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On the Stability of the Quadratic Functional Equation $f(2x+y)+f(x+2y) = 4f(x+y)+f(x)+f(y)$ in 2-Banach Space

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Abstract

In this research paper, we investigate the Hyers-Ulam stability of the functional equation $f(2x+y)+f(x+2y)=4f(x+y)+f(x)+f(y)$ in 2-Banach space.

Keywords: Hyers-Ulam stability; 2-Banach space; Quadratic function.

2020 Mathematics Subject Classification: 46K05, 46H05.

1. Introduction

Stability of function for a function from normed space to Banach space has been studied by Hyers [9]. He has proved that for a function $f : X \longrightarrow Y$, a function between normed space X and Banach space Y satisfying

$$\|f(x+y)-f(x)-f(y)\| \leq \delta$$

for each $x, y \in X$ and $\delta > 0$. Then there exists a unique additive function $T : X \longrightarrow Y$ such that $\|f(x)-T(x)\| \leq \delta$ for each $x \in X$. It is a positive answer to a problem raised by Ulam [16] for a functional equation on metric group. Stability of a functional equation for a function from a normed space to a 2-Banach space have been studied by B.M. Patel and A.B. Patel in [3–5]. In fact several authors have studied the problem for different types of functional equations for functions from normed space to Banach space. (see [1,2,10–15]). Our aim is to study the stability of the functional equation

$$f(2x+y)+f(x+2y)=4f(x+y)+f(x)+f(y) \tag{1}$$

introduced by [7], for a function on 2-normed space to 2-Banach space.

Definition 1.1. *Let X and Y be real vector spaces, and let $f : X \longrightarrow Y$ be a function. Then the functional*

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equation

$$f(x+y)+f(x-y)=f(x)+f(y) \tag{2}$$

is said to be quadratic functional equation and every solution of the quadratic equation is said to be a quadratic function.

Theorem 1.2 ([7]). *Let X and Y be real vector spaces. A function $f : X \longrightarrow Y$ satisfies the functional equation* (1) *if and only if function $f : X \longrightarrow Y$ satisfies the functional equation* (2). *Therefore, every solution of functional equation* (1) *is also a quadratic function.*

In the 1960s, S. Gähler introduced the concept of linear 2-normed spaces.

Definition 1.3. *Let X be a linear space over $\mathbb{R}$ with $\dim X > 1$ and let $\|\cdot,\cdot\| : X \times X \longrightarrow \mathbb{R}$ be a function satisfying the following properties:*

1. $\|x,y\| = 0$ *if and only if x and y are linearly dependent,*
2. $\|x,y\| = \|y,x\|$,
3. $\|ax,y\| = |a|\|x,y\|$,
4. $\|x,y+z\| \leq \|x,y\| + \|x,z\|$

for each $x,y,z \in X$ and $a \in \mathbb{R}$. Then the function $\|\cdot,\cdot\|$ is called a 2*-norm on X and $(X,\|\cdot,\cdot\|)$ is called a* 2*-normed space.*

We introduce a basic property of 2-normed spaces as follows. Let $(X,\|\cdot,\cdot\|)$ be a linear 2-normed space, $x \in X$ and $\|x,y\|$ = 0 for each $y \in X$. Suppose $x \neq 0$, since $\dim X > 1$, choose $y \in X$ such that $\{x,y\}$ is linearly independent so we have $\|x,y\| \neq 0$, which is a contradiction. Therefore, we have the following lemma.

Lemma 1.4. *Let $(X,\|\cdot,\cdot\|)$ be a* 2*-normed space. If $x \in X$ and $\|x,y\| = 0$, for each $y \in X$, then $x = 0$.*

Remark 1.5. *Let $(X,\|\cdot,\cdot\|)$ be a* 2*-normed space. Note that the conditions* (2) *and* (4) *imply that $\|x+y,z\| \leq \|x,z\| + \|y,z\|$ for each $x,y,z \in X$. Putting $w = x+y$, we get $\|w,z\| \leq \|x,z\| + \|w-x,z\|$, for each $x,y,z \in X$. So $\|w,z\| - \|x,z\| \leq \|w-x,z\|$, for each $x,z,w \in X$. Replacing w by x and x by w in the above inequality, we get $\|x,z\| - \|w,z\| \leq \|x-w,z\|$ for each $x,z,w \in X$. Thus, we have*

$$|\|x,z\| - \|y,z\|| \leq \|x-y,z\| \tag{3}$$

for each $x,y,z \in X$. Hence the function $x \rightarrow \|x,y\|$ is continuous from X into $\mathbb{R}$, for each fixed $y \in X$.

