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Exact soliton solutions of some Nonlinear Dispersive Equations

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Abstract

The functional variable method provides an efficient means of deriving exact soliton solutions for nonlinear partial differential equations. In this study we applied the functional variable method to find soliton solutions of nonlinear dispersive equations. The obtained results are novel and have significant applications in contemporary research areas of mathematical physics.

Keywords: Solitary wave solution; Nonlinear dispersive equation; Functional Variable Method.
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1. Introduction

Many real-world phenomena are modelled using nonlinear evolution equations (NLEEs). For better understanding of the physical mechanisms of these natural processes, it is essential to obtain the exact solutions of NLEEs. The obtained solutions play a fundamental role in analyzing the qualitative behavior of various phenomena across different scientific fields. A wide range of equations in engineering, physics, chemistry, and biology include empirically determined parameters or functions. Exact solutions provide a foundation for designing and conducting experiments under controlled conditions, enabling researchers to determine these parameters with greater precision. As a result, the pursuit of exact solutions for NLEEs has become an essential aspect of studying physical phenomena and is now recognized as a key challenge in mathematical physics.

But, not all equations formulated within these models are explicitly solvable, necessitating the development of new analytical methods for obtaining exact solutions. To address this, researchers have proposed several advanced methods, such as Hirota's bilinear transformation technique [1–5], the Exp-function method [6,7], the Homotopy analysis method [8], the Darboux transformation [9,10], the (G′/G)-expansion method [11–13], the first integral method [14,15], the modified extended tanh-function method [16], the Kudryashov expansion method [17–19], the sine–cosine method [20], the F-expansion method [21] the modified simple equation method [22–24], among others

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The functional variable method, originally proposed by Zerarka et al. in [25,26], is a powerful and efficient approach for obtaining exact solutions to nonlinear evolution equations. Recently, Babajanov [27–29] utilized this method to derive soliton solutions for various NLEEs. The purpose of the paper is to apply the functional fariable method (FVM) to establish exact solitory wave solutions for the following nonlinear dispersive equations [30]:

$$\frac{\partial}{\partial t}(u^n) + \alpha \frac{\partial}{\partial x}(u^{kn}) + \beta \frac{\partial}{\partial xxx}(u^n) = 0, \; k > 1, n > 0, ... \tag{1}$$

$$u^n \frac{\partial}{\partial t}(u^n) + \alpha \frac{\partial}{\partial x}(u^{kn}) + \beta u^n \frac{\partial}{\partial xxx}(u^n) = 0, \; k > 2, n > 0, ... \tag{2}$$

$$\frac{\partial}{\partial tt}(u^n + \alpha \frac{\partial}{\partial xx}(u^{kn}) + \beta \frac{\partial}{\partial xxxx}(u^n) = 0, \; k > 1, n > 0, ... \tag{3}$$

where α and β are constants.

Rest part of the paper is organized as follows: Section 2 introduces the fundamental concept of the method employed to obtain exact solitory wave solutions of nonlinear evolution equations (NLEEs). In Section 3, the FVM is applied to three different models of nonlinear dispersive equations. Section 4 provides graphical representations of the obtained solutions, and finally Section 5 provides conclusions of the study.

2. The Functional Variable Method

Consider the following form of a nonlinear partial differential equation (NLPDE):

$$\mathcal{P}(u, u_t, u_x, u_y, u_{tt}, u_{xx}, u_{xy}, u_{xt}, u_{yt}, u_{yy}...) = 0 \tag{4}$$

In this expression, $\mathcal{P}$ represents a polynomial involving $u = u(x, y, t)$ and its partial derivatives. The principal steps of the method can be outlined as:

Step 1: To construct the traveling wave solution of Equation 4, we make use of the wave variable $\zeta = ax + by - ct$, where a, b, and c are constants, such that:

$$u(x, y, t) = U(\zeta) \tag{5}$$

Consequently, the nonlinear partial differential equation (NLPDE) is converted into an ordinary differential equation (ODE) of the form:

$$\mathcal{Q}(U, U_\xi, U_{\xi\xi}, U_{\xi\xi\xi}, ...) = 0 \tag{6}$$

where, $\mathcal{Q}$ is a polynomial in $U = U(\zeta)$ and $U_\zeta = \frac{dU}{d\zeta}, U_{\zeta\zeta} = \frac{d^2U}{d\zeta^2}$ and so on. Equation 6 is integrated under the assumption that all terms involve derivatives, with the integration constants set to zero.

