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Generalized Weighted Composition Operator on Weighted Hardy Spaces

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Abstract

Let φ be an analytic self-map of the open unit disc $\mathbb{D}$ in the finite complex plane $\mathbb{C}$ and ψ be an analytic map of the open unit disc $\mathbb{D}$ to $\mathbb{C}$. Let C_φ , M_ψ *and* D^n be the composition, multiplication and differentiation operators defined by $C_\varphi f = f \circ \varphi$, $M_\psi f = \psi.f$ *and* $D^n f = f^n$ respectively. In this paper, we shall study the boundedness and compactness of the generalized weighted composition operator $W_{\psi\,\varphi}D^n$ defined by $W_{\psi\,\varphi}D^n f = \psi.(f^n \circ \phi)$ on weighted Hardy spaces by using the orthonormal basis of the weighted Hardy spaces $H^2(\beta)$.

Keywords: Weighted composition operator; multiplication operator; differentiation operator; weighted Hardy spaces.

2020 Mathematics Subject Classification: 30G35, 47A10, 47A60.

1. Introduction

Throughout this paper, by $\mathbb{D}$ we shall denote the open unit disc of the finite complex plane $\mathbb{C}$; by $\partial\mathbb{D}$ the boundary of $\mathbb{D}$; by $H(\mathbb{D})$ the set of all complex valued analytic functions on $\mathbb{D}$ and by φ, the analytic self-map of $\mathbb{D}$. Let $\beta = \{\beta_n\}_{n=0}^{\infty}$ be the sequence of positive numbers such that $\beta_0 = 1$ and $\lim\limits_{n\to\infty} \frac{\beta_{n+1}}{\beta_n} = 1$. Then, the weighted Hardy spaces $H^2(\beta)$ is the Banach space of all analytic functions f on the open unit disk $\mathbb{D}$ defined by

$$H^2(\beta) = \left\{ f : z \to \sum_{n=0}^{\infty} a_n z^n \quad s.t \quad \|f\|^2_{H^2(\beta)} = \sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 < \infty \right\}$$

where $\|.\|_{H^2(\beta)}$ is a norm on $H^2(\beta)$. If $\beta \equiv 1$, then $H^2(\beta)$ becomes the classical Hardy space $H^2(D)$. Also, $H^2(\beta)$ is a Hilbert space w.r.t the inner product defined as $\langle f, g\rangle = \sum\limits_{n=0}^{\infty} a_n.\bar{b_n}\beta_n^2$ where $f, g \in H^2(\beta)$. For a detailed discussion on $H^2(\beta)$ one can see [12]. Let ψ be an analytic function from the open unit disc $\mathbb{D}$ to $\mathbb{C}$. Associated with φ and ψ, the linear operator $W_{\psi,\,\varphi}D^n : H(\mathbb{D}) \to H(\mathbb{D})$ is defined by $f \to \psi f^n o \varphi$ and this operator is called the generalised weighted composition operator induced by self-map φ and ψ. Also associated with ψ, the multiplication operator $M_\psi f$ is defined by

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$M_\psi f = \psi f$. It has been known that the composition operator C_φ is bounded on almost all spaces of analytic functions for example see [1–3], and D is usually unbounded on spaces of analytic functions. Recently, the above defined operators has received the attention of many researcher see, for example [5,7–11,16]. In [5], Hibschweiles and Portony defined the product $C_\varphi D$ and DC_φ and studied the boundedness and compactness of these operators between Bergman and Hardy spaces by using the carleson-type measure, where as in [10], the author studied the boundedness and compactness of $C_\varphi D$ and DC_φ between Hardy type spaces. This paper is organised as follows. In the second section, we shall discuss the boundedness of the operator $W_{\psi,\,\varphi}D^n$ on weighted Hardy spaces $H^2(\beta)$, whereas in the third section, we shall study the compactness of the operators $W_{\psi,\,\varphi}D^n$ on weighted Hardy spaces $H^2(\beta)$ and in the final section, we shall give necessary and sufficient condition for the operators $W_{\psi,\,\varphi}D^n$ to be the Hilbert-Schmidt operator on weighted Hardy spaces $H^2(\beta)$.

2. Boundedness of $W_{\psi,\,\varphi}D^n$

In this section, we shall give the necessary and sufficient condition for the boundedness of the operators $W_{\psi,\,\varphi}D^n$ on weighted Hardy spaces $H^2(\beta)$. Recall that a linear operator T on a Hilbert space X is bounded if it takes every bounded set in X into a bounded set in X.