Let $(X,\|\cdot,\cdot\|)$ be a 2-normed space. For $x,z \in X$, let $p_z(x) = \|x,z\|$, $x \in X$. Then for each $z \in X$, p_z is a real-valued function on X such that $p_z(x) = \|x,z\| \geq 0$, $p_z(\alpha x) = |\alpha|\|x,z\| = |\alpha|p_z(x)$ and

$p_z(x+y) = \|x+y,z\| = \|z,x+y\| \le \|z,x\| + \|z,y\| = \|x,z\| + \|y,z\| = p_z(x) + p_z(y)$, for each $\alpha \in \mathbb{R}$ and all $x, y \in X$. Thus p_z is a a semi-norm for each $z \in X$.

For $x \in X$, let $\|x,z\| = 0$, for each $z \in X$. By Lemma 1.4, $x = 0$. Thus for $0 \neq x \in X$, there is $z \in X$ such that $p_z(x) = \|x,z\| \neq 0$. Hence the family $\{p_z(x) : z \in X\}$ is a separating family of semi-norms. Let $x_0 \in X$, for $\varepsilon > 0$, $z \in X$, let $U_{z,\varepsilon}(x_0) := \{x \in X : p_z(x - x_0) < \varepsilon\} = \{x \in X : \|x - x_0, z\| < \varepsilon\}$. Let $S(x_0) := \{U_{z,\varepsilon}(x_0) : \varepsilon > 0, z \in X\}$ and $\beta(x_0) := \{\cap\mathcal{F} : \mathcal{F}$ is a finite subcollection of $S(x_0)\}$. Define a topology τ on X by saying that a set U is open if for every $x \in U$, there is some $N \in \beta(x)$ such that $N \subset U$. That is, τ is the topology on X that has subbase $\{U_{z,\varepsilon}(x_0) : \varepsilon > 0, x_0 \in X, z \in X\}$. The topology τ on X makes X a topological vector space. Since for $x \in X$ collection $\beta(x)$ is a local base whose members are convex, X is locally convex. In the 1960s, S. Gähler and A. White introduced the concept of 2-Banach spaces.

Definition 1.6. *A sequence $\{x_n\}$ in a 2-normed space X is called a 2-Cauchy sequence if $\lim_{m,n\to\infty} \|x_n - x_m, x\| = 0$ for each $x \in X$.*

Definition 1.7. *A sequence $\{x_n\}$ in a 2-normed space X is called a 2-convergent sequence if there is an $x \in X$ such that $\lim_{n\to\infty} \|x_n - x, y\| = 0$ for each $y \in X$. If $\{x_n\}$ converges to x, we write $\lim_{n\to\infty} x_n = x$.*

Definition 1.8. *We say that a 2-normed space $(X, \|\cdot,\cdot\|)$ is a 2-Banach space if every 2-Cauchy sequence in X is 2-convergent in X.*

Following shows that $\|\cdot,\cdot\|$ is continuous in each component.

Lemma 1.9. *For a convergent sequence $\{x_n\}$ in a 2-normed space X, $\lim_{n\to\infty} \|x_n, y\| = \|\lim_{n\to\infty} x_n, y\|$ for each $y \in X$.*

Proof. Since $\{x_n\}$ is a 2-convergent sequence in the 2-normed space X, there is an $x \in X$ such that $\lim_{n\to\infty} \|x_n - x, y\| = 0$ for each $y \in X$. By (3), we have

$$\lim_{n\to\infty} |\|x_n, y\| - \|x, y\|| \le \lim_{n\to\infty} \|x_n - x, y\| = 0$$

for each $y \in X$. Hence

$$\lim_{n\to\infty} \|x_n, y\| = \|x, y\| = \|\lim_{n\to\infty} x_n, y\|$$

for each $y \in X$. □

2. Hyers-Ulam Stability of a Functional Equation for Function $f : (X, \|\cdot,\cdot\|) \longrightarrow (X, \|\cdot,\cdot\|)$

Throughout this section, consider X a real normed linear space. We also consider that there is a 2-norm on X which makes $(X, \|\cdot,\cdot\|)$ a 2-Banach space. For a function $f : (X, \|\cdot,\cdot\|) \longrightarrow (X, \|\cdot,\cdot\|)$, define $D_f : X \times X \longrightarrow X$ by

$$D_f(x,y) = f(2x+y) + f(x+2y) - 4f(x+y) - f(x) - f(y)$$

for each $x, y, \in X$.

