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Hilbert Graceful Labeling on the Eight Sprocket Graph

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Abstract

Let G be a simple, finite, connected, undirected, non-trivial graph with p vertices and q edges. Let $V(G)$ be the vertex set and $E(G)$ be the edge set of G. The n^{th} Hilbert number is denoted by H_n and is defined by $H_n = 4(n-1)+1, \quad$ where $n \geq 1$. A *Hilbert graceful labeling* is an injective function $H : V(G) \to \{x : x = 4(i-1)+1, 1 \leq i \leq 2q\}$ which induces a bijective function $H^* : E(G) \to \{1,2,3,4,\ldots,q\}$ defined by

$$H^*(uv) = \frac{1}{4}\,|H(u) - H(v)|, \quad \forall\, uv \in E(G),\, u,v \in V(G).$$

A graph that admits a Hilbert graceful labeling is called a *Hilbert graceful graph*. This paper focuses on the Eight Sprocket Graph SC_n and demonstrates its Hilbert gracefulness. It also investigates related graph families formed from copies of SC_n, proving that the path union, cycle, and star of the Eight Sprocket Graph are all Hilbert graceful.

Keywords: Hilbert numbers; hilbert graceful labelling; eight sprocket graph.
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1. Introduction

Graph labeling is a well-established area of graph theory with numerous applications in computer science, communication networks, coding theory, and combinatorial design. A graph labeling assigns numbers to the vertices or edges (or both) of a graph according to specific rules. Over the years, several types of labelings have been introduced, including graceful, harmonious, cordial, and sequential labelings, each possessing distinct structural properties and theoretical significance. Among them, graceful and cordial labelings have received particular attention for their elegant combinatorial properties and mathematical appeal.

In recent developments, J. C. Kanani and V. J. Kaneria [5] introduced a novel graph structure known as the Eight Sprocket Graph and proved that it is cordial. They further extended their study to show that

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the path union, cycle, and star of the Eight Sprocket Graph are also cordial. In addition, they examined the gracefulness of the Eight Sprocket Graph and its variants, contributing significantly to the study of labeling in newly constructed graph classes [6].

In our recent work, we introduced a new labeling scheme called *Hilbert Graceful Labeling*. This labeling is inspired by the concept of graceful labeling but defined under a modified numerical structure based on Hilbert numbers. We also proved that certain classes of complete bipartite graphs admit Hilbert graceful labeling [11], thus extending classical results in the field. Furthermore, in this paper, we prove that the Eight Sprocket Graph, as well as its path union, cycle, and star forms, are all Hilbert graceful.

Definition 1.1 (Path Union and Cycle of Graphs)**.** *Let G be a graph and let $G_1, G_2, \ldots, G_n$ ($n \geq 2$) be n copies of the graph G. Let $v \in V(G)$. The graph obtained by joining vertex v of G_i with the same vertex of G_{i+1} by an edge, for all $i = 1, 2, 3, \ldots, n-1$, is called a* path union *of n copies of G. If the same vertex v of G_n is also joined by an edge with v of G_1, then such a graph is known as the* cycle graph *of n copies of G. These are denoted by $P(n : G)$ and $C(n : G)$, respectively [12].*

Definition 1.2 (Star of a Graph)**.** *Let G be a graph on n vertices. The graph obtained by replacing each vertex of the star $K_{1,n}$ with a copy of G is called the* star of *G and is denoted by G^* [12].*

Definition 1.3 (Eight Sprocket Graph)**.** *An* Eight Sprocket Graph *is defined as the union of eight copies of C_{4n}. Let $V_{(i,j)}$ ($\forall i = 1, 2, \ldots, 8$; $\forall j = 1, 2, \ldots, 4n$) be the vertices of the i^{th} copy of C_{4n}. We combine the vertices as follows:*

$$V_{(1,4n)} \text{ and } V_{(2,1)}, \quad V_{(2,4n)} \text{ and } V_{(3,1)}, \quad V_{(1,4n)} \text{ and } V_{(4,1)}, \quad V_{(1,4n)} \text{ and } V_{(5,1)},$$

$$V_{(1,4n)} \text{ and } V_{(6,1)}, \quad V_{(1,4n)} \text{ and } V_{(7,1)}, \quad V_{(1,4n)} \text{ and } V_{(8,1)}, \quad V_{(1,4n)} \text{ and } V_{(1,1)}$$

by a single vertex, where $n \in \mathbb{N} - \{1\}$.