Step 2: We introduce a transformation by representing the unknown function $U(\zeta)$ as a functional variable of the form

$$U_\zeta = F(U) \tag{7}$$

and some successive derivatives of U are

$$\begin{aligned} U_{\zeta\zeta} &= \frac{1}{2}(F^2(U))' \\ U_{\zeta\zeta\zeta} &= \frac{1}{2}(F^2(U))''\sqrt{F^2(U)} \\ U_{\zeta\zeta\zeta\zeta} &= \frac{1}{2}\left[(F^2(U))'''F^2 + \frac{1}{2}(F^2(U))''(F^2(U))'\right] \end{aligned} \tag{8}$$

and so on, where $' = d/dU$

Step 3: Substituting Equation 7 and 8 into 6, we obtain the following ODE

$$\mathcal{R}(U, F, F', F'', F''', ...) = 0 \tag{9}$$

Integrating, Equation 9 yields the expression for F, which, in combination with Equation 7, provides the appropriate solutions for Equation 4. To demonstrate the effectiveness of the proposed method, we analyze several examples previously treated using other approaches.

3. Applications

3.1 The nonlinear Dispersive Equation: Model I

In this subsection, the Functional Variable Method (FVM) is applied to obtain general exact solutions of the nonlinear dispersive equation:

$$\frac{\partial}{\partial t}(u^n) + \alpha\frac{\partial}{\partial x}(u^{kn}) + \beta\frac{\partial}{\partial xxx}(u^n) = 0,\ k > 1, n > 0, ... \tag{10}$$

where α and β are constants. Setting $u^n = v$ into Equation 10, we get

$$v_t + \alpha(v^k)_x + \beta(v)_{xxx} = 0 \tag{11}$$

Using the wave variable $\zeta = x - ct$ (c is the wave speed) and setting $V(\zeta) = v(x,t)$, Equation 11 converts to the following ODE

$$-cV' + \alpha(V^k)' + \beta V''' = 0 \tag{12}$$

Integrating Equation 12 twice with respect to ζ and taking the constants of integration equal to zero, we get

$$-cV + \alpha V^k + \beta V'' = 0 \tag{13}$$

$$V'' = \frac{c}{\beta}V - \frac{\alpha}{\beta}V^k \tag{14}$$

$$\frac{(F^2(V))'}{2} = \frac{c}{\beta}V - \frac{\alpha}{\beta}V^k \tag{15}$$

Integrating the Equation 15 with respect to V with zero constants of integration, we have

$$F(V) = \pm\sqrt{\frac{c}{\beta}}V\sqrt{\left[1 - \frac{2\alpha}{c(k+1)}V^{k-1}\right]} \tag{16}$$

From Equation 7 and 16 we deduce that

$$\int \frac{dV}{V\sqrt{\left[1 - \frac{2\alpha}{c(k+1)}V^{k-1}\right]}} = \pm\sqrt{\frac{c}{\beta}}(\zeta + \zeta_0) \tag{17}$$

where ζ_0 is a constant of integration. After integrating Equation 17, The case $\frac{c}{\beta} > 0$ gives the hyperbolic travelling wave solutions:

$$u_{1,1}(x,t) = \pm\left[\frac{c(k+1)}{2\alpha}sech^2\left(\frac{(k-1)}{2}\sqrt{\frac{c}{\beta}}(x-ct)+\zeta_0\right)\right]^{\frac{1}{(k-1)^2}} \tag{18}$$

$$u_{1,2}(x,t) = \pm\left[\frac{c(k+1)}{2\alpha}csch^2\left(\frac{(k-1)}{2}\sqrt{\frac{c}{\beta}}(x-ct)+\zeta_0\right)\right]^{\frac{1}{(k-1)^2}} \tag{19}$$