Theorem 2.1. *Let $\varepsilon \geq 0, 0 < p < 2$. Assume that the function $f : X \longrightarrow X$ satisfies*

$$\|D_f(x,y), z\| \leq \varepsilon\big[\|x,z\|^p + \|y,z\|^p\big] \tag{4}$$

for each $x, y, z \in X$. Then there exists a unique quadratic function $Q : X \longrightarrow X$ satisfying (1) and

$$\|f(x) - Q(x), z\| \leq \frac{3\,\varepsilon\,\|x,z\|^p}{9 - 3^p} \tag{5}$$

for each $x, z \in X$.

Proof. Letting $x = y = 0$ in (4), we have $\|4f(0), z\| = 0$ for each $z \in X$, so we have $f(0) = 0$. Putting $x = y$ in (4), we get

$$\|2f(3x) - 4f(2x) - 2f(x), z\| \leq 2\varepsilon\|x,z\|^p$$

for each $x, z \in X$. Therefore

$$\|f(3x) - 2f(2x) - f(x), z\| \leq \varepsilon\|x,z\|^p \tag{6}$$

for each $x, z \in X$. Putting $y = 0$ in (4), we get

$$\|f(2x) - 4f(x), z\| \leq \varepsilon\|x,z\|^p \tag{7}$$

for each $x, z \in X$. Multiplying (7) by 2, we get

$$\|2f(2x) - 8f(x), z\| \leq 2\varepsilon\|x,z\|^p \tag{8}$$

for each $x, z \in X$. By adding (6) and (8), we get

$$\|f(3x) - 9f(x), z\| \leq 3\varepsilon\|x,z\|^p \tag{9}$$

for each $x, z \in X$. Therefore

$$\left\|\frac{f(3x)}{9} - f(x), z\right\| \leq \frac{\varepsilon\|x,z\|^p}{3} \tag{10}$$

for each $x, z \in X$. Replacing x by $3x$ in (10), we get

$$\left\| \frac{f(9x)}{9} - f(3x), z \right\| \le \frac{\varepsilon\, 3^p \|x\|^p}{3} \tag{11}$$

for each $x, z \in X$. By (10) and (11), we get

$$\begin{aligned}\left\| \frac{f(9x)}{9^2} - f(x), z \right\| &\le \left\| \frac{f(9x)}{9^2} - \frac{f(3x)}{9}, z \right\| + \left\| \frac{f(3x)}{9} - f(x), z \right\| \\ &\le \frac{1}{9} \frac{\varepsilon\, 3^p \|x, z\|^p}{3} + \frac{\varepsilon \|x, z\|^p}{3} \\ &= \frac{\varepsilon}{3}\left[1 + \frac{3^p}{9}\right] \|x, z\|^p\end{aligned}$$

for each $x, z \in X$. By using induction on n, we have

$$\begin{aligned}\left\| f(x) - \frac{1}{9^n} f(3^n x), z \right\| &\le \frac{\varepsilon \|x, z\|^p}{3} \sum_{j=0}^{n-1} \frac{3^{pj}}{9^j} \\ &= \frac{\varepsilon \|x, z\|^p}{3} \sum_{j=0}^{n-1} 3^{(p-2)j} \\ &= \frac{\varepsilon \|x, z\|^p}{3} \left[\frac{1 - 3^{(p-2)n}}{1 - 3^{p-2}}\right]\end{aligned} \tag{12}$$