The resulting graph becomes sprocket-shaped, consisting of eight sprockets, hence the name Eight Sprocket Graph. *It is denoted by SC_n of size n, where $n \in \mathbb{N} - \{1\}$. The order and size of the Eight Sprocket Graph are given by:*

$$|V(SC_n)| = 16n - 8, \qquad |E(SC_n)| = 16n.$$

The coordinates of the Eight Sprocket Graph, its path union, cycle, and star forms are already well-defined by J. C. Kanani and V. J. Kaneria [5].

2. Main Results

Theorem 2.1. *An Eight Sprocket Graph SC_n is a Hilbert graceful graph, where $n \geq 2$.*

Proof. Consider eight copies of the cycle C_{2n}, where $n \geq 2$. The vertices of the k^{th} copy of the cycle C_{2n} are denoted by $x_{(k,i)}$, where $1 \leq k \leq 8$ and $1 \leq i \leq 2n$. Now, identify (or fuse) the vertex $x_{(k,2n)}$ of the k^{th} copy of the cycle C_{2n} with the vertex $x_{(k+1,1)}$ of the $(k+1)^{\text{th}}$ copy of C_{2n}, for $1 \leq k \leq 7$. Similarly, the vertex $x_{(8,2n)}$ of the eighth cycle is identified with the vertex $x_{(1,1)}$ of the first cycle. The resulting graph is denoted by G and is called the *Eight Sprocket Graph*, represented as SC_n. The vertex and edge sets of G are defined as:

$$V(G) = \{x_{(k,i)} : 1 \leq k \leq 8,\ 1 \leq i \leq 2n\},$$
$$E(G) = \{x_{(k,i)}x_{(k,i+1)} : 1 \leq k \leq 8,\ 1 \leq i \leq 2n-1\} \cup \{x_{(k,1)}x_{(k+1,1)} : 1 \leq k \leq 7\} \cup \{x_{(8,1)}x_{(1,1)}\}.$$

Hence, the total number of vertices and edges in G are:

$$|V(G)| = 16n - 8, \qquad |E(G)| = 16n.$$

We define a vertex labeling function

$$f : V(G) \to \{x : x = 4(i-1)+1,\ 1 \leq i \leq 2q\}$$

as follows:

$$f(x_{(1,i)}) = \begin{cases} 64n + 3 - 2i, & \text{for } i = 1, 3, 5, \ldots, 2n-1, \\ 2i - 3, & \text{for } i = 2, 4, 6, \ldots, n, \\ 2i + 1, & \text{for } i = n+2, n+4, \ldots, 2n. \end{cases}$$

$$f(x_{(2,i)}) = \begin{cases} 4n - 1 + 2i, & \text{for } i = 1, 3, 5, \ldots, 2n-1, \\ 60n + 5 - 2i, & \text{for } i = 2, 4, 6, \ldots, n, \\ 60n + 1 - 2i, & \text{for } i = n+2, n+4, \ldots, 2n. \end{cases}$$

$$f(x_{(3,i)}) = \begin{cases} 56n + 3 - 2i, & \text{for } i = 1, 3, 5, \ldots, 2n-1, \\ 8n - 3 + 2i, & \text{for } i = 2, 4, 6, \ldots, n, \\ 8n + 1 + 2i, & \text{for } i = n+2, n+4, \ldots, 2n. \end{cases}$$

$$f(x_{(4,i)}) = \begin{cases} 12n - 1 + 2i, & \text{for } i = 1, 3, 5, \ldots, n+1, \\ 12n + 3 + 2i, & \text{for } i = n+3, n+5, \ldots, 2n-1, \\ 52n + 1 - 2i, & \text{for } i = n+2, n+4, \ldots, 2n, \\ 52n + 5 - 2i, & \text{for } i = 2, 4, 6, \ldots, n. \end{cases}$$