Theorem 2.1. *Let $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ be an analytic map such that $\{\phi^n : n \geq 0\}$ be an orthogonal family. Then, the operator $W_{\psi,\,\varphi}D^n : H^2(\beta) \longrightarrow H^2(\beta)$ is bounded if and only if*

$$||\psi||| \leq \frac{M}{n!}\beta(n).$$

Proof. First, suppose that the operator $W_{\psi,\,\varphi}D^n : H^2(\beta) \longrightarrow H^2(\beta)$ is bounded. Then, there exist $M > 0$ such that

$$||W_{\psi,\,\varphi}D^n f||_{H^2(\beta)} \leq M||f||_{H^2(\beta)} \quad \forall f \in H^2(\beta). \tag{1}$$

Let $f(z) = z^n$ then $f \in H^2(\beta)$ so, from equation (1), we have

$$||W_{\psi,\,\varphi}D^n z^n||_{H^2(\beta)} = ||\psi.n!|| \leq M||z^n||_{H^2(\beta)}.$$

This implies that

$$||\psi|| \leq \frac{M}{n!}\beta(n).$$

Conversely, suppose that

$$||\psi|| \leq \frac{M}{n!}\beta(n). \tag{2}$$

To prove that $W_{\psi,\,\varphi}D^n$ is bounded. Let $f \in H^2(\beta)$ be any element such that $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Then, we have

$$
\begin{aligned}
||W_{\psi,\,\varphi}D^n f||^2_{H^2(\beta)} &= ||\sum_{n=0}^{\infty} a_n \psi\; n!||^2_{H^2(\beta)} \\
&= \sum_{n=0}^{\infty} |a_n|^2 (n!)^2 ||\psi||^2_{H^2(\beta)} \\
&\leq \sum_{n=0}^{\infty} |a_n|^2 (n!)^2 \frac{M^2}{(n!)^2}.\beta^2(n) \\
&= M^2 \sum_{n=0}^{\infty} |a_n|^2 (\beta(n))^2 \\
&= M^2 ||f||^2
\end{aligned}
$$

This implies that $||W_{\psi,\,\varphi}D^n f||_{H^2(\beta)} \leq M||f||_{H^2(\beta)}$ and hence the operator $W_{\psi,\,\varphi}D^n$ is bounded. □

Corollary 2.2. *Let $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ be an analytic self such that $\{\varphi^n : n \geq 0\}$ be an orthogonal family. Then, the composition operator $M_\psi C_\varphi D : H^2(\beta) \longrightarrow H^2(\beta)$ is bounded if and only if*

$$\|\psi\; \phi^{n-1}\| \leq \frac{M}{n}\beta(n)$$

Proof. Let $n = 1$, then $W_{\psi\;\varphi}D^n = M_\psi C_\varphi D$ and proof follows by taking $D^n = D$ in Theorem 2.1. □

Corollary 2.3. *Let $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ be an analytic map such that $\{\varphi^n : n \geq 0\}$ be an orthogonal family. Then the composition operator $C_\varphi D : H^2(\beta) \longrightarrow H^2(\beta)$ is bounded if and only if*

$$||\phi^{n-1}|| \leq \frac{M}{n}\beta(n)$$

Proof. Let $\psi(z) = 1, n = 1$ then, we have $W_{\psi\;\varphi}D^n = C_\varphi D$ and proof follows by taking $\psi(z) = 1,\ D^n = D$ in Theorem 2.1. □

Corollary 2.4. *Let $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ be an analytic map such that $\{\varphi^n : n \geq 0\}$ be an orthogonal family. Then, the composition operator $DC_\varphi : H^2(\beta) \longrightarrow H^2(\beta)$ is bounded if and only if*

$$||\phi^{n-1}.\phi'|| \leq \frac{M}{n}\beta(n)$$

Proof. Let $\psi(z) = \phi'(z)\ , n = 1$ then $W_{\psi\;\varphi}D^n = DC_\phi$ and proof follows by taking $\psi(z) = \phi'(z)\ , D^n = D$ in Theorem 2.1. □

3. Compactness of $W_{\psi,\,\varphi}D^n$

In this section, we shall study the compactness of the operator $W_{\psi,\,\varphi}D^n$ on the weighted Hardy spaces $H^2(\beta)$. For this, we shall need the following Lemma.

