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New Separation Axioms in Generalized Bitopological Space

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Abstract

The main purpose of this paper was to continue the study of separation axioms which is introduced in part I [1]. Whereas the part I [2] was devoted to the axioms iPT-ordered spaces, $i = 0, 1, 2$, in the part II the axioms iPT^{-} -ordered spaces, $i = 3, 4, 5$ and i PR^{-}-ordered spaces, $j = 2, 3, 4$ are introduced and studied. Clearly, if $I = \{\$\}$ in these axioms, then the previous axioms [3] coincide with the present axioms. Therefore, the current work is a generalization of the previous one. In addition, the relationships between these axioms and the previous one axioms have been obtained. Some examples are given to illustrate the concepts. Moreover, some important results related to these separations have been obtained.

Keywords: Ideal bitopological ordered spaces; I-increasing (I-decreasing) sets; IP-regular ordered spaces; IP-normal ordered spaces; IP-completely normal ordered spaces.

1. Introduction

A bitopological space (X, τ_1, τ_2) was introduced by Kelly in 1963 [4], as a method of generalizes topological spaces (X, τ). Every bitopological space (X, τ_1, τ_2) can be regarded as a topological space (X, τ) if $\tau_1 = \tau_2 = \tau$. Furthermore, he extended some of the standard results of separation axioms of topological spaces to bitopological spaces. Thereafter, a large number of papers have been written to generalize topological concepts to bitopological setting. In 1971 Singal and Singal [3] presented and studied the bitopological ordered space (X, τ_1, τ_2, R). It was a generalization of the study of general topological space, bitopological space and topological ordered space. Every bitopological ordered space (X, τ_1, τ_2, R) can be regarded as a bitopological space (X, τ_1, τ_2) if R is the equality relation "Δ". Singal and Singal [5] studied separation axioms PT$_i$-ordered spaces, $i = 0, 1, 2, 3, 4$ and PR$_j$-ordered spaces, $j = 2, 3$ in bitopological ordered spaces. After that time many authors have already been studied the bitopological ordered spaces.

Abo Elhamayel Abo Elwafa introduced separation axioms P-completely normal ordered spaces, PT$_j$-ordered spaces and PR$_j$-ordered spaces, $j = 0, 1$ on the bitopological ordered spaces. Kandil et

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al. studied the bitopological ordered spaces by using the supra-topological ordered spaces. They introduced new separations axioms Pτ_i-ordered spaces, $i = 0, 1, 2$ which was a generalization of previous one. In 2014 Kandil et al. [1] used the concept of ideal I to introduce and study the ideal bitopological ordered spaces $(X, \tau_1, \tau_2, \mathrm{R}, \mathrm{I})$. Clearly, if $\mathrm{I} = \{\phi\}$, then every ideal bitopological ordered space is bitopological ordered space. Therefore, these spaces are generalization of the bitopological ordered spaces and bitopological spaces. They used the notion of I-increasing (decreasing) sets and introduced separation axioms IPT$_i$-ordered spaces, $(i = 0, 1, 2)$ in ideal bitopological ordered spaces. The present paper is a continuation of So, the aim of the present paper was to study the separation axioms IPT$_i$-ordered spaces, $i = 3, 4, 5$ and IPR$_j$-ordered spaces, $j = 2, 3, 4$ on ideal bitopological ordered space $(X, \tau_1, \tau_2, \mathrm{R}, \mathrm{I})$. The current separation axioms are based on the notion of I-increasing (decreasing) sets. Comparisons between these axioms and the axioms in have been obtained. The importance of the current study is that the new spaces are more general because the old one can be obtained from the current spaces when $\mathrm{I} = \{\phi\}$. Finally, we show that the properties of being IPT$_i$-ordered spaces, $i = 3, 4, 5$ and IPR$_j$-ordered spaces, $j = 2, 3, 4$ are preserved under a bijective, P-open and order (reverse) embedding mappings.

2. Preliminaries

Definition 2.1. *A relation R on a non-empty set X is said to be:*

1. *reflexive if $(x, x) \in R$, for every $x \in X$,*
2. *symmetric if $(x, y) \in R \implies (y, x) \in R$, for every $x, y \in X$,*
3. *transitive if $(x, y) \in R$ and $(y, z) \in R \implies (x, z) \in R$, for every $x, y, z \in X$,*
4. *antisymmetric if $(x, y) \in R$ and $(y, x) \in R \implies x = y$, for every $x, y \in X$,*
5. *preorder relation if it is reflexive and transitive,*
6. *partial order relation if it is reflexive, antisymmetric and transitive, and the pair (X, R) is said to be a partially ordered set (or poset, for short).*

Definition 2.2. *For a non-empty set X and a partially order relation R on X, the pair (X, R) is said to be a partially ordered set (or poset, for short).*

