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Atom Bond Sum Connectivity Downhill and Multiplicative Atom Bond Sum Connectivity Downhill Indices of Graphs

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Abstract

In this study, we introduce the atom bond sum connectivity downhill and multiplicative atom bond sum connectivity downhill indices of a graph. Furthermore, we compute these newly defined atom bond sum connectivity downhill indices for some standard graphs, wheel graphs, gear graphs, helm graphs and tadpole graphs.

Keywords: atom bond sum connectivity downhill index; multiplicative atom bond sum connectivity downhill index; graphs.
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1. Introduction

In this paper, G denotes a finite, simple, connected graph, $V(G)$ and $E(G)$ denote the vertex set and edge set of G. The degree $d_G(u)$ of a vertex u is the number of vertices adjacent to u. A topological index is a numerical parameter mathematically derived from the graph structure. Several topological indices were defined by using vertex degree concept [1]. Topological indices have their applications in various disciplines in Science and Technology [2]. A $u-v$ path P in G is a sequence of vertices in G, starting with u and ending at v, such that consecutive vertices in P are adjacent, and no vertex is repeated. A path $\pi = v_1, v_2, \ldots v_{k+1}$ in G is a downhill path if for every $i, 1 \leq i \leq k, d_G\left(v_i\right) \geq d_G\left(v_{i+1}\right)$. A vertex v is downhill dominates a vertex u if there exists a downhill path originated from u to v. The downhill neighborhood of a vertex v is denoted by $N_{dn}(v)$ and defined as: $N_{dn}(v) = \{u : v$ downhill dominates $u\}$. The downhill degree $d_{dn}(v)$ of a vertex v is the number of downhill neighbors of v, see [3]. Recently, some downhill indices were studied in [4-13]. The uphill domination is introduced by Deering in [14].

A $u-v$ path P in G is a sequence of vertices in G, starting with u and ending at v, such that consecutive vertices in P are adjacent, and no vertex is repeated. A path $\pi = v_1, v_2, \ldots v_{k+1}$ in G is a uphill path

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if for every $i, 1 \leq i \leq k, d_G(v_i) \leq d_G(v_{i+1})$. A vertex v is uphill dominates a vertex u if there exists an uphill path originated from u to v. The uphill neighborhood of a vertex v is denoted by $N_{up}(v)$ and defined as: $N_{up}(v) = \{u : v \text{ uphill dominates } u\}$. The uphill degree $d_{up}(v)$ of a vertex v is the number of uphill neighbors of v, see [15, 16]. The atom bond sum connectivity index [17] of a graph G is defined as

$$ABS(G) = \sum_{uv \in E(G)} \sqrt{\frac{d_G(u) + d_G(v) - 2}{d_G(u) + d_G(v)}}.$$

Recently, some atom bond sum connectivity indices were studied in [18-29]. We introduce the atom bond sum connectivity downhill index of a graph and it is defined as

$$\text{ABSDW}(G) = \sum_{uv \in E(G)} \sqrt{\frac{d_{dn}(u) + d_{dn}(v) - 2}{d_{dn}(u) + d_{dn}(v)}}$$

Also we introduce the multiplicative atom bond sum connectivity downhill index of a graph and it is defined as

$$\text{ABSDWII}(G) = \prod_{uv \in E(G)} \sqrt{\frac{d_{dn}(u) + d_{dn}(v) - 2}{d_{dn}(u) + d_{dn}(v)}}.$$

In this paper, the atom bond sum connectivity downhill index and multiplicative atom bond sum connectivity downhill index of certain graphs are computed.

