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Comparative Growth of Composite Entire and Meromorphic Functions and Wronskians Generated by them on the Basis of their (α, β, γ)-order

Biswajit Saha[1,*], Chinmay Biswas[2]

[1] *Department of Mathematics, Government General Degree College at Pedong, Pedong, Kalimpong, West Bengal, India*
[2] *Department of Mathematics, Nabadwip Vidyasagar College, Nabadwip, Nadia, West Bengal, India*

Abstract

In this paper, we have established some results relating to the comparative growth properties of composite transcendental entire or meromorphic functions and Wronskians generated by one of the factors on the basis of (α, β, γ)-order and (α, β, γ)-lower order, where α, β, γ are continuous non-negative functions defined on $(-\infty, +\infty)$.

Keywords: Entire function; meromorphic function; growth; Wronskians; (α, β, γ)-order; (α, β, γ)-lower order.
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1. Introduction, Definitions and Notations

Throughout this manuscript,we have used the fundamental results and the standard notations of the Nevanlinna theory of meromorphic functions which are available in [3,5,7,8]. We have also used the standard notations and definitions of the theory of entire functions which are available in [6] and therefore we do not explain those in details. Let f be an entire function defined in the open complex plane $\mathbb{C}$ and $M_f(r) = \max\{|f(z)| : |z| = r\}$. When f is meromorphic, one may introduce another function $T_f(r)$, known as Nevanlinna's characteristic function of f (see [3, p.4]), playing the same role as $M_f(r)$, which is defined as

$$T_f(r) = N_f(r) + m_f(r),$$

where $m_f(r)$ and $N_f(r)$ are respectively called as the proximity function of f and the counting function of poles of f in $|z| \leq r$. For details about $T_f(r)$, $m_f(r)$ and $N_f(r)$ one may see [3].
If f is entire, then the Nevanlinna's characteristic function $T_f(r)$ of f is defined as

$$T_f(r) \quad = \quad m_f(r) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f(re^{i\theta})\right| d\theta,$$

*Corresponding author (sahaanjan11@gmail.com)

$$\text{where, } \log^+ x = \max(\log x, 0) \text{ for all } x \geqslant 0.$$

For a meromorphic function f, the Wronskian determinant $W(f) = W(a_1, a_2, ...a_k, f)$ is defined as

$$W(f) = \begin{vmatrix} a_1 & a_2 & . & . & . & a_k & f \\ a_1^{'} & a_2^{'} & . & . & . & a_k^{'} & f^{'} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ a_1^{(k)} & a_2^{(k)} & . & . & . & a_k^{(k)} & f^{(k)} \end{vmatrix}$$

where a_1, a_2, ... a_k are linearly independent meromorphic functions and small with respect to f i.e., $T_{a_i}(r) = S_f(r)$ for $i = 1, 2, 3 \ldots k$. From the Nevanlinna's second fundamental theorem, it follows that the set of values of $a \in \mathbb{C} \cup \{\infty\}$ for which $\delta(a; f) > 0$ is countable and $\sum\limits_{a \neq \infty} \delta(a; f) + \delta(\infty; f) \leq 2$ [3, p.43], where, $\delta(a; f) = 1 - \limsup\limits_{r \to \infty} \frac{N(r,a;f)}{T_f(r)} = \liminf\limits_{r \to \infty} \frac{m(r,a;f)}{T_f(r)}$. If in particular $\sum\limits_{a \neq \infty} \delta(a; f) + \delta(\infty; f) = 2$, we say that f has the maximum deficiency sum.

Now, first of all, let L be a class of continuous non-negative on $(-\infty, +\infty)$ function α such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ with $\alpha(x) \uparrow +\infty$ as $x_0 \leq x \to +\infty$. We say that $\alpha \in L_1$, if $\alpha \in L$ and $\alpha(a + b) \leq \alpha(a) + \alpha(b) + c$ for all $a, b \geq R_0$ and fixed $c \in (0, +\infty)$. Further we say that $\alpha \in L_2$, if $\alpha \in L$ and $\alpha(x + O(1)) = (1 + o(1))\alpha(x)$ as $x \to +\infty$. Finally, $\alpha \in L_3$, if $\alpha \in L$ and $\alpha(a + b) \leq \alpha(a) + \alpha(b)$ for all $a, b \geq R_0$, i.e., α is subadditive. Clearly $L_3 \subset L_1$.