$$f(x_{(5,i)}) = \begin{cases} 48n + 3 - 2i, & \text{for } i = 1, 3, 5, \ldots, 2n-1, \\ 16n + 5 + 2i, & \text{for } i = 2, 4, 6, \ldots, n, \\ 16n + 9 + 2i, & \text{for } i = n+2, n+4, \ldots, 2n. \end{cases}$$

$$f(x_{(6,i)}) = \begin{cases} 20n+7+2i, & \text{for } i = 1,3,5,\dots,2n-1, \\ 44n+5-2i, & \text{for } i = 2,4,6,\dots,n, \\ 44n+1-2i, & \text{for } i = n+2,n+4,\dots,2n. \end{cases}$$

$$f(x_{(7,i)}) = \begin{cases} 40n+3-2i, & \text{for } i = 1,3,5,\dots,2n-1, \\ 24n+5+2i, & \text{for } i = 2,4,6,\dots,n, \\ 24n+9+2i, & \text{for } i = n+2,n+4,\dots,2n. \end{cases}$$

$$f(x_{(8,i)}) = \begin{cases} 28n+7+2i, & \text{for } i = 1,3,5,\dots,2n-1, \\ 36n+5-2i, & \text{for } i = 2,4,6,\dots,2n-2. \end{cases}$$

The above labeling f is injective (one-to-one). The induced function $f^* : E(G) \to \{1,2,3,\dots,q\}$ defined by

$$f^*(uv) = \frac{1}{4}|f(u) - f(v)|$$

is bijective. Hence, f is a Hilbert graceful labeling, and therefore, the Eight Sprocket Graph SC_n is a Hilbert graceful graph. □

Example 2.2. *The Hilbert graceful graph of the Eight Sprocket Graph SC_8 is shown in Figure 1.*

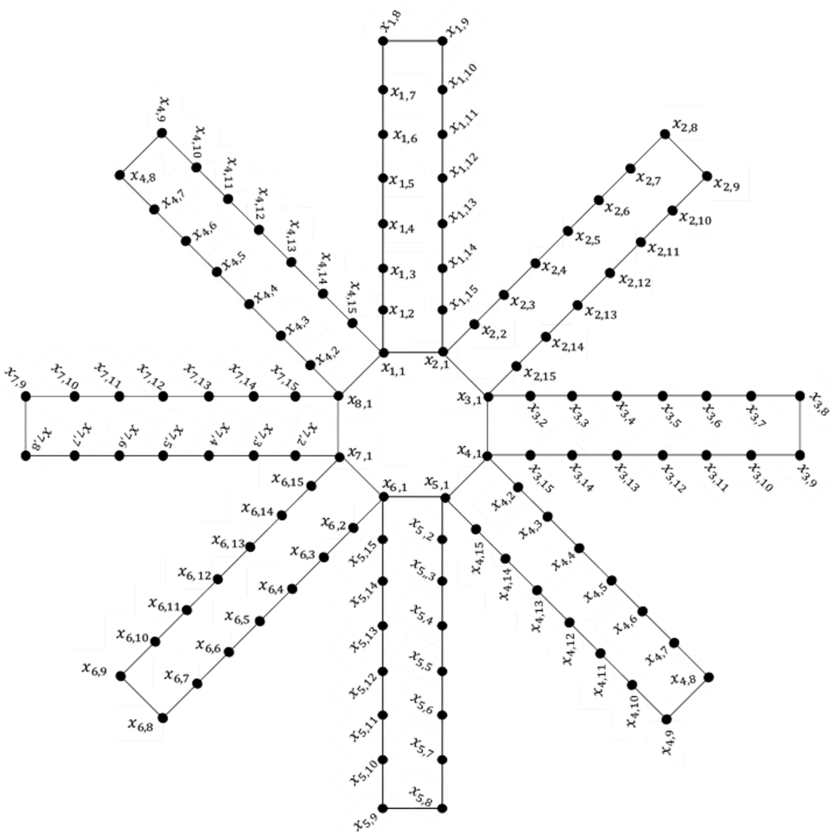

Figure 1: The structure of the Eight Sprocket Graph SC_8.