Definition 2.3. *Let (X, R) be a poset. A set $A \subseteq X$ is said to be:*

1. *decreasing if for every $a \in A$ and $x \in X$, $xRa \implies x \in A$,*
2. *increasing if for every $a \in A$ and $x \in X$, $aRx \implies x \in A$.*

Definition 2.4. *A mapping $f : (X, R) \to (Y, R^*)$ is said to be:*

1. *increasing (decreasing) if for every* $x_1, x_2 \in X$, $x_1 R x_2 \implies f(x_1) R^* f(x_2) (f(x_2) R^* f(x_1))$,

2. *order embedding if for every* $x_1, x_2 \in X$, $x_1 R x_2 \iff f(x_1) R^* f(x_2)$,

3. *order reverse embedding if for every* $x_1, x_2 \in X$, $x_1 R x_2 \iff f(x_2) R^* f(x_1)$.

Definition 2.5. *Let X be a non-empty set. A class τ of subsets of X is called a topology on X if τ satisfies the following axioms:*

1. $X, \phi \in \tau$,

2. *arbitrary union of members of τ is in τ,*

3. *the intersection of any two sets in τ is in τ.*

The members of τ are then called τ-open sets, or simply open sets. The pair (X, τ) is called a topological space. A subset A of a topological space (X, τ) is called a closed set if its complement A^c is an open set.

Definition 2.6. *Let (X, τ) be a topological space and $A \subseteq X$. Then τ-$cl(A) = \bigcap\{F \subseteq X : A \subseteq F$ and F is closed$\}$, is called the τ-closure of a subset $A \subseteq X$.*

Definition 2.7. *A bitopological space (bts, for short) is a triple (X, τ_1, τ_2), where τ_1 and τ_2 are arbitrary topologies for a set X.*

Definition 2.8. *A function $f : (X, \tau_1, \tau_2) \to (X_2, \eta_1, \eta_2)$ is said to be:*

1. *P. continuous (respectively P. open, P. closed) if $f : (X_1, \tau_i) \to (X_2, \eta_i)$, $i = 1, 2$ are continuous (respectively open, closed).*

2. *P. homeomorphism if $f : (X_1, \tau_i) \to (X_2, \eta_i)$, $i = 1, 2$ are homeomorphism.*

Definition 2.9. *A bitopological ordered space (bto-space, for short) has the form (X, τ_1, τ_2, R), where (X, R) is a poset and (X, τ_1, τ_2) is a bts. The notion a R b means that a not related to b, i.e., $a \not R\, b \iff (a, b) \notin R$.*

Definition 2.10. *A bto-space (X, τ_1, τ_2, R) is said to be:*

1. *Lower pair wise T_1(LPT_1, for short) - ordered space if for every $a, b \in X$ such that $a\mathrm{R}b$, there exists an increasing τ_i-open set U contains a such that $b \notin U$, $i = 1$ or 2.*

2. *Upper pair wise T_1(UPT_1, for short) - ordered space if for every $a, b \in X$ such that $a\mathrm{R}b$, there exists a decreasing τ_i-open set V contains b such that $a \notin V$, $i = 1$ or 2.*

3. *Pair wise T_1(PT_1, for short), if it is LPT_1 and UPT_1-ordered space.*

Definition 2.11. *A* $\mathbf{b\tau o}$*-space $(X, \tau_1, \tau_2, \mathrm{R})$ is said to be:*

1. *Lower pair wise regular (LPR_2, for short) ordered space if for all decreasing τ_i-closed set F and for all $a \notin F$, there exist increasing τ_i-open set U and decreasing τ_j-open set V such that $a \in U, F \subseteq V$ and $U \cap V = \phi$.*

2. *Upper pair wise regular* (PUR$_2$, *for short) ordered space if for all increasing* τ_i*-closed set* F *and for all* $a \notin F$*, there exist decreasing* τ_i*-open set* U *and increasing* τ_j*-open set* V *such that* $a \in U, F \subseteq V$ *and* $U \cap V = \phi$.

3. *Pair wise regular* (PR$_2$*, for short) ordered space if it is* LPR$_2$ *and* UPR$_2$.

Definition 2.12. *A* PR$_2$*-ordered space which is also* PT$_1$*- ordered space is said to be* PT$_3$*-ordered space.*

Definition 2.13. *A* **bτo**-*space* $(X, \tau_1, \tau_2, \mathrm{R})$ *is said to be* PR$_3$*-ordered space if for all increasing* τ_i*-closed set* F_1 *and decreasing* τ_j*-closed set* F_2 *such that* $F_1 \cap F_2 = \phi$ *there exist an increasing* τ_i*-open set* U *and a decreasing* τ_j*-open set* V *such that* $F_1 \subseteq U, F_2 \subseteq V$ *and* $U \cap V = \phi$.