The case $\frac{c}{\beta} < 0$ gives the periodic travelling wave solutions:

$$u_{1,3}(x,t) = \pm\left[\frac{c(k+1)}{2\alpha}sec^2\left(\frac{(k-1)}{2}\sqrt{-\frac{c}{\beta}}(x-ct)+\zeta_0\right)\right]^{\frac{1}{(k-1)^2}} \tag{20}$$

$$u_{1,4}(x,t) = \pm\left[\frac{c(k+1)}{2\alpha}csc^2\left(\frac{(k-1)}{2}\sqrt{-\frac{c}{\beta}}(x-ct)+\zeta_0\right)\right]^{\frac{1}{(k-1)^2}} \tag{21}$$

3.2 The nonlinear Dispersive Equation: Model II

In this subsection, we investigate exact solutions of the nonlinear dispersive equation of the form:

$$u^n\frac{\partial}{\partial t}(u^n) + \alpha\frac{\partial}{\partial x}(u^{kn}) + \beta u^n\frac{\partial}{\partial xxx}(u^n) = 0,\ k > 2, n > 0, ... \tag{22}$$

here α and β denote constants. Substituting $u^n = v$ into Equation 22 and dividing by v, gives

$$v_t + \alpha\frac{k}{k-1}(v^{k-1})_x + \beta(v)_{xxx} = 0 \tag{23}$$

By employing the wave variable $\zeta = x - ct$ (c is the wave speed) and setting $V(\zeta) = v(x,t)$, Equation 23 converts to the following ODE

$$-cV' + \alpha\frac{k}{k-1}(V^{k-1})' + \beta V''' = 0 \tag{24}$$

Equation 24, integrated once with respect to ζ under zero integration constant, gives

$$-cV + \alpha \frac{k}{k-1} V^{k-1} + \beta V'' = 0 \tag{25}$$

$$V'' = \frac{c}{\beta} V - \frac{\alpha k}{(k-1)\beta} V^{k-1} \tag{26}$$

$$\frac{(F^2(V))'}{2} = \frac{c}{\beta} V - \frac{\alpha k}{(k-1)\beta} V^{k-1} \tag{27}$$

Integrating the Equation 27 with respect to V with zero constants of integration, we have

$$F(V) = \pm \sqrt{\frac{c}{\beta}} V \sqrt{\left[1 - \frac{2\alpha}{c(k-1)} V^{k-2}\right]} \tag{28}$$

From Equation 7 and 28 we deduce that

$$\int \frac{dV}{V\sqrt{\left[1 - \frac{2\alpha}{c(k-1)} V^{k-2}\right]}} = \pm \sqrt{\frac{c}{\beta}} (\zeta + \zeta_0) \tag{29}$$

where ζ_0 is a constant of integration. After integrating Equation 29, The case $\frac{c}{\beta} > 0$ gives the hyperbolic travelling wave solutions:

$$u_{2,1}(x,t) = \pm \left[\frac{c(k-1)}{2\alpha} sech^2 \left(\frac{(k-2)}{2} \sqrt{\frac{c}{\beta}} (x - ct) + \zeta_0\right)\right]^{\frac{1}{(k-2)^2}} \tag{30}$$

$$u_{2,2}(x,t) = \pm \left[\frac{c(k-1)}{2\alpha} csch^2 \left(\frac{(k-2)}{2} \sqrt{\frac{c}{\beta}} (x - ct) + \zeta_0\right)\right]^{\frac{1}{(k-2)^2}} \tag{31}$$

The case $\frac{c}{\beta} < 0$ gives the periodic travelling wave solutions:

$$u_{2,3}(x,t) = \pm \left[\frac{c(k-1)}{2\alpha} sec^2 \left(\frac{(k-2)}{2} \sqrt{-\frac{c}{\beta}} (x - ct) + \zeta_0\right)\right]^{\frac{1}{(k-2)^2}} \tag{32}$$

$$u_{2,4}(x,t) = \pm \left[\frac{c(k-1)}{2\alpha} csc^2 \left(\frac{(k-2)}{2} \sqrt{-\frac{c}{\beta}} (x - ct) + \zeta_0\right)\right]^{\frac{1}{(k-2)^2}} \tag{33}$$