2. Results for Some Standard Graphs

Proposition 2.1. *Let G be r-regular with n vertices and $r \geq 2$. Then*

$$\text{ABSDW}(G) = \frac{nr}{2}\sqrt{\frac{n-2}{n-1}}.$$

Proof. Let G be an r-regular graph with n vertices and $r \geq 2$ and $\frac{nr}{2}$ edges. Then $d_{dn}(v) = n - 1$ for every v in G.

$$\begin{aligned}\text{ABSDW}(G) &= \sum_{uv \in E(G)} \sqrt{\frac{d_{dn}(u) + d_{dn}(v) - 2}{d_{dn}(u) + d_{dn}(v)}} \\ &= \frac{nr}{2}\sqrt{\frac{(n-1)+(n-1)-2}{(n-1)+(n-1)}} \\ &= \frac{nr}{2}\sqrt{\frac{n-2}{n-1}}\end{aligned}$$

□

Corollary 2.2. *Let C_n be a cycle with at least 3 vertices. Then*

$$\text{ABSDW}(C_n) = n\sqrt{\frac{n-2}{n-1}}$$

Corollary 2.3. *Let K_n be a complete graph with at least 3 vertices. Then*

$$\operatorname{ABSDW}(K_n) = \frac{n}{2}\sqrt{(n-1)(n-2)}$$

Proposition 2.4. *Let G be r-regular with n vertices and $r \geq 2$. Then*

$$\operatorname{ABSDWII}(G) = \left(\sqrt{\frac{n-2}{n-1}}\right)^{\frac{nr}{2}}$$

Proof. Let G be an r-regular graph with n vertices and $r \geq 2$ and $\frac{nr}{2}$ edges. Then $d_{dn}(v) = n-1$ for every v in G.

$$\begin{aligned}
\operatorname{ABSDWII}(G) &= \prod_{uv\in E(G)} \sqrt{\frac{d_{dn}(u)+d_{dn}(v)-2}{d_{dn}(u)+d_{dn}(v)}} \\
&= \left(\sqrt{\frac{(n-1)+(n-1)-2}{(n-1)+(n-1)}}\right)^{\frac{nr}{2}} \\
&= \left(\sqrt{\frac{n-2}{n-1}}\right)^{\frac{nr}{2}}
\end{aligned}$$

□

Corollary 2.5. *Let C_n be a cycle with at least 3 vertices. Then*

$$\operatorname{ABSDWII}(C_n) = \left(\sqrt{\frac{n-2}{n-1}}\right)^{n}$$

Corollary 2.6. *Let K_n be a complete graph with at least 3 vertices. Then*

$$\operatorname{ABSDWII}(K_n) = \left(\sqrt{\frac{n-2}{n-1}}\right)^{\frac{n(n-1)}{2}}$$

Proposition 2.7. *Let P be a path with at least 3 vertices. Then*

$$\operatorname{ABSDW}(P_n) = 2\sqrt{\frac{n-3}{n-1}} + (n-3)\sqrt{\frac{n-2}{n-1}}.$$

Proof. Let P be a path with at least 3 vertices. We obtain two partitions of the edge set of P as follows:

$$E_1 = \left\{uv \in E(P) \mid d_{dn}(u) = 0, d_{up}(v) = n-1\right\}, |E_1| = 2.$$

$$E_2 = \left\{uv \in E(P) \mid d_{up}(u) = d_{up}(v) = n-1\right\}, \quad |E_2| = n-3.$$

$$\begin{aligned}
\text{ABSDW}\left(P_n\right) &= \sum_{uv \in E(P_n)} \sqrt{\frac{d_{dn}(u)+d_{dn}(v)-2}{d_{dn}(u)+d_{dn}(v)}} \\
&= 2\sqrt{\frac{0+(n-1)-2}{0+(n-1)}} + (n-3)\sqrt{\frac{(n-1)+(n-1)-2}{(n-1)+(n-1)}} \\
&= 2\sqrt{\frac{n-3}{n-1}} + (n-3)\sqrt{\frac{n-2}{n-1}}
\end{aligned}$$

□

Proposition 2.8. *Let P be a path with at least 3 vertices. Then*

$$\text{ABSDWII}\left(P_n\right) = \left(\sqrt{\frac{n-3}{n-1}}\right)^2 \times \left(\sqrt{\frac{n-2}{n-1}}\right)^{n-3}$$

Proof. Let P be a path with at least 3 vertices.