Definition 2.14. *A* PR$_3$*-ordered space which is also a* PT$_1$*- ordered space is said to be a* PT$_4$*- ordered space.*

Definition 2.15. *Two sets* A *and* B *in* (X, τ_1, τ_2) *are said to be* P*-separated sets if* $A \cap \tau_i$*-cl* $(B) = \phi$ *and* τ_j*-cl* $(A) \cap B = \phi$, $i, j = 1, 2$, $i \neq j$.

Definition 2.16. $(X, \tau_1, \tau_2, \mathrm{R})$ *is said to be a* P*-completely normal ordered spaces* (PR$_5$*-ordered spaces, for short) if for any two* P*-separated subsets* A *and* B *of* X *such that* A *is an increasing and* B *is a decreasing there exist an increasing* τ_i*-open set* $U, A \subseteq U$ *and a decreasing* τ_j*-open set* $V, B \subseteq V$ *such that* $U \cap V = \phi$.

Definition 2.17. *A* PR$_4$*-ordered space which is a* PT$_1$*- ordered space is called a* PT$_5$*-ordered space.*

Definition 2.18. *A non-empty collection* $\mathbb{I}$ *of subsets of a set* X *is called an ideal on* X*, if it satisfies the following conditions:*

1. $A \in I$ *and* $B \in I \implies A \cup B \in I$.

2. $A \in I$ *and* $B \subseteq A \implies B \in I$.

Definition 2.19. *Let* (X, R) *be a poset and* I *be an ideal on* X*. A set* $A \subseteq X$ *is said to be:*

1. *I-decreasing if* $Ra \cap A \in I$ *for every* $a \in A$*, where* $Ra = \{h : (h, a) \in R\}$.

2. *I-increasing if* $aR \cap A \in I$ *for every* $a \in A$*, where* $aR = \{b : (a, b) \in R\}$.

Theorem 2.20. *Let* $f : (X, R, I) \to (Y, R^*, f(I))$ *be an objective function and order embedding. Then for every I-increasing (decreasing) subset* A *of* X*,* $f(A)$ *is a* $f(I)$*-increasing (decreasing) subset of* Y.

Corollary 2.21. *Let* $f : (X, R, I) \to (Y, R^*, f(I))$ *be an objective function and order embedding. If* B *is a* $f(I)$*-increasing (decreasing) subsets of* Y*, then* $f^{-1}(B)$ *is an I-increasing (decreasing) subset of* X.

Theorem 2.22. *Let* $f : (X, R, I) \to (Y, R^*, f(I))$ *be an objective function and reverse embedding. Then for every I-increasing (decreasing) subset* A *of* X*,* $f(A)$ *is a* $f(I)$*-decreasing (increasing) subset of* Y.

Corollary 2.23. *Let* $f : (X, R, I) \to (Y, R^*, f(I))$ *be an objective function and order reverse embedding. If* B *is a* $f(I)$*-increasing (decreasing) subsets of* Y*, then* $f^{-1}(B)$ *is an I-decreasing (increasing) subset of* X.

Definition 2.24. *A space $(X, \tau_1, \tau_2, R, I)$ is called an ideal bitopological ordered space if (X, τ_1, τ_2, R) is a bitopological ordered space and I is an ideal on X.*

Definition 2.25. *An ideal bitopological ordered space $(X, \tau_1, \tau_2, R, I)$ is said to be:*

1. *I lower $PT_i(ILPT_i$, for short) ordered space if for every $a, b \in X$ such that aRb, there exists an I-increasing τ_i-open set U such that $a \in U$ and $b \in U$, $i = 1$ or 2.*

2. *I upper $PT_i(IUPT_i$, for short) ordered space if for every $a, b \in X$ such that $a\bar{R}b$, there exists an I-decreasing τ_i-open set V such that $h \in V$ and $a \notin V$, $i = 1$ or 2.*

3. *IPT_i-ordered space if it is $ILPT_i$ and $IUPT_i$ ordered space.*

3. IP-regularity and IP-normality Ordered Spaces in Ideal Bitopological Ordered Spaces

The aim of this section was to use (the notion of I-increasing (I-decreasing) sets which based on the ideal I, to introduce new separation axioms IPT$_i$-ordered spaces , $(i = 3, 4, 5)$ and IPR$_j$-ordered spaces, $j = 2, 3, 4$ on the space $(X, \tau_1, \tau_2, R, I)$. Moreover, the relationship between these axioms and the axioms in has been obtained. Some examples are given to illustrate the concepts. Furthermore, some important results related these separations have been studied.