for each $x, z \in X$. For $m, n \in \mathbb{N}$, we get

$$\begin{aligned}\left\| \frac{1}{9^m} f(3^m x) - \frac{1}{9^n} f(3^n x), z \right\| &= \left\| \frac{1}{9^{m+n-n}} f(3^{m+n-n} x) - \frac{1}{9^n} f(3^n x), z \right\| \\ &= \frac{1}{9^n} \left\| \frac{1}{9^{m-n}} f(3^{m-n} \cdot 3^n x) - f(3^n x), z \right\| \\ &\le \frac{\varepsilon\, \|3^n x, z\|^p}{3 \cdot 9^n} \sum_{j=0}^{m-n-1} 3^{(p-2)j} \\ &= \frac{\varepsilon \|x, z\|^p}{3} 3^{(p-2)n} \sum_{j=0}^{m-n-1} 3^{(p-2)j} \\ &= \frac{\varepsilon \|x, z\|^p}{3} \sum_{j=0}^{m-n-1} 3^{(p-2)(n+j)} \\ &= \frac{\varepsilon \|x, z\|^p}{3} \frac{3^{(p-2)n}\left(1 - 3^{(p-2)(m-n)}\right)}{1 - 3^{p-2}} \\ &\longrightarrow 0 \text{ as } m, n \to \infty\end{aligned}$$

for each $x, z \in X$. Therefore, $\{\frac{1}{9^n} f(3^n x)\}$ is a 2-Cauchy sequence in X, for each $x \in X$. Since X is a 2-Banach space, $\{\frac{1}{9^n} f(3^n x)\}$ 2-converges, for each $x \in X$. Define the function $Q : X \longrightarrow X$ as

$$Q(x) := \lim_{n \to \infty} \frac{1}{9^n} f(3^n x)$$

for each $x \in X$. Now, from (12), we have

$$\lim_{n\to\infty} \left\| f(x) - \frac{1}{9^n} f(3^n x), z \right\| \leq \frac{\varepsilon \|x,z\|^p}{3} \left[\frac{1}{1-3^{p-2}} \right]$$

for each $x, z \in X$. Therefore

$$\|f(x) - Q(x), z\| \leq \frac{3\varepsilon \|x,z\|^p}{9-3^p}$$

for each $x, z \in X$. Next we show that Q satisfies (1).

$$\begin{aligned}
\|D_Q(x,y), z\| &= \lim_{n\to\infty} \frac{1}{9^n} \|D_f(3^n x, 3^n y), z\| \\
&\leq \lim_{n\to\infty} \frac{\varepsilon}{9^n} (\|3^n x, z\|^p + \|3^n y, z\|^p) \\
&= \lim_{n\to\infty} \varepsilon \big[3^{(p-2)n} \|x,z\|^p + 3^{(p-2)n} \|y,z\|^p \big] \\
&= 0
\end{aligned}$$

for each $x, z \in X$. Therefore $\|D_Q(x,y), z\| = 0$, for each $z \in X$. So we get $D_Q(x,y) = 0$. Next we prove the uniqueness of Q. Let Q' be another quadratic function satisfying (1) and (5). Since Q and Q' are quadratic, $Q(3^n x) = 9^n Q(x)$, $Q'(3^n x) = 9^n Q'(x)$, for each $x \in X$.

$$\begin{aligned}
\|Q(x) - Q'(x), z\| &= \frac{1}{9^n} \|Q(3^n x) - Q'(3^n x), z\| \\
&\leq \frac{1}{9^n} [\|Q(3^n x) - f(3^n x), z\| + \|f(3^n x) - Q'(3^n x), z\|] \\
&\leq \frac{1}{9^n} \frac{6\varepsilon \|3^n x, z\|^p}{9-3^p} \\
&= 3^{(p-2)n} \, \frac{3\varepsilon \, \|x,z\|^p}{9-3^p} \\
&\longrightarrow 0 \text{ as } n \to \infty
\end{aligned}$$

for each $x, z \in X$. Therefore $\|Q(x) - Q'(x), z\| = 0$, for each $z \in X$. Therefore $Q(x) = Q'(x)$, for each $x \in X$. This proves the uniqueness of Q. □

Theorem 2.2. *Let $\varepsilon \geq 0, p > 2$. Assume that the function $f : X \longrightarrow X$ satisfies*

$$\|D_f(x,y), z\| \leq \varepsilon \big[\|x,z\|^p + \|y,z\|^p \big] \tag{13}$$

for each $x, y, z \in X$. Then there exists a unique quadratic function $Q : X \longrightarrow X$ satisfying (1) and

$$\|f(x) - Q(x), z\| \leq \frac{3\varepsilon \|x,z\|^p}{3^p - 9} \tag{14}$$

for each $x, z \in X$.