$$\begin{aligned}
\text{ABSDWII}\left(P_n\right) &= \prod_{uv \in E(P_n)} \sqrt{\frac{d_{dn}(u)+d_{dn}(v)-2}{d_{dn}(u)+d_{dn}(v)}} \\
&= \left(\sqrt{\frac{0+(n-1)-2}{0+(n-1)}}\right)^2 \times \left(\sqrt{\frac{(n-1)+(n-1)-2}{(n-1)+(n-1)}}\right)^{n-3} \\
&= \left(\sqrt{\frac{n-3}{n-1}}\right)^2 \times \left(\sqrt{\frac{n-2}{n-1}}\right)^{n-3}
\end{aligned}$$

□

3. Results for Wheel Graphs

The wheel W_n is the join of C_n and K_1. Clearly W_n has $n+1$ vertices and $2n$ edges. Then W_n has two types of edges based on the uphill degree of the vertices of each edge as follows:

$$E_1 = \left\{uv \in E\left(W_n\right) \mid d_{dn}(u) = n, dn_p(v) = n-1\right\}, |E_1| = n.$$

$$E_2 = \left\{uv \in E\left(W_n\right) \mid d_{dn}(u) = d_{dn}(v) = n-1\right\}, \quad |E_2| = n.$$

Theorem 3.1. *Let W_n be a wheel graph with $n+1$ vertices and $2n$ edges, $n \geq 4$. Then the atom bond sum connectivity downhill index of W_n is*

$$ABSDWW_n = n\sqrt{\frac{2n-3}{2n-1}} + n\sqrt{\frac{n-2}{n-1}}.$$

Proof. We deduce

$$\begin{aligned}
ABSDW\left(W_{n}\right) &= \sum_{uv \in E(W_n)} \sqrt{\frac{d_{dn}(u)+d_{dn}(v)-2}{d_{dn}(u)+d_{dn}(v)}} \\
&= n\sqrt{\frac{n+n-1-2}{n+n-1}} + n\sqrt{\frac{n-1+n-1-2}{n-1+n-1}} \\
&= n\sqrt{\frac{2n-3}{2n-1}} + n\sqrt{\frac{n-2}{n-1}}
\end{aligned}$$

□

Theorem 3.2. *Let W_n be a wheel graph with $n+1$ vertices and $2n$ edges, $n \geq 4$. Then the multiplicative atom bond sum connectivity downhill index of W_n is*

$$ABSDWII\ W_n = \left(\sqrt{\frac{2n-3}{2n-1}}\right)^n + \left(\sqrt{\frac{n-2}{n-1}}\right)^n.$$

Proof. We deduce

$$\begin{aligned}
\text{ABSDWII}\left(W_{n}\right) &= \prod_{uv \in E(W_n)} \sqrt{\frac{d_{dn}(u)+d_{dn}(v)-2}{d_{dn}(u)+d_{dn}(v)}} \\
&= \left(\sqrt{\frac{n+n-1-2}{n+n-1}}\right)^n \times \left(\sqrt{\frac{n-1+n-1-2}{n-1+n-1}}\right)^n \\
&= \left(\sqrt{\frac{2n-3}{2n-1}}\right)^n \times \left(\sqrt{\frac{n-2}{n-1}}\right)^n
\end{aligned}$$

□

4. Results for Helm Graphs

The helm graph H_n is a graph obtained from W_n (with $n+1$ vertices) by attaching an end edge to each rim vertex of W_n. Clearly, $|V(H_n)| = 2n+1$ and $|E(H_n)| = 3n$. A graph H_n is shown in Figure 1.

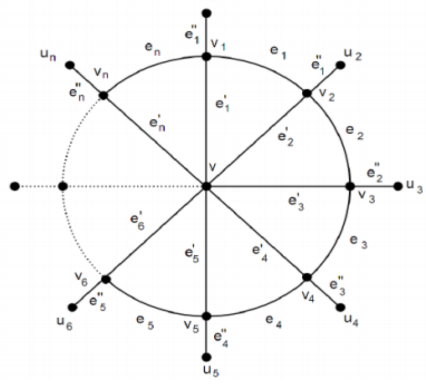

Figure 1: Helm graph H_n

Let H_n be a helm graph with $3n$ edges, $n \geq 5$. Then H_n has three types of edges based on the uphill degree of the vertices of each edge as follows:

$$E_1 = \{uv \in E(H_n) \mid d_{dn}(u) = 2n, d_{dn}(v) = 2n-1\}, \qquad |E_1| = n$$
$$E_2 = \{uv \in E(H_n) \mid d_{dn}(u) = d_{dn}(v) = 2n-1\}, \qquad |E_2| = n$$
$$E_3 = \{uv \in E(H_n) \mid d_{dn}(u) = 2n-1, d_{dn}(v) = 0\}, \qquad |E_3| = n$$