Lemma 3.1. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic self-map of $\mathbb{D}$. Then, the generalised composition operators $W_{\psi,\,\varphi}D^n$: $H^2(\beta) \to H^2(\beta)$ is compact if and only if for any bounded sequence $\{f_n\}_{n=0}^{\infty}$ converges to zero locally uniformly on $\mathbb{D}$, we have*

$$\|W_{\psi,\,\varphi}D^n f_n\|_{H^2(\beta)} \to 0$$

Proof. The proof of this Lemma can be written by using the similar argument as in [2]. □

Theorem 3.2. *Let $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ be an analytic self map of $\mathbb{D}$ such that $\{\varphi^n : n \geq 0\}$ be an orthogonal family.Then, the operator $W_{\psi,\,\varphi}D^n : H^2(\beta) \longrightarrow H^2(\beta)$ is compact if and only if*

$$\frac{||\psi.n!||}{\beta_n} \longrightarrow 0 \quad as \quad n \quad \longrightarrow \infty.$$

Proof. First, suppose that $W_{\psi,\,\varphi}D^n : H^2(\beta) \longrightarrow H^2(\beta)$ is compact and $\{\frac{z^n}{\beta_n}\}_{n=0}^{\infty}$ converges uniformly to zero on $\mathbb{D}$. Then, by using the Lemma 3.1, we have

$$||W_{\psi,\,\varphi}D^n\{\frac{z^n}{\beta_n}\}|| \to 0 \quad as \;\; n \longrightarrow \infty$$

that is

$$\frac{||\psi.n!||}{\beta_n} \longrightarrow 0 \quad as \quad n \longrightarrow \infty.$$

Conversely, suppose that

$$\frac{||\psi.n!||}{\beta_n} \to 0 \quad as \quad n \to \infty.$$

Then, for every $\varepsilon > 0$, there exist a positive integer m such that

$$\frac{||\psi.n!||}{\beta_n} < \varepsilon \;\; \forall \;\; n \geq m.$$

Now, let $f \in H^2(\beta)$ be such that $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Define an operator T_k on $H^2(\beta)$ as

$$\begin{aligned} T_k f &= \sum_{n=0}^{k} a_n (W_{\psi,\,\varphi}D^n z^n) \\ &= \sum_{n=0}^{k} a_n \psi \; n! \end{aligned}$$

Then, T_k is a finite rank operator on $H^2(\beta)$ and

$$\begin{aligned} ||(W_{\psi,\,\varphi}D^n - T_k)f||^2_{H^2(\beta)} &= ||\sum_{n=k+1}^{\infty} a_n \psi.n!||^2_{H^2(\beta)} \\ &\leq \sum_{n=k+1}^{\infty} |a_n|^2 (n!)^2 ||\psi||^2_{H^2(\beta)} \\ &< \sum_{n=k+1}^{\infty} |a_n|^2 (n!)^2 \frac{\varepsilon^2}{(n!)^2} (\beta_n)^2 \end{aligned}$$

$$= \varepsilon^2||f||^2_{H^2(\beta)}$$

Thus

$$||W_{\psi,\,\varphi}D^n - T_k|| < \varepsilon \quad \forall \;\; k \geq m.$$

This proves that the operator $W_{\psi,\,\varphi}D^n$ is compact. □

Corollary 3.3. *Let $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ be an analytic map such that $\{\phi^n : n \geq 0\}$ be an orthogonal family. Then the composition operator $M_\psi C_\varphi D : H^2(\beta) \longrightarrow H^2(\beta)$ is compact if and only if*

$$\frac{||\psi.n\varphi^{n-1}||}{\beta_n} \to 0 \quad as \quad n \to \infty$$

Proof. Let $n = 1$, then $W_{\psi,\,\varphi}D^n = M_\psi C_\varphi D$ and so the proof follows by taking $D^n = D$ in Theorem 3.2. □

Corollary 3.4. *Let $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ be an analytic map such that $\{\phi^n : n \geq 0\}$ be an orthogonal family. Then the composition operator $C_\varphi D : H^2(\beta) \longrightarrow H^2(\beta)$ is compact if and only if*

$$\frac{||n\varphi^{n-1}||}{\beta_n} \to 0 \quad as \quad n \to \infty$$

Proof. Let $\psi(z) = 1, n = 1$ then $W_{\psi,\,\varphi}D^n = C_\varphi D$ so the proof follows by taking $\psi(z) = 1, D^n = D$ in Theorem 3.2. □

Corollary 3.5. *Let $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ be an analytic map such that $\{\phi^n : n \geq 0\}$ be an orthogonal family. Then the composition operator $DC_\varphi : H^2(\beta) \longrightarrow H^2(\beta)$ is compact if and only if*

$$\frac{||\varphi^{n-1}n\varphi'||}{\beta_n} \to 0$$

Proof. Let $\psi(z) = \varphi'(z), n = 1$ then $W_{\psi,\,\varphi}D^n = DC_\varphi$ and so the proof follow by taking $\psi(z) = \varphi'(z), D^n = D$ in Theorem 3.2. □