Proof. By (9) of Theorem 2.1, we get

$$\|f(3x) - 9f(x), z\| \le 3\varepsilon \|x, z\|^p \tag{15}$$

for each $x, z \in X$. Replacing x by $\frac{x}{3}$ in (15), we get

$$\left\|f(x) - 9f\left(\frac{x}{3}\right), z\right\| \le 3\varepsilon 3^{-p} \|x, z\|^p \tag{16}$$

for each $x, z \in X$. Replacing x by $\frac{x}{3}$ in (16), we get

$$\left\|f\left(\frac{x}{3}\right) - 9f\left(\frac{x}{9}\right), z\right\| \le 3\varepsilon 3^{-2p} \|x, z\|^p \tag{17}$$

for each $x, z \in X$. Now, by (16) and (17)

$$\begin{aligned}\left\|f(x) - 9^2 f\left(\frac{x}{9}\right), z\right\| &\le \left\|f(x) - 9f\left(\frac{x}{3}\right), z\right\| + \left\|9f\left(\frac{x}{3}\right) - 9^2 f\left(\frac{x}{9}\right), z\right\| \\ &\le 3\varepsilon 3^{-p} \|x, z\|^p + 9 \cdot 3\varepsilon 3^{-p} \|x, z\|^p \\ &= 3\varepsilon \|x, z\|^p \left[3^{-p} + 9 \cdot 3^{-2p}\right]\end{aligned}$$

for each $x, z \in X$. By using induction on $n \in \mathbb{N}$, we get

$$\begin{aligned}\left\|f(x) - 9^n f\left(\frac{x}{3^n}\right), z\right\| &\le 3\varepsilon \|x, z\|^p \sum_{j=0}^{n-1} 3^{-p(j+1)} \cdot 9^j \\ &= 3\varepsilon \|x, z\|^p \sum_{j=0}^{n-1} 3^{(-p+2)j-p} \\ &= 3\varepsilon \|x, z\|^p \frac{3^{-p}\left(1 - 3^{(-p+2)n}\right)}{1 - 3^{-p+2}}\end{aligned} \tag{18}$$

for each $x, z \in X$. Now, for $m, n \in \mathbb{N}$, we get

$$\begin{aligned}\left\|9^m f\left(\frac{x}{3^m}\right) - 9^n f\left(\frac{x}{3^n}\right), z\right\| &= \left\|9^{m+n-n} f\left(\frac{x}{3^{m+n-n}}\right) - 9^n f\left(\frac{x}{3^n}\right), z\right\| \\ &= 9^n \left\|9^{m-n} f\left(\frac{x}{3^{m-n} \cdot 3^n}\right) - f\left(\frac{x}{3^n}\right), z\right\| \\ &\le 3\varepsilon 9^n \left\|\frac{x}{3^n}, z\right\|^p \sum_{j=0}^{m-n-1} 3^{(-p+2)j-p} \\ &= 3\varepsilon 3^{(-p+2)n} \|x, z\|^p \sum_{j=0}^{m-n-1} 3^{(-p+2)j-p} \\ &= 3\varepsilon \|x, z\|^p \sum_{j=0}^{m-n-1} 3^{(-p+2)(n+j)-p} \\ &= 3\varepsilon \|x, z\|^p \frac{3^{(-p+2)n}\left(1 - 3^{(-p+2)(m-n)}\right)}{1 - 3^{-p+2}} \\ &\longrightarrow 0 \text{ as } n \to \infty\end{aligned}$$

for each $x, z \in X$. Therefore $\{9^n f(\frac{x}{3^n})\}$ is a 2-Cauchy sequence in X, for each $x \in X$. Since X is a 2-Banach space,$\{9^n f(\frac{x}{3^n})\}$ 2-converges in X, for each $x \in X$. Define $Q : X \longrightarrow X$ as

$$Q(x) := \lim_{n\to\infty} 9^n f\Big(\frac{x}{3^n}\Big)$$

for each $x \in X$. Now, by (18), we get

$$\begin{aligned}\lim_{n\to\infty} \Big\| f(x) - 9^n f\Big(\frac{x}{3^n}\Big), z\Big\| &\le 3\varepsilon\|x, z\|^p \frac{3^{-p}}{1 - 3^{-p+2}}\\ &= 3\varepsilon\|x, z\|^p \frac{1}{3^p - 9}\end{aligned}$$

for each $x, z \in X$. Therefore

$$\|f(x) - Q(x), z\| \le 3\varepsilon\|x, z\|^p \frac{1}{3^p - 9}$$

for each $x, z \in X$. The rest of the proof is similar to the proof of Theorem 2.1. □