Value of i	$f(x_{(1,i)})$	$f(x_{(2,i)})$	$f(x_{(3,i)})$	$f(x_{(4,i)})$	$f(x_{(5,i)})$	$f(x_{(6,i)})$	$f(x_{(7,i)})$	$f(x_{(8,i)})$
1	513	33	449	97	385	169	321	233
2	1	481	65	417	137	353	201	289
3	509	37	445	101	381	173	317	237
4	5	477	69	413	141	349	205	285
5	505	41	441	105	377	177	313	241
6	9	473	73	409	145	345	209	281
7	501	45	437	109	373	181	309	245
8	13	469	77	405	149	341	213	277
9	497	49	433	113	369	185	305	249
10	21	461	85	397	157	333	221	273
11	493	53	429	121	365	189	301	253
12	25	457	89	393	161	329	225	269
13	489	57	425	125	361	193	297	257
14	29	453	93	389	165	325	229	265
15	485	61	421	129	357	197	293	261

Table 1: Vertex labeling of an Eight Sprocket Graph SC_8

Value of i	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$f^*(x_{(1,i)}, x_{(1,i+1)})$	128	127	126	125	124	123	122	121	119	118	117	116	115	114
$f^*(x_{(2,i)}, x_{(2,i+1)})$	112	111	110	109	108	107	106	105	103	102	101	100	99	98
$f^*(x_{(3,i)}, x_{(3,i+1)})$	96	95	94	93	92	91	90	89	87	86	85	84	83	82
$f^*(x_{(4,i)}, x_{(4,i+1)})$	80	79	78	77	76	75	74	73	71	69	68	67	66	65
$f^*(x_{(5,i)}, x_{(5,i+1)})$	62	61	60	59	58	57	56	55	53	52	51	50	49	48
$f^*(x_{(6,i)}, x_{(6,i+1)})$	46	45	44	43	42	41	40	39	37	36	35	34	33	32
$f^*(x_{(7,i)}, x_{(7,i+1)})$	30	29	28	27	26	25	24	23	21	20	19	18	17	16
$f^*(x_{(8,i)}, x_{(8,i+1)})$	14	13	12	11	10	9	8	7	6	5	4	3	2	1
$f^*(x_{(i,15)}, x_{(i+1,1)})$	113	97	81	64	47	31	15							
$f^*(x_{(i,15)}, x_{(1,1)})$								63						
$f^*(x_{(i,1)}, x_{(i+1,1)})$	120	104	88	72	54	38	22							
$f^*(x_{(i,1)}, x_{(1,1)})$								70						

Table 2: Edge labels of an Eight Sprocket Graph SC_8

From Tables 2 and 2, we observe that the function

$$f : V(G) \to \{x : x = 4(i-1)+1,\ 1 \le i \le 2q\}$$

is injective (one-to-one), and the induced function

$$f^* : E(G) \to \{1, 2, 3, \ldots, q\}, \quad f^*(uv) = \frac{1}{4}|f(u) - f(v)|$$

is bijective. Hence, f is a Hilbert graceful labeling, and therefore, the Eight Sprocket Graph SC_8 is a Hilbert graceful graph.

Theorem 2.3. *The path union of finitely many copies of the Eight Sprocket Graph SC_n is a Hilbert graceful graph, where $n \ge 2$.*

Proof. Consider r copies of the Eight Sprocket Graphs. Let $\{x_{(k,i,j)} : 1 \le i \le 8, 1 \le j \le 2n\}$ be the vertex set of the k^{th} copy of the Eight Sprocket Graph. The graph $P(r : SC_n)$ is obtained by joining the vertices

$$\{x_{(k,1,n+1)}x_{(k+1,1,n+1)} : 1 \leq k \leq r-1\}.$$

Let $G = P(r : SC_n)$ denote the path union of r copies of the Eight Sprocket Graph SC_n, where $n \geq 2$. The vertex set and edge set of G are defined as:

$$V(G) = \{x_{(k,i,j)} : 1 \leq k \leq r,\ 1 \leq i \leq 8,\ 1 \leq j \leq 2n\},$$
$$E(G) = \{x_{(k,i,j)}x_{(k,i,j+1)} : 1 \leq k \leq r,\ 1 \leq i \leq 8,\ 1 \leq j \leq 2n-1\} \cup \{x_{(k,i,1)}x_{(k,i+1,1)} : 1 \leq k \leq r,\ 1 \leq i \leq 7\}$$
$$\cup \{x_{(k,8,1)}x_{(k,1,1)} : 1 \leq k \leq r\} \cup \{x_{(k,1,n+1)}x_{(k+1,1,n+1)} : 1 \leq k \leq r-1\}.$$