3.3 The nonlinear Dispersive Equation: Model III

In this subsection, we investigate exact solutions of the nonlinear dispersive equation of the form:

$$\frac{\partial}{\partial tt}(u^n + \alpha \frac{\partial}{\partial xx}(u^{kn}) + \beta \frac{\partial}{\partial xxxx}(u^n) = 0,\ k > 1, n > 0, ... \tag{34}$$

here α and β are constants. Setting $u^n = v$ into Equation 34, we get

$$v_{tt} + \alpha(v^k)_{xx} + \beta(v)_{xxxx} = 0 \tag{35}$$

By employing the wave variable $\zeta = x - ct$ (c is the wave speed) and setting $V(\zeta) = v(x,t)$, Equation 35 converts to the following ODE

$$c^2V'' + \alpha(V^k)'' + \beta V'''' = 0 \tag{36}$$

Equation 36, integrated twice with respect to ζ under zero integration constants, gives

$$c^2V + \alpha V^k + \beta V'' = 0 \tag{37}$$

$$V'' = -\frac{c^2}{\beta}V - \frac{\alpha}{\beta}V^k \tag{38}$$

$$\frac{(F^2(V))'}{2} = -\frac{c^2}{\beta}V - \frac{\alpha}{\beta}V^k \tag{39}$$

Integrating the Equation 39 with respect to V with zero constants of integration, we have

$$F(V) = \pm\sqrt{\frac{-c^2}{\beta}}V\sqrt{\left[1 + \frac{2\alpha}{c^2(k+1)}V^{k-1}\right]} \tag{40}$$

From Equation 7 and 40 we deduce that

$$\int \frac{dV}{V\sqrt{\left[1 + \frac{2\alpha}{c^2(k+1)}V^{k-1}\right]}} = \pm\sqrt{\frac{-c^2}{\beta}}(\zeta + \zeta_0) \tag{41}$$

where ζ_0 is a constant of integration. After integrating Equation 41, The case $\frac{1}{\beta} > 0$ gives the periodic travelling wave solutions:

$$u_{3,1}(x,t) = \pm\left[\frac{c^2(k+1)}{2\alpha}sech^2\left(\frac{(k-1)}{2}\sqrt{\frac{-c^2}{\beta}}(x-ct)+\zeta_0\right)\right]^{\frac{1}{(k-1)^2}} \tag{42}$$

$$u_{3,2}(x,t) = \pm\left[\frac{c^2(k+1)}{2\alpha}csch^2\left(\frac{(k-1)}{2}\sqrt{\frac{-c^2}{\beta}}(x-ct)+\zeta_0\right)\right]^{\frac{1}{(k-1)^2}} \tag{43}$$

The case $\frac{1}{\beta} < 0$ gives the periodic travelling wave solutions:

$$u_{3,3}(x,t) = \pm\left[\frac{c^2(k+1)}{2\alpha}sec^2\left(\frac{(k-1)}{2}\sqrt{\frac{c^2}{\beta}}(x-ct)+\zeta_0\right)\right]^{\frac{1}{(k-1)^2}} \tag{44}$$

$$u_{3,4}(x,t) = \pm\left[\frac{c^2(k+1)}{2\alpha}csc^2\left(\frac{(k-1)}{2}\sqrt{\frac{c^2}{\beta}}(x-ct)+\zeta_0\right)\right]^{\frac{1}{(k-1)^2}} \tag{45}$$