Theorem 2.3. *Let $\varepsilon \ge 0, 0 < p < 2$. Assume that the function $f : X \longrightarrow X$ satisfies*

$$\|D_f(x, y), z\| \le \varepsilon\big[\|x, z\|^p + \|y, z\|^p\big] \tag{19}$$

for each $x, y, z \in X$. Then there exists a unique quadratic function $Q : X \longrightarrow X$ satisfying (1) and

$$\|f(x) - Q(x), z\| \le \frac{\varepsilon\|x, z\|^p}{4 - 2^p} \tag{20}$$

for each $x, z \in X$.

Proof. By (7) of Theorem 2.1, we get

$$\|f(2x) - 4f(x), z\| \le \varepsilon\|x, z\|^p$$

for each $x, z \in X$. Therefore

$$\Big\|\frac{f(2x)}{4} - f(x), z\Big\| \le \frac{\varepsilon}{4}\|x, z\|^p \tag{21}$$

for each $x, z \in X$. Replacing x by $2x$ in (21), we get

$$\Big\|\frac{f(4x)}{4} - f(2x), z\Big\| \le \frac{\varepsilon}{4}2^p\|x, z\|^p \tag{22}$$

for each $x, z \in X$. By (21) and (22), we get

$$\Big\|\frac{f(4x)}{4^2} - f(x), z\Big\| \le \Big\|\frac{f(4x)}{4^2} - \frac{f(2x)}{4}, z\Big\| + \Big\|\frac{f(2x)}{4} - f(x), z\Big\|$$

$$\begin{aligned}
&\leq \frac{1}{4}\cdot\frac{\varepsilon}{4}2^p\|x,z\|^p+\frac{\varepsilon}{4}\|x,z\|^p\\
&=\frac{\varepsilon}{4}\|x,z\|^p\Big[1+\frac{2^p}{4}\Big]
\end{aligned}$$

for each $x, z \in X$. Now, by using induction on $n \in \mathbb{N}$, we get

$$\begin{aligned}
\Big\|\frac{f(2^n x)}{4^n}-f(x),z\Big\| &\leq \frac{\varepsilon}{4}\|x\|^p\|z\|^r\sum_{j=0}^{n-1}\frac{2^{pj}}{4^j}\\
&=\frac{\varepsilon}{4}\|x,z\|^p\sum_{j=0}^{n-1}2^{(p-2)j}\\
&=\frac{\varepsilon}{4}\|x,z\|^p\Big[\frac{1-2^{(p-2)n}}{1-2^{p-2}}\Big] \qquad (23)
\end{aligned}$$

for each $x, z \in X$. For $m, n \in \mathbb{N}$, we get

$$\begin{aligned}
\Big\|\frac{f(2^m x)}{4^m}-\frac{f(2^n x)}{4^n},z\Big\| &= \Big\|\frac{f(2^{m+n-n}x)}{4^{m+n-n}}-\frac{f(2^n x)}{4^n},z\Big\|\\
&=\frac{1}{4^n}\Big\|\frac{f(2^{m-n}\cdot 2^n x)}{4^{m-n}}-f(2^n x),z\Big\|\\
&\leq \frac{1}{4^n}\frac{\varepsilon}{4}\|2^n x,z\|^p\sum_{j=0}^{m-n-1}2^{(p-2)j}\\
&=\frac{\varepsilon}{4}2^{(p-2)n}\|x,z\|^p\sum_{j=0}^{m-n-1}2^{(p-2)j}\\
&=\frac{\varepsilon}{4}\|x,z\|^p\sum_{j=0}^{m-n-1}2^{(p-2)(n+j)}\\
&=\frac{\varepsilon}{4}\|x,z\|^p\frac{2^{(p-2)n}\big(1-2^{(p-2)(m-n)}\big)}{1-2^{p-2}}\\
&\longrightarrow 0 \; as \; n\rightarrow\infty
\end{aligned}$$