Definition 3.1. *An ideal bitopological space (X, τ_1, τ_2, I) is said to be:*

1. *I lower pair wise regular ($ILPR_2$, for short) ordered space if for every I-τ_2-decreasing τ_1-closed set F and for every $a \notin F$, there exist an I-τ_2-increasing τ_1-open set U and a I-τ_2-decreasing τ_1-open set V such that $a \in U, F \subseteq V \subseteq I$ and $U \cap V \subseteq I$.*

2. *I upper pair wise regular ($IUPR_2$, for short) ordered space if for every I-τ_1-increasing τ_2-closed set F and for every $a \notin F$, there exist a I-τ_1-decreasing τ_2-open set U and an I-τ_1-increasing τ_2-open set V such that $a \in U, F \subseteq V \subseteq I$ and $U \cap V \subseteq I$.*

3. *I pair wise regular (IPR_2, for short) ordered space if it is $ILPR_2$ and $IUPR_2$.*

Definition 3.2. *A IPR_2-ordered space is which also IPT_1-ordered space is said to be IPT_2-ordered space.*

Remark 3.3. *It Should be Noted that $I = \{\phi\}$ in Definitions 3.1 and 3.2 then we get Definition 2.11 and 2.12, So the current Definition 3.1 and 3.2 more general.*

Example 3.4. *Let $X = \{1,2,3,4\}$, $R = \Delta \cup \{(1,4),(1,3),(2,3),(4,3)\}$, $I = \{\phi, \{1\}, \{3\}, \{1,3\}\}$, $\tau_1 = \{X, \phi, \{1\}, \{4\}, \{1,4\}, \{1,3,4\}\}$, $\tau_2 = \{X, \phi, \{2,3\}\}$.*

It is clear that $(X, \tau_1, \tau_2, R, I)$ is ILPR$_2$-ordered space but it is not IUPR$_2$-ordered space.

Example 3.5. *In Example 3.1 take $I = \{\phi, \{1\}, \{2\}, \{4\}, \{1,2\}, \{1,4\}, \{2,4\}, \{1,2,4\}\}$, $\tau_1 = \{X, \phi, \{2\}, \{3\}, \{2,3\}, \{1,2,3\}\}$, $\tau_2 = \{X, \phi, \{1,4\}\}$. It is clear that $(X, \tau_1, \tau_2, R, I)$ is $IUPR_2$-ordered space but it is not $ILPR_2$-ordered space.*

The following example shows that $(X, \tau_1, \tau_2, R, I)$ is IPR$_2$- ordered space, but it is not IPT$_2$-ordered space.

Example 3.6. *In Example 3.1 take* $I = \{\phi, \{1\}, \{2\}, \{4\}, \{1,2\}, \{1,4\}, \{2,4\}, \{1,2,4\}\}$, $\tau_1 = \{X, \phi, \{1,2\}, \{1,2,3\}, \{1,2,4\}\}$, $\tau_2 = \{X, \phi, \{2,3\}\}$. *It is clear that* $(X, \tau_1, \tau_2, R, I)$ *is IPR$_2$-ordered space.*

Example 3.7. *In Example 3.1 take* $I = \{\phi, \{1\}, \{2\}, \{4\}, \{1,2\}, \{1,4\}, \{2,4\}, \{1,2,4\}\}$, $\tau_1 = \{X, \phi, \{1\}, \{3\}, \{1,2\}, \{1,3\}, \{1,4\}, \{3,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}\}$, $\tau_2 = \{X, \phi, \{2,3\}\}$.

The following proposition studies the relationship between the current Definitions 3.1 and 3.2 and the previous Definitions 2.11 and 2.12.

Proposition 3.8. *Let* $(X, \tau_1, \tau_2, R, I)$ *be an ideal bitopological ordered space. Then*

$$PR_2 \wedge PT_1\text{-ordered space} \implies IPR_2 \wedge IPT_1\text{-ordered space}$$
$$PR_2\text{-ordered space} \implies IPR_2\text{-ordered space}.$$

Proof. The proof follows directly from the definitions of IPT$_\tau$-ordered spaces, PT$_\tau$-ordered spaces, IPR$_{\tau_2}$-ordered spaces and PR$_{\tau_2}$-ordered spaces. □

Example 3.9. *Shows that* $(X, \tau_1, \tau_2, R, I)$ *is IPT$_{\tau_2}$-ordered space, but it is PT$_{\tau_1}$-ordered space.*

The following example shows that $(X, \tau_1, \tau_2, R, I)$ is IPR$_{\tau_2}$-ordered space, but it is PR$_{\tau_2}$-ordered space.