4. Graphical Representation of the Nonlinear Dispersive Equations

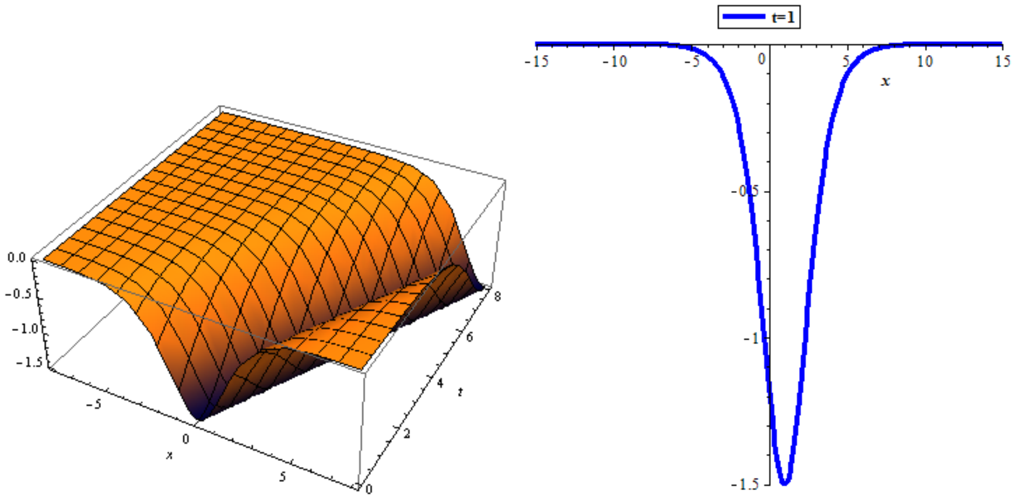

Figure 1: Three-dimensional view of the solution $u_{1,1}(x,t)$ along with its 2D- projection at $t = 1$ with $c = 1, k = 2, \alpha = -1, \beta = 1, \zeta_0 = 0$.

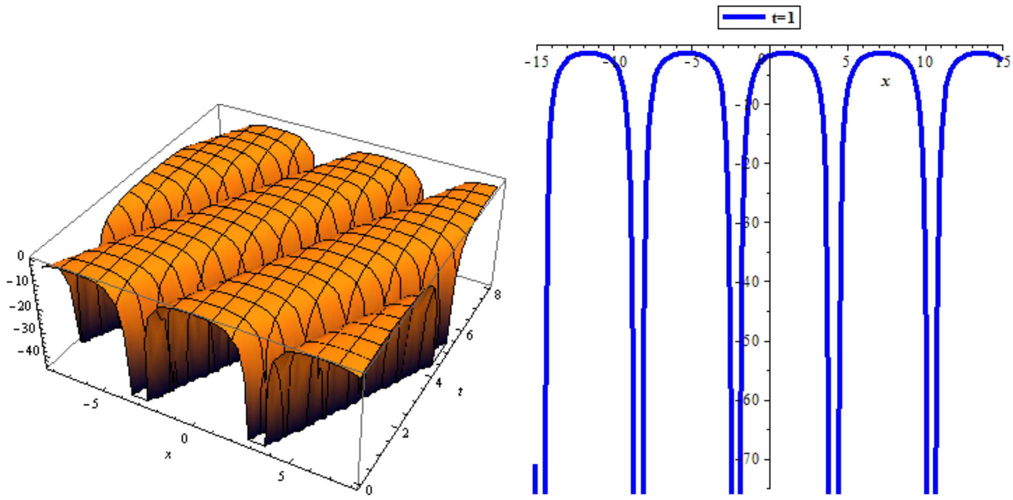

Figure 2: Three-dimensional view of the solution $u_{1,3}(x,t)$ along with its 2D-projection at $t = 1$ with $c = 1, k = 2, \alpha = -1, \beta = -1, \zeta_0 = 0$.

