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An Analytical Solution of Abel Integral Equation of the Second Kind in Coupling of Various Transform Methods via Homotopy Perturbation Transform Method

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Abstract

In this paper analytical solution for Abel integral equation of the second kind in the coupling of the Laplace-Stieltjes, Kamal, Laplace-Carson and Aboodh transform method with homotopy perturbation transform method is introduced. Abel integral equation of the second occurs in several models in physical sciences, astrophysics, applied sciences etc. The numerical approach of the method is very simple and illustrates the accuracy, validity, and stability of the solution in form of the exact solution.

Keywords: Abel integral equation; Laplace-Stieltjes transform; Kamal transform; Laplace-Carson transform; Aboodh transform; Homotopy perturbation transform method.
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1. Introduction

Niels Henrik Abel was the first mathematician, who took the initiative of integral equations of singular type in 1823. Abel integral equation [1] is one of the integral equations which is derived directly from a concrete problem of physics, without passing through a differential equation. This equation appears in several models in astrophysics, solid mechanics, physical sciences and applied sciences. Zeilon N. [2] in 1924, gave an idea of the solution of Abel integral equation on a finite segment. The various methods for solving Abel integral equation and fractional differential equations are given in [3-5]. He [6-7] developed the homotopy perturbation transform method. Numerical solutions of Abel integral equations are given in [8,24,28,29] by using the different Wavelets methods. The homotopy perturbation transform method is used in [9-11,19]. The different polynomial methods are used in [13,14]. Various transform methods are used in [12,18,21-23]. Analytical solutions are given in [16,17]. Fractional calculus is used in [20]. The Variational Iteration method is used in [27]. The Laplace decomposition method is used in [15]. The Taylor-Collocation method is used in [25]. The solution of Abel integral

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using the differential transform method is given in [26]. The solution of generalized Abel integral equation of second kind using various transform methods is given in [30]. The solution of Abel integral equation of second kind in coupling of Laplace transform method and Leibnitz linear differential equation is discussed in [31]. The main purpose of this paper is to produce an analytical solution for Abel integral equation of the second kind via various transform methods using the homotopy perturbation transform method.

2. Some Basic Definitions and Terminologies

Definition 2.1. *The Laplace-Stieltjes transform [22] of a function $f(t)$ defined for $t \geq 0$ is defined by the improper integral*

$$L_S[f(t)] = \int_0^\infty e^{-st} d[f(t)], \tag{1}$$

where L_S is the Laplace-Stieltjes transform operator.

Definition 2.2. *The Kamal transform [22] over the set of functions $\{f(t) : \exists \ M > 0$ (finite) $, \sigma_i > 0$, (finite or infinite) with $|f(t)| < Me^{\frac{|t|}{\sigma_i}}$ if $t\epsilon(-1)^i \times [0,\infty), i = 1,2\}$ is defined by the improper integral*

$$K[f(t)] = \int_0^\infty e^{\frac{-t}{s}} f(t)\, dt, \tag{2}$$

where K is the Kamal transform operator.

Definition 2.3. *The Laplace-Carson transform (Mahgoub transform) [23] of a function $f(t)$ defined for $t \geq 0$ is defined by the improper integral*

$$L_c[f(t))] = s\int_0^\infty e^{-st} f(t)\, dt, \tag{3}$$

where L_c is the Laplace-Carson transform operator.

Definition 2.4. *The Aboodh transform [18] over the set of functions $\{f(t) : \exists \ M > 0$ (finite) $, u, v > 0$ (finite or infinite) such that $|f(t)| < Me^{-st}$ and $u \leq s \leq v\}$ is defined by the improper integral*

$$A[f(t)] = \frac{1}{s}\int_0^\infty e^{-st} f(t)\, dt, \tag{4}$$

where A is the Aboodh transform operator.