Example 3.10. *In Example 3.1 take* $I = \{\phi, \{1\}, \{2\}, \{4\}, \{1,2\}, \{1,4\}, \{2,4\}, \{1,2,4\}$, $\tau_1 = \{X, \phi, \{2\}, \{1,2\}, \{2,3\}, \{1,2,3\}, \{1,2,4\}\}$ *and* $\tau_2 = \{X, \phi, \{2,3\}\}$. *It is clear that* $(X, \tau_1, \tau_2, R, I)$ *is IPR$_{\tau_2}$-ordered space, but it is not PR$_{\tau_2}$-ordered space.*

The following theorem shows that the property of being IPR$_{\tau_2}$-ordered space is preserved by a objective, ordered embedding (order reverse embedding) and P-homeomorphism mapping.

Theorem 3.11. *If* $(X, \tau_1, \tau_2, R, I)$ *is IPR$_{\tau_2}$-ordered space,* $f : (X, \tau_1, \tau_2, R, I) \to (Y, \eta_1, \eta_2, R^*, f(I))$ *is a objective, ordered embedding (order reverse embedding) and P-homeomorphism mapping. Then* $(Y, \eta_1, \eta_2, R^*, f(I))$ *is* $f(I)PR_{\tau_2}$*-ordered space.*

Proof. We prove the theorem in the case of ordered embedding and the other case is similar. Let H be a I-decreasing (increasing) η_1-closed subset of Y, $y \in H$. Since, f is an onto function, then there exists $x \in X$ such that $x = f^{-1}(y)$. Since, f is P-continuous, $f^{-1}(H)$ is τ_1-closed. By Corollary 2.1, $f^{-1}(H)$ is a I-decreasing (increasing) τ_1-closed subset of X and $x \notin f^{-1}(H)$. As $(X, \tau_1, \tau_2, R, I)$ is IPR$_{\tau_2}$-ordered space, there exist an I-increasing (decreasing) τ_1-open set U contains x and a I-decreasing (increasing) τ_2-open set V such that $f^{-1}(H) - V \in I$ and $U \cap V \in I$. Since, f is P-open and by Theorem 2.1, $f(U)$ is a $f(I)$-increasing (decreasing) η_1-open set contains $y = f(x)$, $f(V)$ is a $f(I)$-decreasing (increasing)

η_2-open set such that $f(f^{-1}(H) - V) = H - f(V) \in f(I)$ and $f(U) \cap f(V) = f(U \cap V) \in f(I)$. Hence, $(Y, \eta_1, \eta_2, R^*, f(I))$ is $f(I)\mathrm{PR}_{\tau_2}$-ordered space. □

The following corollary shows that the property of being IPT_{τ_2}-ordered spaces is preserved by a objective, ordered embedding (order reverse embedding) and P-homeomorphism mapping.

Corollary 3.12. *If $(X, \tau_1, \tau_2, R, I)$ is IPT_{τ_2}-ordered space, $f : (X, \tau_1, \tau_2, R, I) \rightarrow (Y, \eta_1, \eta_2, R^*, f(I))$ is a bijective, ordered embedding (order reverse embedding) and P-homeomorphism mapping. Then $(Y, \eta_1, \eta_2, R^*, f(I))$ is $f(I)T_{\tau_2}$-ordered space.*

Definition 3.13. *$(X, \tau_1, \tau_2, R, I)$ is said to be IP-normal ordered space (IPR_{τ_2}-ordered space, for short) if for all I-increasing τ_1-closed set F_1 and I-decreasing τ_2-closed set F_2 such that $F_1 \cap F_2 \in I$, there exists an I-increasing τ_1-open set U and a I-decreasing τ_2-open set V such that $F_1 - U \in I$, $F_2 - V \in I$ and $U \cap V \in I$.*

Example 3.14. *In Example 3.1 take $I = \{\phi, \{1\}, \{2\}, \{4\}, \{1,2\}, \{1,4\}, \{2,4\}, \{1,2,4\}\}$, $\tau_1 = \{X, \phi, \{1,4\}, \{1,3,4\}\}, \tau_2 = \{X, \phi, \{1\}, \{2,4\}, \{1,2,4\}, \{2,3,4\}\}$. It is clear that $(X, \tau_1, \tau_2, R, I)$ is IPR_3-ordered space.*

Definition 3.15. *A IPR_5-ordered space which is also a IPT_5-ordered space is said to be a IPT_5_ordered space.*

Example 3.16. *Shows that $(X, \tau_1, \tau_2, R, I)$ is IPR_5-ordered space, but it is not IPT_5-ordered space.*

The following example shows that if $(X, \tau_1, \tau_2, R, I)$ is IPR_5-ordered space, then it is not necessary to be IPT_5-ordered space.