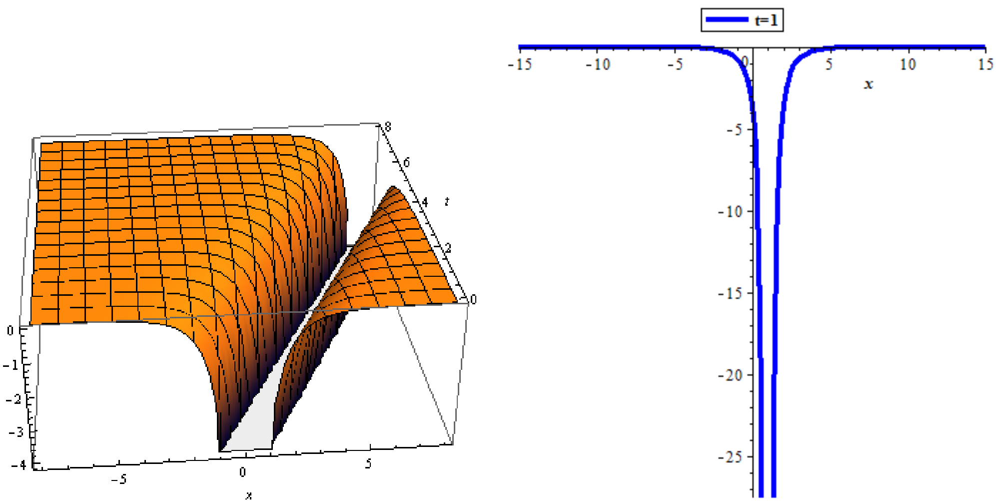

Figure 3: Three-dimensional view of the solution $u_{2,2}(x,t)$ along with its 2D-projection at $t = 1$ with $c = 1, k = 3, \alpha = -1, \beta = 1, \zeta_0 = 0$.

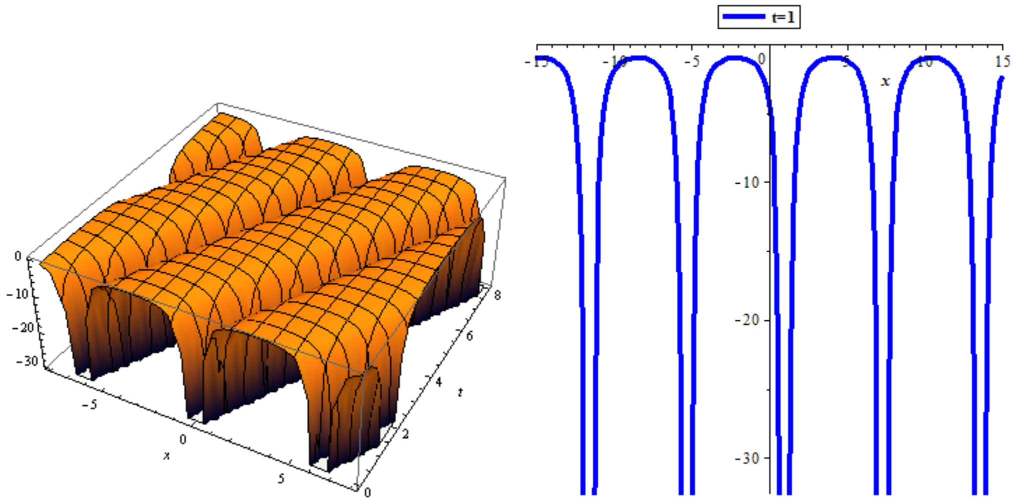

Figure 4: Three-dimensional view of the solution $u_{2,4}(x,t)$ along with its 2D-projection at $t = 1$ with $k = 3, c = 1, \alpha = -1, \beta = -1, \zeta_0 = 0$.