3. Basic Idea of the Proposed Method for an Analytical Solution for Abel Integral Equation of the Second Kind

To demonstrate the idea of the solution by using HPTM, we consider the following Abel integral equation of the second kind as

$$f(t) = g(t) + \int_0^t \frac{f(u)}{(t-u)^{\frac{1}{2}}} du, \quad where \ \ 0 \leq t \leq 1 \tag{5}$$

3.1 Coupling of Laplace-Stieltjes transform and homotopy perturbation transform method on Abel integral equation of second kind

Operating the Laplace-Stieltjes transform on both sides of the equation (5), we have

$$L_S[f(t)] = L_S[g(t)] + L_S\left\{\int_0^t \frac{f(u)}{(t-u)^{\frac{1}{2}}}du\right\}. \tag{6}$$

By using convolution property of the Laplace-Stieltjes transform, in equation (6), we get

$$\begin{aligned} L_S[f(t)] &= L_S[g(t)] + \frac{1}{s}L_S[f(t)]\,L_S[t^{\frac{-1}{2}}] \\ &= L_S[g(t)] + \frac{1}{s}L_S[f(t)]\sqrt{\pi s} \\ \Rightarrow L_S[f(t)] &= L_S[g(t)] + \sqrt{\frac{\pi}{s}}L_S[f(t)]. \end{aligned} \tag{7}$$

On operating the inverse Laplace- Stieltjes transform, in equation (7), we have

$$f(t) = g(t) + L_S{}^{-1}\left\{\sqrt{\frac{\pi}{s}}L_S[f(t)]\right\} \tag{8}$$

We assume the solution of Abel integral equation of the second kind (5) in the series form as $\varphi(t) = \sum_{n=0}^{\infty} p^n\varphi_n(t)$, where $\varphi_n(t)$ are to be determined by the iterative scheme of HPTM. Consider the following convex homotopy in order to solve the equation (5),

$$\sum_{n=0}^{\infty} p^n\varphi_n(t) = g(t) + p\left[L_S{}^{-1}\left\{\sqrt{\frac{\pi}{s}}L_S\left(\sum_{n=0}^{\infty} p^n\varphi_n(t)\right)\right\}\right]. \tag{9}$$

This is a combination of the Laplace-Stieltjes transform and the homotopy perturbation transform method. On equating the coefficients of same powers of p of (9), we obtained the following approximations;

$$p^0 : \varphi_0(t) = g(t) \;\; and \;\; p^n : \varphi_n(t) = L_S{}^{-1}\left\{\sqrt{\frac{\pi}{s}}L_S[\varphi_{n-1}(t)]\right\}; n \in \mathbb{N}. \tag{10}$$

The approximate analytical solution of the equation (5) is given by

$$f(t) = \lim_{p\to 1}\varphi(t) = \sum_{n=0}^{\infty}\varphi_n(t). \tag{11}$$

3.2 Coupling of Kamal transform and homotopy perturbation transform method on Abel integral equation of second kind

Operating the Kamal transform on both sides of the equation (5), we have

$$K[f(t)] = K[g(t)] + K\left\{\int_0^t \frac{f(u)}{(t-u)^{\frac{1}{2}}}du\right\}. \tag{12}$$

By using convolution property of the Kamal transform, in the equation (12), we get

$$\begin{aligned} K[f(t)] &= K[g(t)] + K[f(t)]\, K\left[t^{\frac{-1}{2}}\right] \\ &= K[g(t)] + K[f(t)]\ \sqrt{\pi s} \\ \Rightarrow K[f(t)] &= K[g(t)] + \sqrt{\pi s} K[f(t)]. \end{aligned} \tag{13}$$

On operating the inverse Kamal transform, in the equation (13), we have

$$f(t) = g(t) + K^{-1}\{\sqrt{\pi s} K[f(t)]\} \tag{14}$$

We assume the solution of Abel integral equation of the second kind (5) in the series form as $\varphi(t) = \sum_{n=0}^{\infty} p^n \varphi_n(t)$, where $\varphi_n(t)$ are to be determined by the iterative scheme of HPTM. Consider the following convex homotopy in order to solve (5)

$$\sum_{n=0}^{\infty} p^n \varphi_n(t) = g(t) + p\left[K^{-1}\left\{\sqrt{\pi s}\ K\left(\sum_{n=0}^{\infty} p^n \varphi_n(t)\right)\right\}\right]. \tag{15}$$