Theorem 4.1. *Let H_n be a helm graph with $2n+1$ vertices, $n \geq 5$. Then the atom bond sum connectivity downhill index of H_n is*

$$\text{ABSDW } H_n = n\sqrt{\frac{4n-3}{4n-1}} + n\sqrt{\frac{2n-2}{2n-1}} + n\sqrt{\frac{2n-3}{2n-1}}.$$

Proof. We obtain

$$\begin{aligned}
\text{ABSDW}(H_n) &= \sum_{uv \in E(H_n)} \sqrt{\frac{d_{dn}(u)+d_{dn}(v)-2}{d_{dn}(u)+d_{dn}(v)}} \\
&= n\sqrt{\frac{2n+2n-1-2}{2n+2n-1}} + n\sqrt{\frac{2n-1+2n-1-2}{2n-1+2n-1}} + n\sqrt{\frac{2n-1+0-2}{2n-1+0}} \\
&= n\sqrt{\frac{4n-3}{4n-1}} + n\sqrt{\frac{2n-2}{2n-1}} + n\sqrt{\frac{2n-3}{2n-1}}
\end{aligned}$$

□

Theorem 4.2. *Let H_n be a helm graph with $2n+1$ vertices, $n \geq 5$. Then the multiplicative atom bond sum connectivity downhill index of H_n is*

$$ABSDWII\ H_n = \left(\sqrt{\frac{4n-3}{4n-1}}\right)^n \times \left(\sqrt{\frac{2n-2}{2n-1}}\right)^n \times \left(\sqrt{\frac{2n-3}{2n-1}}\right)^n.$$

Proof. We obtain

$$\begin{aligned}
\text{ABSDWII}(H_n) &= \prod_{uv \in E(H_n)} \sqrt{\frac{d_{dn}(u)+d_{dn}(v)-2}{d_{dn}(u)+d_{dn}(v)}} \\
&= \left(\sqrt{\frac{2n+2n-1-2}{2n+2n-1}}\right)^n \times \left(\sqrt{\frac{2n-1+2n-1-2}{2n-1+2n-1}}\right)^n \times \left(\sqrt{\frac{2n-1+0-2}{2n-1+0}}\right)^n \\
&= \left(\sqrt{\frac{4n-3}{4n-1}}\right)^n \times \left(\sqrt{\frac{2n-2}{2n-1}}\right)^n \times \left(\sqrt{\frac{2n-3}{2n-1}}\right)^n
\end{aligned}$$

□

5. Results for Tadpole Graphs

Let $T_{n,m}$ be a tadpole graph with $n+m$ vertices, $n,m \geq 3$. Then $T_{n,m}$ has five types of edges based on the downhill degree of the vertices of each edge as follows:

$$E_1 = \{uv \in E(T_{n,m}) \mid d_{dn}(u) = n+m-1, d_{dn}(v) = m-1\}, \qquad |E_1| = 1$$
$$E_2 = \{uv \in E(T_{n,m}) \mid d_{dn}(u) = n+m-1, d_{dn}(v) = n-2\}, \qquad |E_2| = 2$$
$$E_3 = \{uv \in E(T_{n,m}) \mid d_{dn}(u) = n-2, d_{dn}(v) = n-2\}, \qquad |E_3| = n-2$$
$$E_4 = \{uv \in E(T_{n,m}) \mid d_{dn}(u) = m-1, d_{dn}(v) = m-1\}, \qquad |E_4| = m-2$$
$$E_5 = \{uv \in E(T_{n,m}) \mid d_{dn}(u) = m-1, d_{dn}(v) = 0\}, \qquad |E_5| = 1$$