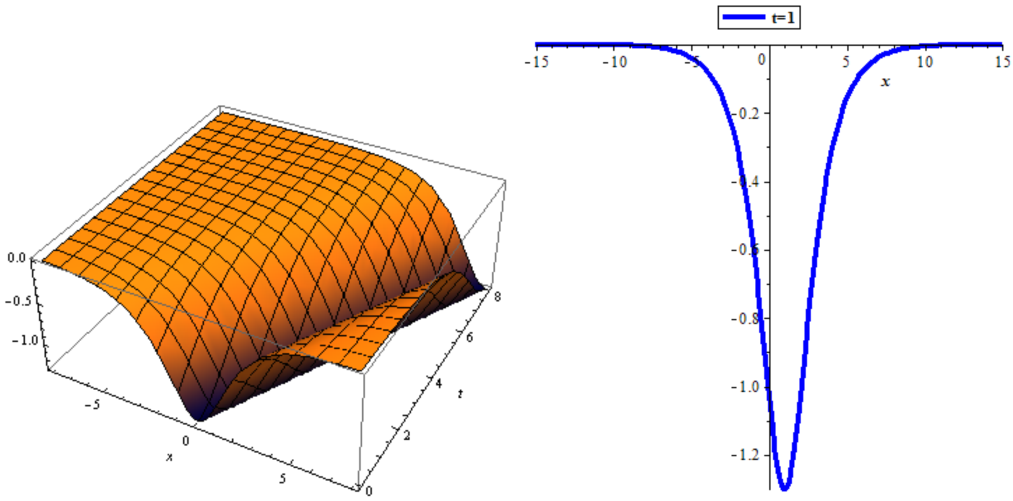

Figure 5: Three-dimensional view of the solution $u_{3,1}(x,t)$ along with its 2D-projection at $t = 1$ with $k = 1 + \sqrt{2}, c = 1, \alpha = 1, \beta = -1, \zeta_0 = 0$.

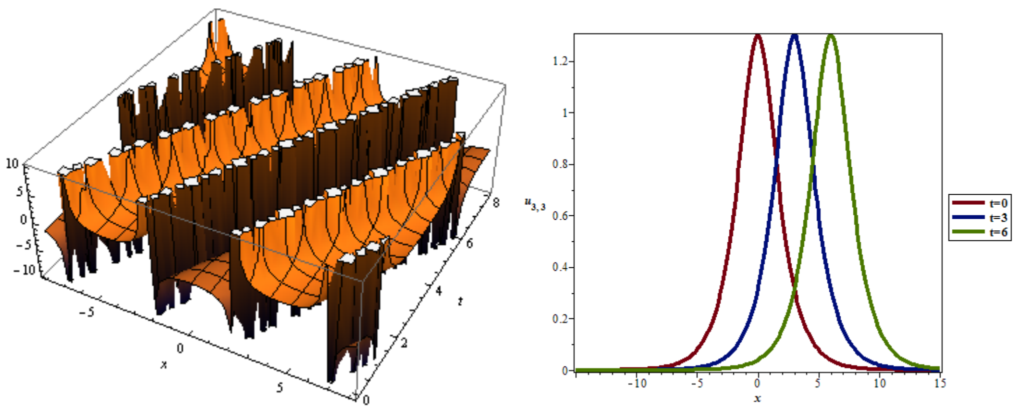

Figure 6: Three-dimensional plot of the exact solution $u_{3,3}(x,t)$ with $k = 1 + \sqrt{2}, c = 1, \alpha = 1, \beta = -1, \zeta_0 = 0$ and its projection.

Solitary and periodic wave solutions are central to the study of nonlinear partial differential equations, as many such equations admit diverse solitary wave forms. Among these, solitons—localized solutions characteristic of weakly nonlinear systems—are particularly important owing to their capacity to model a broad spectrum of physical phenomena. In contrast, the existence of periodic traveling wave solutions generally depends on specific parameters in the governing equations, which determine both the amplitude and the propagation speed of the wave.

Graphical representations of solitary waves corresponding to Eqs. 18, 20, 31, 33, 42, and 44 have been generated by assigning suitable values to the free parameters, thereby facilitating a clearer understanding of the underlying physical phenomena. These graphs provide an effective visualization of the solutions and offer insight into the behavior of the equations. The corresponding plots are shown in Figs. 1–6.

5. Conclusions

The Functional Variable Method was effectively utilized to obtain exact traveling wave solutions for three distinct models of nonlinear dispersive equations. This method is more efficient and straightforward compared to traditional approaches, making it particularly suitable for computer implementation. Complex algebraic calculations were efficiently handled using the symbolic computation software Mathematica. Consequently, this approach can be further extended to address various nonlinear problems in soliton theory and related fields.

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