Example 3.17. *In Example 3.1 take $I = \{\phi, \{1\}, \{2\}, \{4\}, \{1,2\}, \{1,4\}, \{2,4\}, \{1,2,4\}\}$, $\tau_1 = \{X, \phi, \{1\}, \{4\}, \{2,4\}, \{1,3,4\}, \{1,2,3,4\}\}$, $\tau_2 = \{X, \phi, \{1\}, \{1,2\}, \{1,2,3\}, \{1,2,4\}\}$. It is clear that $(X, \tau_1, \tau_2, R, I)$ is IPT_4-ordered space.*

Remark 3.18. *It should be noted that if $L(\phi)$ in Definitions 3.3 and 3.4, then we get Definitions 2.13 and 2.14 given by signal and signal and for Definitions 2.13 and 2.14 due to signal and signal are a special case of the current Definitions 3.3 and 3.4.*

The following proposition studies the relationship between Definitions 3.3 and 3.4, and the previous Definitions 2.13 and 2.14.

Proposition 3.19. *Let $(X, \tau_1, \tau_2, R, I)$ be an ideal bitopological ordered space. Then*

$$PR_3 \wedge PT_1\text{- ordered space } \Rightarrow IPR_3 \wedge IPT_1\text{- ordered space.}$$

$$PR_3\text{- ordered space } \rightarrow IPR_3\text{- ordered space}$$

Proof. The proof follows directly from the definitions of IPT_4-ordered space, PT_4-ordered space, PR_3-ordered space and IPR_3-ordered spaces. □

Example 3.20. *Shows that $(X, \tau_1, \tau_2, R, I)$ is IPR_3-ordered space, but it is not PR_3-ordered space.*

Example 3.21. *shows that $(X, \tau_1, \tau_2, R, I)$ is IPT_4-ordered space, but it is not PT_4-ordered space.*

The following theorem shows that the property of being IPR_3-ordered space is preserved by a objective, ordered embedding (order reverse embedding) and P-homeomorphism mapping.

Theorem 3.22. *Let $(X, \tau_1, \tau_2, R, I)$ is a IPR_3-ordered space, $f : (X, \tau_1, \tau_2, R, I) \to (Y, \eta_1, \eta_2, R^*, J)$ is a objective, order embedding (order reverse embedding) and P-homeomorphism mapping. Then $(Y, \eta_1, \eta_2, R^*, J)$ is $f(I)PR_3$-ordered space.*

Proof. Let H_1 be a f(I)-decreasing (increasing) $\eta_j$closed subset of Y and H_2 be a f(I)-increasing (decreasing) $\eta_j$closed subset of Y such that $H_1 \cap H_2 \in f(I)$. Since f is P-continuous, $f^{-1}(H_1)$ is a τ_i-closed subset of X and $f^{-1}(H_2)$ is τ_j-closed subsets of X, and by Corollary 2.1, $f^{-1}(H_1)$ is a I-increasing (decreasing)τ_j- closed subset of X. Now, we have $f^{-1}(H_1) \cap f^{-1}(H_2) = f^{-1}(H_1 \cap H_2) \in I$. Since, (X, τ_1, τ_2, R) is IPR_3-ordered space, then there exist an I-increasing (decreasing) τ_j-open set U, $f^{-1}(H_1) - U \in I$ and a I-decreasing (increasing) τ_j-open set V, $f^{-1}(H_2) - V \in I$ such that $U \cap V \in I$. Since, f is P-open and by Theorem 2.1, $f(U)$ is an $f(I)$-increasing (decreasing) η_j-open set , $f(f^{-1}(H_1) - U) = H_1 - f(U) \in f(I)$ and $f(V)$ is a $f(I)$-decreasing (increasing) η_i-open set, $f(f^{-1}(H_2) - V) = H_2 - f(V) \in f(I)$ such that $f(U) \cap f(V) = f(U \cap V) \in f(I)$. Hence, $(Y, \eta_1, \eta_2, R^*, f(I))$ is $f(I)$PRf-ordered space. □

Corollary 3.23. *If $(X, \tau_1, \tau_2, R, I)$ isIPT_4-ordered space, $f : (X, \tau_1, \tau_2, R, I) \to (Y, \eta_1, \eta_2, R^*, f(I))$ is a f-injective, order embedding (order reverse embedding) and P-continuous mapping. Then $(Y, \eta_1, \eta_2, R^*, f(I))$ is f(I) PT_4-ordered space.*

Definition 3.24. *Two sets A and B in (X, τ_1, τ_2, I) are said to be IP-separated sets if $A \cap \tau_j\text{-cl}(B) \in I$ and $\tau_j\text{-cl}(A) \cap B \in I$.*