Hence, the total number of vertices and edges in G are given by:

$$|V(G)| = r(16n-8), \qquad |E(G)| = (r-1) + 16rn.$$

Let f be the Hilbert graceful labeling of SC_n as defined in Theorem 2.1. Each copy of SC_n has $16n-8$ vertices and $16n$ edges. To form the path union of r copies, join the vertex $x_{(k,1,n+1)}$ of the k^{th} copy with the vertex $x_{(k+1,1,n+1)}$ of the $(k+1)^{\text{th}}$ copy by a single edge. We define a vertex labeling function

$$g : V(G) \to \{x : x = 4(i-1)+1,\ 1 \leq i \leq 2q\}$$

as follows, for $1 \leq k \leq r$, $1 \leq i \leq 8$, $1 \leq j \leq 2n$:

$$g(x_{(1,i,j)}) = \begin{cases} f(x_{(i,j)}), & \text{if } f(x_{(i,j)}) \leq 32n+5, \\ f(x_{(i,j)}) + 4(r-1)(1+16n), & \text{if } f(x_{(i,j)}) > 32n+5. \end{cases}$$

$$g(x_{(2,i,j)}) = \begin{cases} g(x_{(1,i,j)}) + 4(r-1)(1+16n), & \text{if } g(x_{(1,i,j)}) \leq 32n+5, \\ g(x_{(1,i,j)}) - 4(r-1)(1+16n), & \text{if } g(x_{(1,i,j)}) > 32n+5. \end{cases}$$

$$g(x_{(k,i,j)}) = \begin{cases} g(x_{(k-2,i,j)}) + 4(16n+1), & \text{if } g(x_{(k-2,i,j)}) \leq 32n+5, \\ g(x_{(k-2,i,j)}) - 4(16n+1), & \text{if } g(x_{(k-2,i,j)}) > 32n+5. \end{cases}$$

The above labeling pattern ensures that g is a one-to-one (injective) mapping. The induced function $g^* : E(G) \to \{1, 2, 3, \ldots, q\}$ defined by

$$g^*(uv) = \frac{1}{4}|g(u) - g(v)|$$

is bijective. Hence, g is a Hilbert graceful labeling, and therefore, the path union $P(r : SC_n)$ of Eight

Sprocket Graphs is a Hilbert graceful graph. □

Theorem 2.4. *The cycle of r copies of the Eight Sprocket Graph, denoted by $C(r : SC_n)$, is a Hilbert graceful graph, where $n \geq 2$ and $r \equiv 0, 3 \pmod 4$.*

Proof. Consider r copies of the Eight Sprocket Graphs. Let $\{x_{(k,i,j)} : 1 \leq i \leq 8, 1 \leq j \leq 2n\}$ be the vertex set of the k^{th} copy of the Eight Sprocket Graph. The graph $C(r : SC_n)$ is obtained by joining the vertices

$$\{x_{(k,1,n+1)}x_{(k+1,1,n+1)} : 1 \leq k \leq r-1\} \cup \{x_{(r,1,n+1)}x_{(1,1,n+1)}\}.$$