Theorem 5.1. *Let $T_{n,m}$ be a tadpole graph with $n+m$ vertices, $n,m \geq 3$. Then the atom bond sum connectivity downhill index of $T_{n,m}$ is*

$$ABSDWT_{n,m} = 1\sqrt{\frac{n+2m-4}{n+2m-2}} + 2\sqrt{\frac{2n+m-5}{2n+m-3}} + (n-2)\sqrt{\frac{n-3}{n-2}} + (m-2)\sqrt{\frac{m-2}{m-1}} + 1\sqrt{\frac{m-3}{m-1}}.$$

Proof. We obtain

$$\begin{aligned}
\text{ABSDW}(T_{n,m}) &= \sum_{uv \in E(T_{n,m})} \sqrt{\frac{d_{dn}(u) + d_{dn}(v) - 2}{d_{dn}(u) + d_{dn}(v)}} \\
&= 1\sqrt{\frac{n+m-1+m-1-2}{n+m-1+m-1}} + 2\sqrt{\frac{n+m-1+n-2-2}{n+m-1+n-2}} + (n-2)\sqrt{\frac{n-2+n-2-2}{n-2+n-2}} \\
&+ (m-2)\sqrt{\frac{m-1+m-1-2}{m-1+m-1}} + 1\sqrt{\frac{m-1+0-2}{m-1+0}} \\
&= 1\sqrt{\frac{n+2m-4}{n+2m-2}} + 2\sqrt{\frac{2n+m-5}{2n+m-3}} + (n-2)\sqrt{\frac{n-3}{n-2}} + (m-2)\sqrt{\frac{m-2}{m-1}} + 1\sqrt{\frac{m-3}{m-1}}
\end{aligned}$$

□

Theorem 5.2. *Let $T_{n,m}H_n$ be a helm graph with $2n+1$ vertices, $n \geq 5$. Then the multiplicative atom bond sum connectivity downhill index of $T_{n,m}H_n$ is*

$$\begin{aligned}
ABSDWII\ T_{n,m} = &\left(\sqrt{\frac{n+2m-4}{n+2m-2}}\right)^1 \times \left(\sqrt{\frac{2n+m-5}{2n+m-3}}\right)^2 \times \left(\sqrt{\frac{n-3}{n-2}}\right)^{n-2} \\
&\times \left(\sqrt{\frac{m-2}{m-1}}\right)^{m-2} \times \left(\sqrt{\frac{m-3}{m-1}}\right)^1
\end{aligned}$$

Proof. We obtain

$$
\begin{aligned}
\mathrm{ABSDWII}\,(T_{n,m}) &= \prod_{uv\in E(T_{n,m})} \sqrt{\frac{d_{dn}(u)+d_{dn}(v)-2}{d_{dn}(u)+d_{dn}(v)}} \\
&= \left(\sqrt{\frac{n+m-1+m-1-2}{n+m-1+m-1}}\right)^{1} \times \left(\sqrt{\frac{n+m-1+n-2-2}{n+m-1+n-2}}\right)^{2} \\
&\times \left(\sqrt{\frac{n-2+n-2-2}{n-2+n-2}}\right)^{n-2} \times \left(\sqrt{\frac{m-1+m-1-2}{m-1+m-1}}\right)^{m-2} \times \left(\sqrt{\frac{m-1+0-2}{m-1+0}}\right)^{1} \\
&= \left(\sqrt{\frac{n+2m-4}{n+2m-2}}\right)^{1} \times \left(\sqrt{\frac{2n+m-5}{2n+m-3}}\right)^{2} \times \left(\sqrt{\frac{n-3}{n-2}}\right)^{n-2} \\
&\times \left(\sqrt{\frac{m-2}{m-1}}\right)^{m-2} \times \left(\sqrt{\frac{m-3}{m-1}}\right)^{1}.
\end{aligned}
$$

□

6. Conclusion

In this research work, the atom bond sum connectivity downhill and multiplicative atom bond sum connectivity downhill indices of a graph are defined. Also these newly defined atom bond sum connectivity downhill and multiplicative atom bond sum connectivity downhill indices of certain graphs are computed.

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