$\begin{matrix} a_{11} & a_{12} & . & . & . & a_{1N} \\ a_{21} & a_{22} & . & . & . & a_{2N} \\ . & . & . & . & . & . \\ a_{k1} & a_{k2} & . & . & . & a_{kN} \end{matrix}$$

with elements belonging to M such that among the differences of the corresponding elements of any two rows, the element 0 occurs λ_1 times and the other non zero elements occur λ_2 times, then by adding the elements of the module to the elements in the above array and reducing mod s, we can generate Ns columns: this constitutes a $BA[N, k, s, 2]$ with parameters $\lambda(x, y) = \lambda_1$ or λ_2 according as $x = y$ or $x \neq y$.

The balanced arrays that can be constructed by Theorem A.20 are completely resolvable. We give the following example to illustrate the application of Theorem A.20.

Example A.21. *Let $M = [0, 1]$. Among the differences of the corresponding elements of any two rows of the* *following array* 0 *occurs twice whereas* 1 *occurs four times*

0	0	0	0	0	0
1	1	0	1	0	1
0	0	1	1	1	1
1	1	1	0	1	0

hence we can construct a $BA[12,4,2,2]$ shown in table 19 below

0	0	0	0	0	0	1	1	1	1	1	1
1	1	0	1	0	1	0	0	1	0	1	0
0	0	1	1	1	1	1	1	0	0	0	0
1	1	1	0	1	0	0	0	0	1	0	1

Table 19: Table $BA[12,4,2,2]$

Parameters of BA[12,4,2,2]

- $\lambda(0,0) = \lambda(1,1) = 2$
- $\lambda(0,1) = \lambda(1,0) = 4$

Definition A.22. *An orthogonal array $OA[N,k,s,2]$ is said to be a-resolvable if it is statistically equivalent to the juxtaposition of $\frac{N}{as}$ arrays such that each factor occurs in each of these arrays a times at each level. A* 1*-resolvable orthogonal array is also called completely resolvable, otherwise it is called Partly resolvable.*

Theorem A.23. *The existence of a partly resolvable $OA[ks^2,ks,s,2]$is equivalent to the existence of a $BA[T][k,s,1]$.*

Proof. If a partly resolvable $OA[ks^2,ks,s,2]$ exists then there exists s assemblies which form $OA[s,ks,s,1]$. We can permute the symbols of the orthogonal array in each row such that these s, assemblies are of the form $(i,i,...,i)'$ for $i = 0,1,...,s-1$. Deleting these assemblies we obtain a $BA[T][k,s,1]$. □

References

[1] S. B. Agarwal and M. N. Das, *Asymmetrical factorial type switch over designs*, Calcutta Statistical Association Bulletin, A035(3-4)(1986), 149-156.

[2] Sunanda Bagchi, *Inter-class orthogonal main effect plans for asymmetrical experiments*, Sankhya B, 81(2019), 93-122.

[3] S. Bahl and A. Dalal, *A method for the construction of asymmetrical factorial experiments with $(p-1)$ replications*, Bulletin of Pure and Applied Sciences E. Mathematics and Statistics, (2008).

[4] K. A. Bush, *Orthogonal arrays of index unity*, Mathematical Statistics, (1952).

[5] Liuping Hu, Kashinath Chatterjee and Jiaqi Liu and Zujun Ou, *New lower bound for lee discrepancy of asymmetrical factorials*, Statistical Papers, 61(2020), 1763-1772.

[6] P. A. Chatterjee, K. Angelopoulos and C. Koukouvinos, *A lower bound to the measure of optimality for main effect plans in the general asymmetric factorial experiments*, Statistics, 47(2)(2013), 1-6.

[7] Ching-Shui Cheng, *Theory of factorial design single-and multi-stratum experiments*, CRC Press, (2013).

[8] Dipa Rani Das and Sanjib Ghosh, *An alternative method of construction and analysis of asymmetrical factorial experiment of the type* 6×22 *in blocks of size 12*, Chittagong University Journal of Science, 40(01)(2018), 137-150.

[9] M. N. Das and N. C. Giri, *Design and analysis of experimets*, Second Edtion, Wiley Eastern: New Delhi, (1960).

[10] Angela Dean, Daniel Voss and Danel Draguljik, *Confounding in general factorial experiments first online*, Springer Texts in Statistics, (2017), 473-493.

[11] Jyoti Divecha and Bharat Tarapara, *Small, balanced, efficient, optimal, and near rotatable response surface designs for factorial experiments asymmetrical in some quantitative, qualitative factors*, Quality Engineering, 29(2)(2016), 196-210.