Let $G = C(r : SC_n)$ denote the cycle of r copies of the Eight Sprocket Graph. The vertex and edge sets of G are given by:

$$\begin{aligned} V(G) &= \{x_{(k,i,j)} : 1 \leq k \leq r,\, 1 \leq i \leq 8,\, 1 \leq j \leq 2n\}, \\ E(G) &= \{x_{(k,i,j)}x_{(k,i,j+1)} : 1 \leq k \leq r,\, 1 \leq i \leq 8,\, 1 \leq j \leq 2n-1\} \\ &\quad \cup \{x_{(k,i,1)}x_{(k,i+1,1)} : 1 \leq k \leq r,\, 1 \leq i \leq 7\} \cup \{x_{(k,8,1)}x_{(k,1,1)} : 1 \leq k \leq r\} \\ &\quad \cup \{x_{(k,1,n+1)}x_{(k+1,1,n+1)} : 1 \leq k \leq r-1\} \cup \{x_{(r,1,n+1)}x_{(1,1,n+1)}\}. \end{aligned}$$

Hence, the total number of vertices and edges in G are:

$$|V(G)| = r(16n-8), \qquad |E(G)| = r + 16nr.$$

Let f be the Hilbert graceful labeling of SC_n as defined in Theorem 2.1. We define a vertex labeling function

$$g : V(G) \to \{x : x = 4(i-1)+1,\, 1 \leq i \leq 2q\}$$

as follows. For $1 \leq i \leq 8$ and $1 \leq j \leq 2n$,

$$g(x_{(1,i,j)}) = \begin{cases} f(x_{(i,j)}), & \text{if } f(x_{(i,j)}) \leq 32n+5, \\ f(x_{(i,j)}) + 64n(r-1) + 4r, & \text{if } f(x_{(i,j)}) > 32n+5. \end{cases}$$

$$g(x_{(2,i,j)}) = \begin{cases} g(x_{(1,i,j)}) + 64n(r-1) + 4r, & \text{if } g(x_{(1,i,j)}) \leq 32n+5, \\ g(x_{(1,i,j)}) - 64n(r-1) + 4r, & \text{if } g(x_{(1,i,j)}) > 32n+5. \end{cases}$$

For $3 \leq k \leq \lceil r/2 \rceil$ and $1 \leq i \leq 8$, $1 \leq j \leq 2n$,

$$g(x_{(\lceil k/2 \rceil+1,i,j)}) = \begin{cases} g(x_{(\lceil k/2 \rceil-1,i,j)}) + 64n + 8, & \text{if } g(x_{(\lceil k/2 \rceil-1,i,j)}) \leq 32n+5, \\ g(x_{(\lceil k/2 \rceil-1,i,j)}) - 64n - 4, & \text{if } g(x_{(\lceil k/2 \rceil-1,i,j)}) > 32n+5. \end{cases}$$

$$g(x_{(\lceil k/2\rceil+2,i,j)}) = \begin{cases} g(x_{(\lceil k/2\rceil,i,j)}) + 64n + 8, & \text{if } g(x_{(\lceil k/2\rceil,i,j)}) \leq 64n + 1, \\ g(x_{(\lceil k/2\rceil,i,j)}) - 64n - 4, & \text{if } g(x_{(\lceil k/2\rceil,i,j)}) > 64n + 1. \end{cases}$$

For $k = \lceil r/2\rceil + 3, \lceil r/2\rceil + 5, \lceil r/2\rceil + 7, \ldots, r$,

$$g(x_{(k,i,j)}) = \begin{cases} g(x_{(k-2,i,j)}) + 64n + 8, & \text{if } g(x_{(k-2,i,j)}) \leq 32n + 5, \\ g(x_{(k-2,i,j)}) - 64n - 4, & \text{if } g(x_{(k-2,i,j)}) > 32n + 5. \end{cases}$$

For $k = \lceil r/2\rceil + 4, \lceil r/2\rceil + 6, \lceil r/2\rceil + 8, \ldots, r$,

$$g(x_{(k,i,j)}) = \begin{cases} g(x_{(k-2,i,j)}) + 64n + 8, & \text{if } g(x_{(k-2,i,j)}) \leq 64n + 1, \\ g(x_{(k-2,i,j)}) - 64n - 4, & \text{if } g(x_{(k-2,i,j)}) > 64n + 1. \end{cases}$$

The above labeling pattern ensures that g is injective (one-to-one). The induced function $g^* : E(G) \to \{1, 2, 3, \ldots, q\}$ defined by

$$g^*(uv) = \frac{1}{4}\left|g(u) - g(v)\right|$$

is bijective. Hence, g is a Hilbert graceful labeling, and therefore, the cycle $C(r : SC_n)$ of the Eight Sprocket Graph is a Hilbert graceful graph. □