Example 3.25. *Let $(R, \tau_1, \tau_U, R, I)$ be an ideal bitopological ordered space in which R is the real numbers and R is the usual order; $\tau_1 = \{(-\infty, a) : a \in R\} \cup \{\emptyset, R\}$ is the usual topology, and $I = \{\emptyset, \}(1, \infty)(a, \infty), (a, b], (a, b), [a, b), [a, b], c, \dots\}$ where $1 < a < b$, $1 < c < \infty$, $a, b, c \in R$ and let$A, B \subseteq R = \emptyset$ such that $a = (1, -\infty)$ and $B = (-\infty, 1)$. It is clear that A and B are IP-separated sets as $\tau_U\text{-cl}(A) = [1, \infty), \tau_1\text{-cl}(B) = R$ and so $\tau_1\text{-cl}(A) \cap B = \emptyset \in I$ and $\tau_U\text{-cl}(A) \cap B = \phi \in I$. and $\tau_1\text{cl}(B) \cap A = (1, \infty) \in I$.*

Definition 3.26. *An ideal bitopological ordered space $(X, \tau_1, \tau_2, R, I)$ is said to be **IP-completely normal ordered space** (**IPR_4-ordered space**, for short) if for any two IP-separated subsets A and B of X such that A is an i-increasing set and B is an i-decreasing set, there exist an i-increasing τ_i-open set U, $A \subseteq U$ and a i-decreasing r_j-open set V, $B \subseteq V$ such that $U \cap V \in I$.*

Example 3.27. *Let $(R, \tau_U, \tau_1, R, I)$ be an ideal bitopological ordered space in which R is the real numbers and R is the usual order; τ_U is the usual topology, $\tau_1 = \{(-\infty, a) : a \in R\} \cup \{\emptyset, R\}$ and*

$I = \{\emptyset, (0, \infty), (a, infinity), (a, b), [a, b), [a, b], c, \dots\}$ where $0 \leq a < b$, $0 \leq \infty$, $a, b, c \in R$. Then, it is clear that $(R, \tau_U, \tau_1, R, I)$ is IPR_4*-ordered space.*

Definition 3.28. *A* IPR_4*-ordered space which is* IPT_1*-ordered space is called a* IPT_5*-ordered space.*

Remark 3.29. *It should be noted that if $I = \{\emptyset\}$ in Definition 3.5-3.7 then we get Definition 2.15-2.17 [3,17,18] are special case of the current Definition 3.5-3.7.*

The following proposition studies the relationship between Definitions 3.5-3.7 and 2.15-2.17 given in [3,17,18].

Proposition 3.30. *Let $(X, \tau_1, \tau_2, R, I)$ be an ideal bitopological ordered space. Then*

$$PR_4 \wedge PT_1\text{-ordered space} \longrightarrow IPR_4l \wedge IPT_1\text{-ordered space}$$
$$PR_4\text{−-ordered space} \longrightarrow IPR_4\text{−-ordered space}$$
$$P\text{-separated} \longrightarrow IP\text{-separated}$$

Proof. The proof follows directly from the definitions of IPT_5-ordered space, PT_5-ordered space, PR_4-ordered space and IPR_4-ordered spaces. □

Example 3.31. *Shows that $A = (1, \infty)$ and $B = (-\infty, 1)$ are IP-separated sets, but it is not P-separated sets as $tau_1 cl(B) \cap A = (1, \infty) = \phi$*

Example 3.32. *Let $(R, \tau_u, \tau_1, R, I)$ is IPR_4-ordered space, but it is not IPT_5-ordered space, as it is not IPT_1-ordered space, for $3R2$ all I-decreasing sets which are containing 2, also, contain 3.*

The following example shows that $(X, \tau_1, \tau_2, R, I)$ is IPR_4-ordered space, but it is not $PR4$−-ordered space.

Example 3.33. *Let $(R, \tau_u, \tau_u, R, I)$ be an ideal bitopological ordered space in which R is the real numbers and R is the usual order, τ_u is the usual topology, $\tau_u = \{(a, \infty) : a \in R\} \cup \{R, \phi\}$ and $I = \{(\phi, (0, \infty), [a, \infty), (a, b), [a, b), (a, b], [a, b], c\}$ where $0 \leq a < b, 0 \leq c < \infty, a, b, c \in R$. It is clear that $(R, \tau_u, \tau_u, R, I)$ is IPR_4-ordered space, but it is not PR_4-ordered space, as there exist $A, B \subseteq R$ where $A = [1, \infty)$ and $B = (-\infty, 0)$ which are two P-separated sets, A is an I-increasing set and B is a decreasing set, but the increasing τ_u-open superset of A is $U = (d, \infty), d \leq 1$, the only decreasing τ_u-open superset of B is $V = R$ and $U \cap V = (d, \infty) \cap R = (d, \infty) \neq \phi, d \leq 1$.*