This is a combination of the Kamal transform and the homotopy perturbation transform method. On equating the coefficients of same powers of p of (15) we obtained the following approximations;

$$p^0 : \varphi_0(t) = g(t) \ \ and \ \ p^n : \varphi_n(t) = K^{-1}\{\sqrt{\pi s} K[\varphi_{n-1}(t)]\}; \ \ n \in \mathbb{N} \tag{16}$$

The approximate analytical solution of the equation (5) is given by

$$f(t) = \lim_{p \to 1} \varphi(t) = \sum_{n=0}^{\infty} \varphi_n(t). \tag{17}$$

3.3 Coupling of Laplace-Carson transform and homotopy perturbation transform method on Abel integral equation of second kind

Operating the Laplace-Carson transform on both sides of the equation (5), we have

$$L_C[f(t)] = L_C[g(t)] + L_C\left\{\int_0^t \frac{f(u)}{(t-u)^{\frac{1}{2}}}du\right\}. \tag{18}$$

By using convolution property of the Laplace-Carson transform, in equation (18), we get

$$\begin{aligned} L_C[f(t)] &= L_C[g(t)] + \frac{1}{s} L_C\,[f\,(t)]\;\; L_C[t^{\frac{-1}{2}}] \\ &= L_C[g(t)] + \frac{1}{s} L_C\,[f\,(t)]\,\sqrt{\pi s} \\ \Rightarrow L_C[f(t)] &= L_C\,[g\,(t)] + \sqrt{\frac{\pi}{s}} L_C\,[f\,(t)]\,. \end{aligned} \tag{19}$$

On operating the inverse Laplace-Carson transform, in the equation (19), we have

$$f\,(t) = g\,(t) + {L_C}^{-1}\left\{\sqrt{\frac{\pi}{s}}\; L_C\,[f\,(t)]\right\}. \tag{20}$$

We assume the solution of Abel integral equation of the second kind (5), in the series form as $\varphi\,(t) = \sum\limits_{n=0}^{\infty} p^n \varphi_n(t)$, where $\varphi_n(t)$ are to be determined by the iterative scheme of HPTM. Consider the following convex homotopy in order to solve the equation (5),

$$\sum_{n=0}^{\infty} p^n \varphi_n(t) = g\,(t) + p\left[{L_C}^{-1}\left\{\sqrt{\frac{\pi}{s}}\; L_C\left(\sum_{n=0}^{\infty} p^n \varphi_n\,(t)\right)\right\}\right]. \tag{21}$$

This is a combination of the Laplace-Carson transform and the homotopy perturbation transform method. On equating the coefficients of same powers of p of (21), we obtained the following approximations;

$$p^0 : \varphi_0(t) = g\,(t) \quad and \quad p^n : \varphi_n(t) = {L_C}^{-1}\left\{\sqrt{\frac{\pi}{s}} L_C\,[\varphi_{n-1}\,(t)]\right\};\;\; n \in \mathbb{N} \tag{22}$$

The analytical solution of the equation (5) is given by

$$f\,(t) = \lim_{p \to 1} \varphi(t) = \sum_{n=0}^{\infty} \varphi_n(t). \tag{23}$$

3.4 Coupling of Aboodh transform and homotopy perturbation transform method on Abel integral equation of second kind

Operating the Aboodh transform on both sides of the equation (5), we have

$$A[f\,(t)] = A[g(t)] + A\left\{\int_0^t \frac{f\,(u)}{(t-u)^{\frac{1}{2}}} du\right\}. \tag{24}$$

By using convolution property of the Aboodh transform, in equation (24), we get

$$\begin{aligned} A\,[f\,(t)] &= A\,[g\,(t)] + sA\,[f\,(t)]\,A[t^{\frac{-1}{2}}] \\ &= A\,[g\,(t)] + sA\,[f\,(t)]\,\sqrt{\pi} s^{\frac{-3}{2}} \end{aligned}$$

$$\Rightarrow A[f(t)] = A[g(t)] + A\left[f\left(t\right)\right]\sqrt{\frac{p}{s}} \tag{25}$$