[12] El-Helbawy, T. Abdalla, Essam Ahmed and Abdullah H. Alharbey, *Optimal designs for asymmetrical factorial paired comparison experiments*, Communications in Statistics-Simulation and Computation, 23(3)(1994).

[13] K. N. Gachii and J. W. Odhiambo, *Theory and methods asymmetrical single replicate design*, South African Statist. J., 32(1998), 1-18.

[14] S. C. Gupta, *Generating generalized cyclic designs with factorial balance*, Commun. Statist. Theor. Math., 16(1987), 1885-1900.

[15] V. K. Gupta, *On orthogonal arrays and their applications*, Journal of the Indian Society of Agricultural Statistics, 62(1)(2008), 1-18.

[16] Klaus Hinkelmann and Oscar Kempthorne, *Design and Analysis of Experiments*, Volume 2, Advanced Experimental Design, (2005), 466-506.

[17] M. A. Jalil, *A monograph on construction method of factorial experiments*, LAP LAMBERT Academic Publishing, (2013).

[18] Prakash Kumar, Krishan Lal, Anirban Mukherjee, Upendra Kumar Pradhan, Mrinmoy Ray and Om Prakash, *Advanced row-column designs for animal feed experiments*, Indian Journal of Animal Sciences, 88(2023).

[19] S. M. Lewis and M. G. Tuck, *Paired comparision designs for factorial experiments*, Journal of the Royal Statistical Society Series C, Royal Statistical Society, 34(3)(1985), 227-234.

[20] Romario A. Conto Lopez, Alexander A. Correa Espinal and Olga C. Usuga Manco, *Run orders in factorial designs: A literature review*, Communications in Statistics - Theory and Methods, 53(13)(2024), 4557-4575.

[21] Mainardi, Pedro Henrique and Ederio Dino Bidoia, *Fundamental concepts and recent applications of factorial statistical designs*, Brazilian Journal of Biometrics, 40(1)(2022).

[22] Metrika, *Choosing an optimal* $\left(s^2\right) s^n$ *design for the practical need, where s is any prime or prime power*, NoP, (2002).

[23] E. R. Muller, *Balanced confounding of factorial experiments*, Biometrics, (1966).

[24] G. S. R. Murthy, *Optimization in* $2^m 3^n$ *factorial experiments algorithmic*, Operations Research, 7(2013).

[25] K. R. Nair and C. R. Rao, *Confounding in asymmetrical factorial experiments*, Journal of the Royal Statistical Society, 10(1)(1948), 109-131.

[26] Ulrike Grömping, *R package doe.base for factorial experiments*, Journal of Statistical Software, 85(5)(2018), 1-41.

[27] C. R. Rao, *A general class of quasi factorial and related design*, Sankhya, (1956).

[28] B. V. Shah, *Balanced factorial experiments*, Annals of Mathematical Statistics, 31(7)(1960), 502-514.

[29] P. R. Sreenath, *On designs for asymmetrical factorial experiments through confounded symmetricals*, Statistics and Applications, 9(7)(2011), 71-81.

[30] P. R. Sreenath, *Designs for assymetrical factorial experiment through confounded symmetrical*, Journal of The Indian Society of Agricuture Statistics, 9(1 and 2)(2011).

[31] C. Y. Suen and I. M Chakravati, *Efficient two factor balanced designs*, J. R. Statist. Soc., (1986).

[32] S. K. Tharthare, *Generalized right angular designs*, Ann. Math. Statist, (1965).

[33] H. R. Thomson and I. D. Dick, *Facorial designs in small bblock derivable from orthogonal latin squares*, J. R. Statist. Soc., (1951).

[34] D. T. Voss, *On generalizations of the classical method of confounding to asy mmetric factorial experiments*, Communications in Statistics-Theory and Methods, 15(4)(1986), 1299-1314.

[35] J. N. Wanyoike, F. Njui and M. M. Manene, *Efficiency factors for linear contrasts in confounded asymmetrical factorial designs*, Far East Journal of Theoretical Statistics, 55(1)(2019), 23-52.

[36] Rahul Mukerjee and C. F. Jeff Wu, *A modern theory of factorial design*, Springer Series in Statistics, (2006).

[37] F. Yates, *The design and analysis of factorial assignment*, Imp. Bur, of Social Sci., Tech., Communication, (1937).

[38] Xue Min Zi, Min Qian Liu and Runchu Zhang, *Asymmetrical factorial designs containing clear effects*, New Delhi Statistics and Applications, 9(1 and 2)(2007), 71-81.