for each $x, z \in X$. Therefore $\{\frac{f(2^n x)}{4^n}\}$ is a 2-Cauchy sequence in X, for each $x \in X$. Since X is a 2-Banach space, $\{\frac{f(2^n x)}{4^n}\}$ 2-converges in X, for each $x \in X$. Define $Q : X \longrightarrow X$ as

$$Q(x) := \lim_{n\rightarrow\infty}\frac{1}{4^n}f(2^n x)$$

for each $x \in X$. Now, by (23), we get

$$\lim_{n\rightarrow\infty}\Big\|\frac{f(2^n x)}{4^n}-f(x),z\Big\| \leq \frac{\varepsilon\|x,z\|^p}{4\big(1-2^{p-2}\big)}$$

for each $x, z \in X$. Therefore

$$\|Q(x)-f(x),z\| \leq \frac{\varepsilon\|x,z\|^p}{4-2^p}$$

for each $x, z \in X$. Next we show that Q satisfies (1).

$$\begin{aligned}
\|D_Q(x,y),z\| &= \lim_{n\to\infty} \frac{1}{4^n}\|D_f(2^n x, 2^n y), z\| \\
&\le \lim_{n\to\infty} \frac{\varepsilon}{4^n}(\|2^n x, z\|^p + \|2^n y, z\|^p) \\
&= \lim_{n\to\infty} \varepsilon\big[2^{(p-2)n}\|x,z\|^p + 2^{(p-2)n}\|y,z\|^p\big] \\
&= 0
\end{aligned}$$

for each $x, z \in X$. Therefore $\|D_Q(x,y),z\| = 0$, for each $z \in X$. So we get $D_Q(x,y) = 0$. Next we prove the uniqueness of Q. Let Q' be another quadratic function satisfying (1) and (20). Since Q and Q' are quadratic, $Q(2^n x) = 4^n Q(x)$, $Q'(2^n x) = 4^n Q'(x)$, for each $x \in X$.

$$\begin{aligned}
\|Q(x) - Q'(x), z\| &= \frac{1}{4^n}\|Q(2^n x) - Q'(2^n x), z\| \\
&\le \frac{1}{4^n}[\|Q(2^n x) - f(2^n x), z\| + \|f(2^n x) - Q'(2^n x), z\|] \\
&\le \frac{1}{4^n}\frac{2\varepsilon\|2^n x, z\|^p}{4 - 2^p} \\
&= 2^{(p-2)n}\,\frac{2\varepsilon\ \|x,z\|^p}{4-2^p} \\
&\longrightarrow 0 \text{ as } n \to \infty
\end{aligned}$$

for each $x, z \in X$. Therefore $\|Q(x) - Q'(x), z\| = 0$, for each $z \in X$. Therefore $Q(x) = Q'(x)$, for each $x \in X$. □

Theorem 2.4. *Let $\varepsilon \ge 0, p > 2$. Assume that a function $f : X \longrightarrow X$ satisfies*

$$\|D_f(x,y),z\| \le \varepsilon\big[\|x,z\|^p + \|y,z\|^p\big] \tag{24}$$

for each $x, y, z \in X$. Then there exists a unique quadratic function $Q : X \longrightarrow X$ satisfying (1) and

$$\|f(x) - Q(x), z\| \le \frac{\varepsilon\|x,z\|^p}{2^p - 4} \tag{25}$$

for each $x, z \in X$.

Proof. By (7) of Theorem 2.1, we get

$$\|f(2x) - 4f(x), z\| \le \varepsilon\|x,z\|^p$$

for each $x, z \in X$. Therefore

$$\|f(2x) - 4f(x), z\| \le \varepsilon\|x,z\|^p \tag{26}$$

for each $x, z \in X$. Replacing x by $\frac{x}{2}$ in (26), we get

$$\left\| f(x) - 4f\left(\frac{x}{2}\right), z \right\| \leq \varepsilon 2^{-p} \|x, z\|^p \tag{27}$$