On operating the inverse Aboodh transform, in the equation (25), we have

$$f\left(t\right) = g\left(t\right) + A^{-1}\left\{\sqrt{\frac{\pi}{s}}A\left[f\left(t\right)\right]\right\} \tag{26}$$

Assume the solution of Abel integral equation of second kind (5), in the series form as $\varphi\left(t\right) = \sum\limits_{n=0}^{\infty} p^n \varphi_n(t)$, where $\varphi_n(t)$ are to be determined by the iterative scheme of HPTM. Consider the following convex homotopy in order to solve the equation (5),

$$\sum_{n=0}^{\infty} p^n \varphi_n(t) = g\left(t\right) + p\left[A^{-1}\left\{\sqrt{\frac{\pi}{s}}A\left(\sum_{n=0}^{\infty} p^n \varphi_n\left(t\right)\right)\right\}\right]. \tag{27}$$

This is a combination of the Aboodh transform and the homotopy perturbation transform method. On equating the coefficients of same powers of p of (27), we obtained the following approximations;

$$p^0 : \varphi_0(t) = g\left(t\right) \quad and \quad p^n : \varphi_n(t)A^{-1}\left\{\sqrt{\frac{\pi}{s}}A\left[\varphi_{n-1}\left(t\right)\right]\right\}; n \in \mathbb{N}. \tag{28}$$

The approximate analytical solution of the equation (5) is given by

$$f(t) = \lim_{p\to 1} \varphi(t) = \sum_{n=0}^{\infty} \varphi_n(t). \tag{29}$$

4. Numerical Implementation of the Method

The approximate analytical solution determined by coupling of the homotopy perturbation transform method and the Laplace-Stieltjes transform method is same as determined by coupling with Kamal, Laplace-Carson and Aboodh transform method, due to their duality relations.

Example 4.1. *Consider the Abel integral equation of second kind as* $f\left(t\right) = \frac{\pi}{2}t + \sqrt{t} - \int_0^t \frac{f(u)}{(t-u)^{\frac{1}{2}}}du$, *where* $0 \le t \le 1$ *with exact solution* $\sqrt{t}$. *By using the convex homotopy perturbation transform method, we have*

$$\sum_{n=0}^{\infty} p^n \varphi_n(t) = \frac{\pi}{2}t + \sqrt{t} - p\left[{L_S}^{-1}\left\{\sqrt{\frac{\pi}{s}}L_S\left(\sum_{n=0}^{\infty} p^n \varphi_n\left(t\right)\right)\right\}\right].$$

On equating the coefficients of the various powers of p on both sides in above equation, we have

$$p^0 : \varphi_0(t) = \frac{\pi}{2}t + \sqrt{t}, p^1 : \varphi_1(t) = -{L_S}^{-1}\left\{\sqrt{\frac{\pi}{s}}L_S\left(\varphi_0\left(t\right)\right)\right\} = -\frac{2}{3}\pi t^{\frac{3}{2}} - \frac{1}{2}\pi t,$$

$$p^2 : \varphi_2(t) = -{L_S}^{-1}\left\{\sqrt{\frac{\pi}{s}}L_S\left(\varphi_1\left(t\right)\right)\right\} = \frac{1}{4}\pi^2 t^2 + \frac{2}{3}\pi t^{\frac{3}{2}},$$

$$p^3 : \varphi_3(t) = -{L_S}^{-1}\left\{\sqrt{\frac{\pi}{s}}L_S\left(\varphi_2\left(t\right)\right)\right\} = -\frac{4}{15}\pi^2 t^{\frac{5}{2}} - \frac{1}{4}\pi^2 t^2,$$

$$p^4 : \varphi_4(t) = -L_S{}^{-1}\left\{\sqrt{\frac{\pi}{s}} L_S\left(\varphi_3\left(t\right)\right)\right\} = \frac{1}{4}\pi^3 t^3 + \frac{4}{15}\pi^2 t^{\frac{5}{2}},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