4. Necessary and Sufficient Condition for $W_{\psi,\,\varphi}D^n$ to be Hilbert-Schmidt Operator on $H^2(\beta)$

In this section, we shall give necessary and sufficient condition for the operator $W_{\psi,\,\varphi}D^n$ to be Hilbert-Schmidt operator on $H^2(\beta)$. Recall that a linear operator T on a Hilbert space H is said to be Hilbert-Schmidt operator if $\sum_{n=0}^{\infty} \|Te_n\|^2 < \infty$ for some orthonormal basis $\{e_n : n \geq 0\}$ of Hilbert spaces H.

Theorem 4.1. *Let φ be an analytic self-map of $\mathbb{D}$. Then the composition operator $W_{\psi,\,\varphi}D^n : H^2_\beta(\mathbb{D}) \longrightarrow H^2_\beta(\mathbb{D})$*

is a Hilbert-Schmidt operator if and only if

$$\sum_{n=0}^{\infty}\left(\frac{n!}{\beta_n}\right)^2 ||\psi||^2_{H^2_\beta(\mathbb{D})} \quad < \infty.$$

Proof. Since $\{\frac{z^n}{\beta_n}; n \geq 0\}$ is an orthonormal basis for $H^2_\beta(\mathbb{D})$. Therefore, the operator $W_{\psi,\,\varphi}D^n$: $H^2_\beta(\mathbb{D}) \longrightarrow H^2_\beta(\mathbb{D})$ is a Hilbert-Schmidt operator

$$\begin{aligned} if\ and\ only\ if\ & \sum_{n=0}^{\infty} ||W_{\psi,\,\varphi}D^n \left\{\frac{z^n}{\beta_n}\right\}||^2_{H^2_\beta(\mathbb{D})} < \infty \\ if\ and\ only\ if\ & \sum_{n=0}^{\infty} \frac{||\psi\ n!||^2_{H^2_\beta(\mathbb{D})}}{(\beta_n)^2} < \infty \\ if\ and\ only\ if\ & \sum_{n=0}^{\infty}\left(\frac{n!}{\beta_n}\right)^2 ||\psi||^2_{H^2_\beta(\mathbb{D})} < \infty \end{aligned}$$

□

Corollary 4.2. *Let φ be an analytic self-map of $\mathbb{D}$. Then the composition operator $M_\psi C_\varphi D$: $H^2(\beta) \longrightarrow H^2(\beta) \longrightarrow H^2_\beta(\mathbb{D})$ is a Hilbert-Schmidt operator if and only if*

$$\sum_{n=0}^{\infty}\left[\frac{n}{\beta_n}\right]^2 \|\psi\ \varphi^{n-1}\|^2_{H^2_\beta(\mathbb{D})} \quad < \infty.$$

Proof. Let $n = 1$, then $W_{\psi,\,\varphi}D^n = M_\psi C_\varphi D$ and so the proof follows by taking $D^n = D$ in Theorem 4.1. □

Corollary 4.3. *Let φ be an analytic self-map of $\mathbb{D}$. Then the composition operator $C_\varphi D : H^2_\beta(\mathbb{D}) \longrightarrow H^2_\beta(\mathbb{D})$ is a Hilbert-Schmidt operator if and only if*

$$\sum_{n=0}^{\infty}\left(\frac{n}{\beta_n}\right)^2 ||\varphi^{n-1}||^2_{H^2_\beta(\mathbb{D})} \quad < \infty.$$

Proof. Let $\psi(z) = 1, n = 1$ then $W_{\psi,\,\varphi}D^n = C_\varphi D$ so the proof follows by taking $\psi(z) = 1, D^n = D$ in Theorem 4.1. □

Corollary 4.4. *Let φ be an analytic self-map of $\mathbb{D}$. Then the composition operator $DC_\varphi : H^2_\beta(\mathbb{D}) \longrightarrow H^2_\beta(\mathbb{D})$ is a Hilbert-Schmidt operator if and only if*

$$\sum_{n=0}^{\infty}\left(\frac{n}{\beta_n}\right)^2 ||\varphi^{n-1}\varphi'||^2_{H^2_\beta(\mathbb{D})} \quad < \infty.$$

Proof. Let $\psi(z) = \varphi'(z), n = 1$ then $W_{\psi,\,\varphi}D^n = DC_\varphi$ and so the proof follows by taking $\psi(z) = \varphi'(z), D^n = D$ in Theorem 4.1. □

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