Particularly, when $\alpha \in L_3$, then one can easily verify that $\alpha(mr) \leq m\alpha(r)$, $m \geq 2$ is an integer. Up to a normalization, subadditivity is implied by concavity. Indeed, if $\alpha(r)$ is concave on $[0, +\infty)$ and satisfies $\alpha(0) \geq 0$, then for $t \in [0, 1]$,

$$\begin{aligned} \alpha(tx) &= \alpha(tx + (1 - t) \cdot 0) \\ &\geq t\alpha(x) + (1 - t)\alpha(0) \geq t\alpha(x), \end{aligned}$$

so that by choosing $t = \frac{a}{a+b}$ or $t = \frac{b}{a+b}$, we obtain

$$\begin{aligned} \alpha(a + b) &= \frac{a}{a + b}\alpha(a + b) + \frac{b}{a + b}\alpha(a + b) \\ &\leq \alpha\left(\frac{a}{a + b}(a + b)\right) + \alpha\left(\frac{b}{a + b}(a + b)\right) \\ &= \alpha(a) + \alpha(b), \ a, b \geq 0. \end{aligned}$$

As a non-decreasing, subadditive and unbounded function, $\alpha(r)$ satisfies

$$\alpha(r) \leq \alpha(r + R_0) \leq \alpha(r) + \alpha(R_0)$$

for any $R_0 \geq 0$. This yields that $\alpha(r) \sim \alpha(r + R_0)$ as $r \to +\infty$. Throughout this paper we assume $\alpha \in L_1$, $\beta \in L_2$, $\gamma \in L_3$.

Heittokangas et al. [2] have introduced a new concept of φ-order of entire and meromorphic functions considering φ as subadditive function. For details one may see [2]. After that, Belaïdi et al. [1] have extended the above idea and have introduced the definitions of (α, β, γ)-order and (α, β, γ)-lower order of a meromorphic function f, which are as follows:

Definition 1.1 ([1])**.** *The (α, β, γ)-order denoted by $\rho_{(\alpha,\beta,\gamma)}[f]$ of a meromorphic function f is defined as:*

$$\rho_{(\alpha,\beta,\gamma)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\log(T_f(r)))}{\beta(\log(\gamma(r)))}.$$

Definition 1.2 ([1])**.** *The (α, β, γ)-lower order denoted by $\lambda_{(\alpha,\beta,\gamma)}[f]$ of a meromorphic function f is defined as:*

$$\lambda_{(\alpha,\beta,\gamma)}[f] = \liminf_{r\to+\infty} \frac{\alpha(\log(T_f(r)))}{\beta(\log(\gamma(r)))}.$$

In this paper, we have established some results depending on the comparative growth properties of composite transcendental entire or meromorphic functions and Wronskians generated by one of the factors on the basis of (α, β, γ)-order and (α, β, γ)-lower order.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1 ([4])**.** *Let f be a transcendental meromorphic function having the maximum deficiency sum. Then*

$$\lim_{r\to+\infty} \frac{T_{W(f)}(r)}{T_f(r)} = 1 + k - k\delta(a; f).$$

Lemma 2.2. *Let f be a transcendental meromorphic function having the maximum deficiency sum. Then the (α, β, γ)-order and (α, β, γ)-lower order of $W(f)$ and that of f are same.*

Proof. From Lemma 2.1 we get $\alpha(\log T_{W(f)}(r)) \sim \alpha(\log T_f(r))$ as $r \to +\infty$. So,

$$\begin{aligned} \frac{\alpha(\log T_{W(f)}(r))}{\beta(\log \gamma(r))} &\sim \frac{\alpha(\log T_f(r))}{\beta(\log \gamma(r))} \text{ as } r \to +\infty, \\ \text{i.e., } \limsup_{r\to+\infty} \frac{\alpha(\log T_{W(f)}(r))}{\beta(\log \gamma(r))} &= \limsup_{r\to+\infty} \frac{\alpha(\log T_f(r))}{\beta(\log \gamma(r))}, \\ \text{i.e., } \varrho_{(\alpha,\beta,\gamma)}[W(f)] &= \varrho_{(\alpha,\beta,\gamma)}[f]. \end{aligned}$$

Similarly,

$$\lambda_{(\alpha,\beta,\gamma)}[W(f)] = \lambda_{(\alpha,\beta,\gamma)}[f].$$

Thus the lemma follows. □

3. Main Results

In this section we present the main results of the paper.