Finally, we approximate the analytical solution $f(t)$ using the truncated series as

$$f\left(t\right) = \sum_{n=0}^{\infty} \varphi_n\left(t\right) = \left(\frac{\pi}{2}t + \sqrt{t}\right) - \left(\frac{2}{3}\pi t^{\frac{3}{2}} + \frac{1}{2}\pi t\right) + \left(\frac{1}{4}\pi^2 t^2 + \frac{2}{3}\pi t^{\frac{3}{2}}\right) - \left(\frac{4}{15}\pi^2 t^{\frac{5}{2}} + \frac{1}{4}\pi^2 t^2\right) + \left(\frac{1}{4}\pi^3 t^3 + \frac{4}{15}\pi^2 t^{\frac{5}{2}}\right) - \ldots$$

Hence $f(t) \to \sqrt{t}$ as $n \to \infty$, which is exact solution.

Example 4.2. *Consider the Abel integral equation of second kind as $f\left(t\right) = t + \frac{4}{3}t^{\frac{3}{2}} - \int_0^t \frac{f(u)}{(t-u)^{\frac{1}{2}}}du$, where $0 \leq t \leq 1$ with exact solution t. By using the convex homotopy perturbation transform method, we have*

$$\sum_{n=0}^{\infty} p^n \varphi_n(t) = t + \frac{4}{3}t^{\frac{3}{2}} - p\left[L_S{}^{-1}\left\{\sqrt{\frac{\pi}{s}} L_S\left(\sum_{n=0}^{\infty} p^n \varphi_n\left(t\right)\right)\right\}\right]$$

Now equating the coefficients of the various power of p on both sides in above equation, we have

$$p^0 : \varphi_0(t) = t + \frac{4}{3}t^{\frac{3}{2}}, p^1 : \varphi_1(t) = -L_S{}^{-1}\left\{\sqrt{\frac{\pi}{s}} L_S\left(\varphi_0\left(t\right)\right)\right\} = -\frac{4}{3}\pi t^{\frac{3}{2}} - \frac{1}{2}\pi t^2,$$

$$p^2 : \varphi_2(t) = -L_S{}^{-1}\left\{\sqrt{\frac{\pi}{s}} L_S\left(\varphi_1\left(t\right)\right)\right\} = \frac{1}{2}\pi t^2 + \frac{8}{15}\pi t^{\frac{5}{2}},$$

$$p^3 : \varphi_3(t) = -L_S{}^{-1}\left\{\sqrt{\frac{\pi}{s}} L_S\left(\varphi_2\left(t\right)\right)\right\} = -\frac{8}{15}\pi t^{\frac{5}{2}} - \frac{1}{6}\pi^2 t^3,$$

$$p^4 : \varphi_4(t) = -L_S{}^{-1}\left\{\sqrt{\frac{\pi}{s}} L_S\left(\varphi_3\left(t\right)\right)\right\} = \frac{1}{6}\pi^2 t^3 + \frac{16}{105}\pi^2 t^{\frac{7}{2}},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

Finally, we approximate the analytical solution $f(t)$ using the truncated series as

$$f\left(t\right) = \sum_{n=0}^{\infty} \varphi_n(t) = \left(t + \frac{4}{3}t^{\frac{3}{2}}\right) - \left(\frac{4}{3}\pi t^{\frac{3}{2}} + \frac{1}{2}\pi t^2\right) + \left(\frac{1}{2}\pi t^2 + \frac{8}{15}\pi t^{\frac{5}{2}}\right) - \left(\frac{8}{15}\pi t^{\frac{5}{2}} + \frac{1}{6}\pi^2 t^3\right) + \left(\frac{1}{6}\pi^2 t^3 + \frac{16}{105}\pi^2 t^{\frac{7}{2}}\right) + \ldots$$

Hence $f(t) \to t$ as $n \to \infty$, which is exact solution.