for each $x, z \in X$. Replacing x by $\frac{x}{2}$ in (27), we get

$$\left\| f\left(\frac{x}{2}\right) - 4f\left(\frac{x}{4}\right), z \right\| \leq \varepsilon 2^{-2p} \|x, z\|^p \tag{28}$$

for each $x, z \in X$. By (27) and (28), we get

$$\begin{aligned}
\left\| f(x) - 16f\left(\frac{x}{4}\right), z \right\| &\leq \left\| f(x) - 4f\left(\frac{x}{2}\right), z \right\| + \left\| 4f\left(\frac{x}{2}\right) - 16f\left(\frac{x}{4}\right), z \right\| \\
&\leq \varepsilon 2^{-p} \|x, z\|^p + 4\varepsilon 2^{-2p} \|x\|^p \\
&= \varepsilon \|x, z\|^p \left[2^{-p} + 4 \cdot 2^{-2p}\right]
\end{aligned}$$

for each $x, z \in X$. By using induction on $n \in \mathbb{N}$, we get

$$\begin{aligned}
\left\| f(x) - 4^n f\left(\frac{x}{2^n}\right), z \right\| &\leq \varepsilon \|x, z\|^p \sum_{j=0}^{n-1} 2^{-p(j+1)} \cdot 4^j \\
&= \varepsilon \|x, z\|^p \sum_{j=0}^{n-1} 2^{(-p+2)j-p} \\
&= \varepsilon \|x, z\|^p \, \frac{2^{-p}\left(1 - 2^{(-p+2)n}\right)}{1 - 2^{-p+2}}
\end{aligned} \tag{29}$$

for each $x, z \in X$. For $m, n \in \mathbb{N}$, we get

$$\begin{aligned}
\left\| 4^m f\left(\frac{x}{2^m}\right) - 4^n f\left(\frac{x}{2^n}\right), z \right\| &= \left\| 4^{m+n-n} f\left(\frac{x}{2^{m+n-n}}\right) - 4^n f\left(\frac{x}{2^n}\right), z \right\| \\
&= 4^n \left\| 4^{m-n} f\left(\frac{x}{2^{m-n} \cdot 2^n}\right) - f\left(\frac{x}{2^n}\right), z \right\| \\
&\leq 4^n \varepsilon \left\| \frac{x}{2^n}, z \right\|^p \sum_{j=0}^{m-n-1} 2^{(-p+2)j-p} \\
&= \varepsilon 2^{(-p+2)n} \|x, z\|^p \sum_{j=0}^{m-n-1} 2^{(-p+2)j-p} \\
&= \varepsilon \|x, z\|^p \sum_{j=0}^{m-n-1} 2^{(-p+2)(n+j)-p} \\
&= \varepsilon \|x, z\|^p \frac{2^{(-p+2)n-p}\left(1 - 2^{(-p+2)(m-n)}\right)}{1 - 2^{(-p+2)}} \\
&\longrightarrow 0 \ as \ n \to \infty
\end{aligned}$$

for each $x, z \in X$. Therefore $\left\{4^n f\left(\frac{x}{2^n}\right)\right\}$ is a Cauchy sequence in X. Since X is a 2-Banach space,

$\left\{4^n f\left(\frac{x}{2^n}\right)\right\}$ converges in X. So, define $Q : X \longrightarrow X$ as

$$Q(x) := \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$$

for each $x \in X$. Now, by (29), we have

$$\left\|f(x) - 4^n f\left(\frac{x}{2^n}\right), z\right\| \leq \varepsilon\|x, z\|^p \frac{2^{-p}\left(1 - 2^{(-p+2)n}\right)}{1 - 2^{-p+2}}$$

for each $x, z \in X$. Therefore

$$\begin{aligned}\lim_{n\to\infty}\left\|f(x) - 4^n f\left(\frac{x}{2^n}\right), z\right\| &\leq \varepsilon\|x, z\|^p \frac{2^{-p}}{1 - 2^{-p+2}} \\ &= \varepsilon\|x, z\|^p \frac{1}{2^p - 4}\end{aligned}$$

for each $x, z \in X$. Therefore

$$\|f(x) - Q(x), z\| \leq \frac{\varepsilon\|x, z\|^p}{2^p - 4}$$

for each $x, z \in X$. The rest of the proof is similar to the proof of Theorem 2.3. □

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