Theorem 2.5. *The star of the Eight Sprocket Graph, denoted by $(SC_n)^*$, is Hilbert graceful, where $n \geq 2$.*

Proof. Consider $(16n + 1) + 1$ copies of the Eight Sprocket Graph SC_n, where $n \geq 2$. We denote the central copy of $(SC_n)^*$ by $(SC_n)^0$, and the other copies by $(SC_n)^k$, where $1 \leq k \leq 16n - 8$. Let $\{x_{(k,i,j)} : 1 \leq i \leq 8, 1 \leq j \leq 2n\}$ be the vertex set of the k^{th} copy of the Eight Sprocket Graph, and $\{x_{(0,i,j)} : 1 \leq i \leq 8, 1 \leq j \leq 2n\}$ be the vertex set of the central copy. The graph $(SC_n)^*$ is obtained by joining the vertices

$$\{x_{(0,i,j)}x_{(k,i,j)} : 1 \leq k \leq 16n - 8, 1 \leq i \leq 8, 1 \leq j \leq 2n\}.$$

Let G denote $(SC_n)^*$. The vertex and edge sets of G are defined as:

$$\begin{aligned} V(G) &= \{x_{(0,i,j)} : 1 \leq i \leq 8, 1 \leq j \leq 2n\} \cup \{x_{(k,i,j)} : 1 \leq k \leq 16n - 8, 1 \leq i \leq 8, 1 \leq j \leq 2n\}, \\ E(G) &= \{x_{(0,i,j)}x_{(0,i,j+1)} : 1 \leq i \leq 8, 1 \leq j \leq 2n - 1\} \\ &\quad \cup \{x_{(0,i,1)}x_{(0,i+1,1)} : 1 \leq i \leq 7\} \cup \{x_{(0,8,1)}x_{(0,1,1)}\} \\ &\quad \cup \{x_{(k,i,j)}x_{(k,i,j+1)} : 1 \leq k \leq 16n - 8, 1 \leq i \leq 8, 1 \leq j \leq 2n - 1\} \\ &\quad \cup \{x_{(k,i,1)}x_{(k,i+1,1)} : 1 \leq k \leq 16n - 8, 1 \leq i \leq 7\} \\ &\quad \cup \{x_{(k,8,1)}x_{(k,1,1)} : 1 \leq k \leq 16n - 8\} \\ &\quad \cup \{x_{(0,i,j)}x_{(k,i,j)} : 1 \leq k \leq 16n - 8, 1 \leq i \leq 8, 1 \leq j \leq 2n\}. \end{aligned}$$

For each $1 \leq k \leq 16n-8$ and $1 \leq i \leq 7$, we fuse the vertex $x_{(k,i,2n)}$ with $x_{(k,i+1,1)}$, and $x_{(k,8,2n)}$ with $x_{(k,1,1)}$, by a single vertex $x_{(k,i+1,1)}$ and $x_{(k,1,1)}$, respectively. Thus, the total number of vertices and edges in G are:

$$|V(G)| = (16n-7)(16n-8), \qquad |E(G)| = (16n)(16n-7) + (16n-8).$$

Let f be the Hilbert graceful labeling of SC_n as defined in Theorem 2.1. We now define a vertex labeling function

$$g : V(G) \to \{x : x = 4(i-1)+1,\ 1 \leq i \leq 2q\}$$

as follows. For $1 \leq i \leq 8$, $1 \leq j \leq 2n$, and $k = 0$:

$$g(x_{(0,i,j)}) = \begin{cases} f(x_{(i,j)}), & \text{if } f(x_{(i,j)}) \leq 32n+5, \\ f(x_{(i,j)}) + 32(2n-1)(16n+1), & \text{if } f(x_{(i,j)}) > 32n+5. \end{cases}$$