Example 4.3. *Consider the Abel integral equation of second kind as $f\left(t\right) = 2\sqrt{t} - \int_0^t \frac{f(u)}{(t-u)^{\frac{1}{2}}}du$, where $0 \leq t \leq 1$, with exact solution $1 - e^{\pi t} erfc(\sqrt{\pi t})$ and the complementary error function $erfc(t)$ is defined as $erfc(t) = \frac{2}{\sqrt{\pi}} \int_t^{\infty} e^{-u^2} du$. By using the convex homotopy perturbation transform method, we have*

$$\sum_{n=0}^{\infty} p^n \varphi_n(t) = 2\sqrt{t} - p\left[L_S{}^{-1}\left\{\sqrt{\frac{\pi}{s}} L_S\left(\sum_{n=0}^{\infty} p^n \varphi_n\left(t\right)\right)\right\}\right]$$

On equating the coefficients of the various powers of p on both sides in above equation, we have

$$
\begin{aligned}
&p^0 : \varphi_0(t) = 2\sqrt{t}, p^1 : \varphi_1(t) = -{L_S}^{-1}\left\{\sqrt{\frac{\pi}{s}}\, L_S\left(f_0\left(t\right)\right)\right\} = -\pi t,\\
&p^2 : \varphi_2(t) = -{L_S}^{-1}\left\{\sqrt{\frac{\pi}{s}}L_S\left(\varphi_1\left(t\right)\right)\right\} = \frac{4}{3}\pi t^{\frac{3}{2}},\\
&p^3 : \varphi_3(t) = -{L_S}^{-1}\left\{\sqrt{\frac{\pi}{s}}L_S\left(\varphi_2\left(t\right)\right)\right\} = -\frac{1}{2}\pi^2 t^2,\\
&p^4 : \varphi_4(t) = -{L_S}^{-1}\left\{\sqrt{\frac{\pi}{s}}L_S\left(f_3\left(t\right)\right)\right\} = \frac{8}{15}\pi^2 t^{\frac{5}{2}},\\
&\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}
$$

Finally, we approximate the analytical solution $f(t)$ using the truncated series as

$$
\begin{aligned}
f\left(t\right) = \sum_{n=0}^{\infty}\varphi_n(t) &= 2\sqrt{t} - \pi t + \frac{4}{3}\pi t^{\frac{3}{2}} - \frac{1}{2}\pi^2 t^2 + \frac{8}{15}\pi^2 t^{\frac{5}{2}} + \ldots\\
&= 1 - \left(1 - 2\sqrt{t} + \pi t - \frac{4}{3}\pi t^{\frac{3}{2}} + \frac{1}{2}\pi^2 t^2 - \frac{8}{15}\pi^2 t^{\frac{5}{2}} + \ldots\right.
\end{aligned}
$$

Hence $f(t) \to 1 - e^{\pi t} erfc(\sqrt{\pi t})$ as $n \to \infty$, which is an exact solution.

Example 4.4. *Consider the Abel integral equation of second kind as $f(t) = t^2 + \frac{16}{15}t^{\frac{5}{2}} - \int_0^t \frac{f(u)}{(t-u)^{\frac{1}{2}}}du$, where $0 \le t \le 1$, with exact solution t^2. Using coupling of Laplace-Stieltjes transform and the convex homotopy perturbation transform method, we have*

$$
\sum_{n=0}^{\infty} p^n\varphi_n(t) = t^2 + \frac{16}{15}t^{\frac{5}{2}} - p\left[{L_S}^{-1}\left\{\sqrt{\frac{\pi}{s}}L_S\left(\sum_{n=0}^{\infty}p^n\varphi_n\left(t\right)\right)\right\}\right]
$$