Theorem 3.34. *Every IPR_4-ordered space is IPR_3-ordered space.*

Proof. Let $(X, \tau_1, \tau_2, R, I)$ be IPR_4-ordered space, A and B be subsets of X such that $A \cap \bar{B} \in I, A$ is an I-increasing τ_i-closed set and B is a I-decreasing τ_j-closed set. Then $\tau_i cl(A) \cap \bar{B} \in I$ and $\bar{A} \cap \tau_j cl(B) \in I$. Consequently, A and B being two IP-separated subsets of the IPR_4-ordered space $(X, \tau_1, \tau_2, R, I)$. Therefore, there exist an I-increasing τ_i-open set $U, A \subseteq U$ and a I-decreasing τ_j-open set $V, B \subseteq V$ such that $U \cap V \in I$. Hence, $(X, \tau_1, \tau_2, R, I)$ be a IPR_3-ordered space. □

The following example shows that $(X, \tau_1, \tau_2, R, I)$ is IPR_3-ordered space, but it is not IPR_4-ordered space.

Example 3.35. *In **Example 3.1** take* $I = \{\phi, \{3\}, \{4\}, \{3,4\}$, $\tau_1 = \{X, \phi, \{1,4\}, \{1,3,4\}\}$, $\tau_2 = \{X, \phi, \{2\}, \{1,2\}, \{2,4\}, \{1,2,4\}, \{2,3,4\}\}$. *It is clear that* $(X, \tau_1, \tau_2, R, I)$ *is* IPR_3*-ordered space, but it is not* IPR_4*-ordered space as* $A = \{2,3\}, B = \{1\}$ *are IP-separated sets, A is an I-increasing set and B is a I-decreasing set, the only I-increasing* τ_1*-open superset of A is X and the I-decreasing* τ_2*-open supersets of B are* $X, \{1,2\}, \{1,2,4\}$ *and their intersection is* $X, \{1,2\}, \{1,2,4\} \notin I$.

Corollary 3.36. *Every* IPT_5*-ordered space is* IPT_4*-ordered space.*

Example 3.37. *shows that* $(X, \tau_1, \tau_2, R, \mathfrak{I})$ *is* IPT_4*-ordered space, but it is not* IPT_5*-ordered space as it is not* IP_4*-ordered space.* $A = (3,4)$, $B = (1,2,4)$ *are IP-separated sets, A is an* $\mathcal{I}$*-increasing set and B is a* $\mathcal{I}$*-decreasing set, the only* $\mathcal{I}$*-increasing* $\tau 2$*-open subset of A is X and the only* $\mathcal{I}$*-decreasing* $\tau 1$*-open super subset of B is X and their intersection is* $X \notin \mathcal{I}$.

Theorem 3.38. *The property of being* IPR_4*-ordered space is preserved by a objective, order embedding and P-homeomorphism mapping.*

Proof. Let $f : (X, \tau_1, \tau_2, R, I) \longrightarrow (Y, \eta_1, \eta_2, R^*, f(I))$ be a objective, order embedding and P-homeomorphism mapping. Let A and B be two $f(I)P$-separated subsets of Y such that A is $f(I)$-increasing set, B is a $f(I)$-decreasing set. Then $A \cap \eta_j - \text{cl}(B) \in f(I)$ and $\eta_i - \text{cl}(A) \cap B \in f(I)$. Now $f^{-1}(A)$ is an I-increasing and $f^{-1}(B)$ is a I-decreasing, $(f^{-1}(A)) \cap \tau_j - clf^{-1}(B) \subseteq f^{-1}A \cap f^{-1}(\eta_j - \text{cl}(B)) = f^{-1}(A) \cap \eta_j - \text{cl}(B) \in I$ and $\tau_i - \text{cl}(f^{-1}(A)) \cap f^{-1}(B) \subseteq f^{-1}(\eta^i - \text{cl}(A)) \cap f^{-1}(B) = f^{-1}(\eta^i - \text{cl}(A) \cap (B) \in I$. Thus, $f^{-1}(A)$ and $f^{-1}(B)$ are IP-separated subsets of X. So, there exist an I-increasing τ_j-open set U, $f^{-1}(A) \subseteq U$ and a I-decreasing τ^i-open set V, $f^{-1}(B) \subseteq V$ such that $U \cap V \in I$. Therefore, $f(f^{-1}(A)) \subseteq f(U)$ and $f(f^{-1}(B)) \subseteq f(V)$. Thus, $A \subseteq f(U)$ and $B \subseteq f(V)$ and $f(U) \cap f(V) = f(U \cap V) \in f(I)$. Consequently, $(Y, \eta_1, \eta_2, R^*, f(I)$ is $f(\text{I})PR_4$-ordered space. □

Corollary 3.39. *The property of being* IPT_5*-ordered space is preserved by an objective, order embedding and P-homeomorphism mapping.*

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