For $1 \leq i \leq 8$, $1 \leq j \leq 2n$, and $k = 1$:

$$g(x_{(1,i,j)}) = \begin{cases} g(x_{(0,i,j)}) + 4(16n-8)(16n+1), & \text{if } g(x_{(0,i,j)}) \leq 32n-15, \\ g(x_{(0,i,j)}) - 4(16n-8)(16n+1), & \text{if } g(x_{(0,i,j)}) > 32n-15. \end{cases}$$

For $2 \leq k \leq 16n-8$, $1 \leq i \leq 8$, $1 \leq j \leq 2n$:

$$g(x_{(k,i,j)}) = \begin{cases} g(x_{(k-2,i,j)}) + 4(16n+1), & \text{if } g(x_{(k-2,i,j)}) \leq 4[n(16n-8)]+1, \\ g(x_{(k-2,i,j)}) - 4(16n+1), & \text{if } g(x_{(k-2,i,j)}) > 4[n(16n-8)]+1. \end{cases}$$

The above labeling pattern ensures that g is injective (one-to-one). The induced function $g^* : E(G) \to \{1, 2, 3, \ldots, q\}$ defined by

$$g^*(uv) = \frac{1}{4}\,|g(u) - g(v)|$$

is bijective. Hence, g is a Hilbert graceful labeling, and therefore, the star of the Eight Sprocket Graph $(SC_n)^*$ is a Hilbert graceful graph. □

3. Conclusion

In this paper, we have investigated the Hilbert graceful labeling of the Eight Sprocket Graph and its associated structures. Building upon the concept of Hilbert graceful labeling, we have proved that the Eight Sprocket Graph, as well as its path union, cycle, and star variants, admit Hilbert graceful labelings. These results extend the earlier work of Kanani and Kaneria, who established the cordial and graceful properties of the same graph structures. Our findings confirm that the Eight Sprocket Graph and its derived forms not only satisfy cordial and graceful labeling conditions but also exhibit Hilbert graceful behavior, thereby broadening the class of graphs known to possess this property. Future research may focus on exploring Hilbert graceful labelings for other graph families, such as wheel graphs, helm graphs, and other sprocket-type constructions. Additionally, algorithmic methods for generating Hilbert graceful labelings and their potential applications in network design and combinatorial optimization present interesting avenues for further study.

References

[1] F. Harary, *Graph theory*, Narosa Publishing House, (2001).

[2] G. S. Bloom and S. W. Golomb, *Application of numbered undirected graphs*, IEEE Transactions on Computers, 65(1977), 562-570.

[3] I. Cahit, *Cordial graphs: A weaker version of graceful and harmonious graphs*, Ars Combinatoria, 23(1987), 201-207.

[4] J. A. Gallian, *A dynamic survey of graph labeling*, The Electronic Journal of Combinatorics, 22(2019).

[5] J. C. Kanani and V. J. Kaneria, *Cordial labeling for eight sprocket graph*, International Journal of Mathematics and its Applications, 9(1)(2021), 67-72.

[6] J. C. Kanani and V. J. Kaneria, *Graceful labeling for eight sprocket graph*, In Recent Advancements in Graph Theory, CRC Press, (2020), 1-8.

[7] S. K. Vaidya and N. H. Shah, *Prime cordial labeling of some wheel related graphs*, Malaya Journal of Matematik, 4(1)(2013), 148-156.

[8] S. K. Vaidya and P. L. Vihol, *Prime cordial labeling for some cycle related graphs*, International Journal of Open Problems in Computer Mathematics, 3(5)(2010).

[9] S. W. Golomb, *How to number a graph*, In R. C. Read (Ed.), Graph Theory and Computing, Academic Press, (1972), 23-37.

[10] V. J. Kaneria, H. P. Chudasama and P. P. Andharia, *Absolute mean graceful labeling in path union of various graphs*, Mathematical Journal of Interdisciplinary Sciences, 7(2018), 51-56.

[11] J. S. Kumar, *Hilbert graceful labeling on complete multipartite graph*, Asian Research Journal of Mathematics, 20(9)(2024), 132-139.

[12] A. A. Sathakathulla and M. F. Kareem, *Prime cordial labeling for eight sprocket graph*, International Journal of Mathematics Trends and Technology, 67(12)(2021), 41-46.