Now equating the coefficients of the various power of p on both sides, in above equation, we have

$$
\begin{aligned}
&p^0 : \varphi_0(t) = t^2 + \frac{16}{15}t^{\frac{5}{2}}, p^1 : \varphi_1(t) = -{L_S}^{-1}\left\{\sqrt{\frac{\pi}{s}}L_S\left(\varphi_0\left(t\right)\right)\right\} = -\frac{16}{15}t^{\frac{5}{2}} - \frac{1}{3}\pi t^3,\\
&p^2 : \varphi_2(t) = -{L_S}^{-1}\left\{\sqrt{\frac{\pi}{s}}L_S\left(\varphi_1\left(t\right)\right)\right\} = \frac{1}{3}\pi t^3 + \frac{32}{105}\pi t^{\frac{7}{2}},\\
&p^3 : \varphi_3(t) = -{L_S}^{-1}\left\{\sqrt{\frac{\pi}{s}}L_S\left(f_2\left(t\right)\right)\right\} = -\frac{32}{105}\pi t^{\frac{7}{2}} - \frac{1}{12}\pi^2 t^4,\\
&p^4 : \varphi_4(t) = -{L_S}^{-1}\left\{\sqrt{\frac{\pi}{s}}L_S\left(f_3\left(t\right)\right)\right\} = \frac{1}{12}\pi^2 t^4 + \frac{64}{945}\pi^2 t^{\frac{9}{2}},\\
&\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}
$$

Finally, we approximate the analytical solution $f(t)$ using the truncated series as

$$
\begin{aligned}
f\left(t\right) = \lim_{p\to 1}\varphi\left(t\right) = \sum_{n=0}^{\infty}\varphi_n(t) = &\left(t^2 + \frac{16}{15}t^{\frac{5}{2}}\right) - \left(\frac{16}{15}t^{\frac{5}{2}} + \frac{1}{3}\pi t^3\right) + \left(\frac{1}{3}\,\pi t^3 + \frac{32}{105}\,\pi t^{\frac{7}{2}}\right)\\
&- \left(\frac{32}{105}\pi t^{\frac{7}{2}} + \frac{1}{12}\pi^2 t^4\right) + \left(\frac{1}{12}\pi^2 t^4 + \frac{64}{945}\pi^2 t^{\frac{9}{2}}\right) - \ldots
\end{aligned}
$$

Hence $f(t) \to t^2$ as $n \to \infty$, which is an exact solution.

5. Conclusion

We have drafted an analytical solution for the Abel integral equation of second kind in coupling of various integral transform methods using homotopy perturbation transform method. The approach of the method is very simple and illustrates the accuracy, validity and stability of the solution obtained in form of an exact solution.

References

[1] N. H. Abel, *Solution de quelques problems à l'aide d' integrals definies(in French) Magazin for Naturvidenskabernc*, Alu-gang I, Bind 2 Christiania, 18(1823), 1-27.

[2] N. Zeilon, *Sur quelques points de la theorie de l'equationintegraled'Abel*, Arkiv. Mat. Astr. Fysik., 18(1924), 1-19.

[3] R. Gorenflo and S. Vessella, *Abel Integral Equations*, Springer, Berlin, (1991).

[4] A. M. Wazwaz, *A First Course in Integral Equations*, World Scientific, New Jersey, (1997).

[5] I. Podlubny, *Fractional Differential Equation*, Academic Press, San Diego, CA, (1999).

[6] J. H. He, *Homotopy perturbation technique*, Computer Methods in Applied Mechanics and Engineering, 178(1999), 257-262.

[7] J. H. He, *A coupling method of homotopy technique and perturbation technique for non-linear problems*, International Journal of Non-linear Mechanics, 35(2000), 37-43.

[8] S. A. Yousefi, *Numerical Solution of Abel Integral Equation by using Legendre Wavelets*, Appl. Math Comput., 175(1)(2006), 574-580.

[9] R. K. Pandey, O. P. Singh and V. K. Singh, *Efficient algorithms to solve singular integral equations of Abel type*, Comput. Math. Appl., 57(2009), 664-676.

[10] S. Kumar, O. P. Singh and S. Dixit, *Homotopy perturbation method for solving system of generalized Abel integral equations*, Appl. Appl. Math., 6(2009), 268-283.

[11] S. Kumar and O. P. Singh, *Numerical inversion of Abel integral equation using homotopy perturbation method*, Z. Naturforsch., 65a(2010), 677-682.

[12] D. Loonker and P. K. Banerji, *On the Solution of Distributional Abel Integral Equation by Distributional Sumudu Transform*, International Journal of Mathematics and Mathematical Sciences, (2011), Article ID 480528.

[13] Z. Avazzadeh, B. Shafiee and G. B. Logmani, *Fractional Calculus for Solving Abel Integral Equations using Chebyshev Polynomials*, Applied Mathematical Sciences, 5(45)(2011), 2207-2216.

[14] M. Alipour and D. Rostamy, *Bernstein Polynomials for solving Abel Integral Equation*, J. Math. Comput. Sci., 3(4)(2011), 403-412.

[15] M. Khan and M. A. Gondal, *A reliable treatment of Abel second kind singular integral equations*, Appl. Math. Lett., 25(11)(2012), 1666-1670.

[16] M. Khan, M. A. Gondal and S. Kumar, *A New Analytical Solution Procedure for Nonlinear Integral Equations*, Mathematical and Computer Modelling, 55(2012), 1892-1897.

[17] K. K. Singh, R. K. Pandey, B. N. Mandal and N. Dubey, *An Analytical Method for Solving Singular Integral Equation of Abel type*, Procedia Engineering, 38(2012), 2726-2738.

[18] K. S. Aboodh, *The New Integral Transform 'Aboodh Transform'*, Global Journal of Pure and Applied Mathematics, 9(1)(2013), 35-43.

[19] S. Kumar, A. Kumar, D. Kumar, J. Singh and A. Singh, *Analytical solution of Abel integral equation arising in astrophysics via Laplace transform*, Journal of the Egyptian Mathematical Society, 23(2015), 102-107.

[20] S. Jahanshahi, E. Babolian, D. F. M. Torres and A. Vahidi, *Solving Abel Integral Equations of first kind via Fractional Calculus*, Journal of King Saud University-Science, 27(2015), 161-167.

[21] K. Abdelilah and H. Sedeeg, *The New Integral Transform 'Kamal Transform'*, Advances in Theoretical and Applied Mathematics, 11(4)(2016), 451-458.

[22] D. Loonker and P. K. Banerji, *Solution of the integral equations and Laplace-Stieltjes transform*, Palestine Journal of Mathematics, 5(1)(2016), 43-49.

[23] A. Mahgoub, *The New Integral Transform 'Mahgoub Transform'*, Advances in the Theoretical and Applied Mathematics, 11(4)(2016), 391-398.

[24] E. Fathizadeh, R. Ezzati and K. Maleknejad, *CAS Wavelet Function Method for Solving Abel Equations with Error Analysis*, International Journal of Research in Industrial Engineering, 6(04)(2017), 350-364.

[25] E. Zarei and S. Noeiaghdam, *Solving Generalized Abel Integral Equations of the first and second kinds via Taylor-Collocation method*, Math N.A, (2018).

[26] S. Mondal and B. N. Mondal, *Solution of Abel Integral Equation using Differential Transform Methods*, Journal of Advances in mathematics, 14(01)(2018), 7521-7532.

[27] Kh. M. Shadimetov and B. S. Daliyev, *Optimal Quadrature Formulas for Approximate Solution of Abel Integral Equation*, Uzbek Mathematical Journal, 2(2020), 157-163.

[28] C. P. Pandey, P. Phukan and K. Moungkang, *Solution of Integral Equations by Bessel Wavelet Transform*, Advances in mathematics: Scientific Journal, 10(04)(2021), 2245-2253.

[29] M. Jyotirmoy, M. M. Panja and B. N. Mandal, *Approximate Solution of Abel Integral Equation in Daubechies Wavelet basis*, CUBO, A Mathematical Journal, 23(02)(2021), 245-264.

[30] N. Sharma and U. Singh, *An Approximate Solution of generalized Abel Integral Equation of second kind in coupling of various transform methods*, Advances and Applications in Mathematical Sciences, 23(04)(2024), 339-352.

[31] N. Sharma and U. Singh, *Solution of the Abel Integral Equation of second kind in coupling of Laplace transform method*, Advances and Applications in Mathematical Sciences 24(7-8)(2025), 57-68.