Theorem 3.1. *Let f be a transcendental meromorphic function with $\sum_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$ and g be an entire function such that $0<\lambda_{(\alpha,\beta,\gamma)}[f]\leq\rho_{(\alpha,\beta,\gamma)}[f]<+\infty$ and $\lambda_{(\alpha,\beta,\gamma)}[f\circ g]=+\infty$, then*

$$\lim_{r\to+\infty}\frac{\alpha(\log(T_{f\circ g}(r)))}{\alpha(\log(T_{W(f)}(r)))}=+\infty.$$

Proof. If possible, let the conclusion of the theorem does not hold. Then we can find a constant $\Delta>0$ such that for a sequence of values of r tending to infinity

$$\alpha(\log(T_{f\circ g}(r)))\leq\Delta\cdot\alpha(\log(T_{W(f)}(r))). \tag{1}$$

It follows from Lemma 2.2 and Definition 1.1, for all sufficiently large values of r that

$$\alpha(\log(T_{W(f)}(r)))\leq(\rho_{(\alpha,\beta,\gamma)}[f]+\epsilon)\beta(\log(\gamma(r))). \tag{2}$$

From (1) and (2), for a sequence of values of r tending to $+\infty$, we have

$$\begin{aligned}
&\alpha(\log(T_{f\circ g}(r)))\leq\Delta(\rho_{(\alpha,\beta,\gamma)}[f]+\epsilon)\beta(\log(\gamma(r))),\\
\text{i.e.,}\quad&\frac{\alpha(\log(T_{f\circ g}(r)))}{\beta(\log(\gamma(r)))}\leq\Delta(\rho_{(\alpha,\beta,\gamma)}[f]+\epsilon),\\
\text{i.e.,}\quad&\liminf_{r\to+\infty}\frac{\alpha(\log(T_{f\circ g}(r)))}{\beta(\log(\gamma(r)))}<+\infty.
\end{aligned}$$

Hence from Definition 1.2, we have

$$\lambda_{(\alpha,\beta,\gamma)}[f\circ g]<+\infty.$$

This is a contradiction. Thus the theorem follows. □

Remark 3.2. *Theorem 3.1 is also valid with "limit superior" instead of "limit" if "$\lambda_{(\alpha,\beta,\gamma)}[f\circ g]=+\infty$" is replaced by "$\rho_{(\alpha,\beta,\gamma)}[f\circ g]=+\infty$" and the other conditions remain the same.*

Theorem 3.3. *Let f be a transcendental meromorphic function with $\sum_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$ and g be an entire function such that $0<\lambda_{(\alpha,\beta,\gamma)}[f]\leq\rho_{(\alpha,\beta,\gamma)}[f]<+\infty$ and $0<\lambda_{(\alpha_1,\beta,\gamma)}[f\circ g]\leq\rho_{(\alpha_1,\beta,\gamma)}[f\circ g]<+\infty$, then*

$$\begin{aligned}
\frac{\lambda_{(\alpha_1,\beta,\gamma)}[f\circ g]}{\rho_{(\alpha,\beta,\gamma)}[f]}&\leq\liminf_{r\to+\infty}\frac{\alpha_1(\log(T_{f\circ g}(r)))}{\alpha(\log(T_{W(f)}(r)))}\\
&\leq\min\left\{\frac{\lambda_{(\alpha_1,\beta,\gamma)}[f\circ g]}{\lambda_{(\alpha,\beta,\gamma)}[f]},\frac{\rho_{(\alpha_1,\beta,\gamma)}[f\circ g]}{\rho_{(\alpha,\beta,\gamma)}[f]}\right\}
\end{aligned}$$

$$\leq \max\left\{\frac{\lambda_{(\alpha_1,\beta,\gamma)}[f\circ g]}{\lambda_{(\alpha,\beta,\gamma)}[f]}, \frac{\rho_{(\alpha_1,\beta,\gamma)}[f\circ g]}{\rho_{(\alpha,\beta,\gamma)}[f]}\right\}$$
$$\leq \limsup_{r\to+\infty}\frac{\alpha_1(\log(T_{f\circ g}(r)))}{\alpha(\log(T_{W(f)}(r)))} \leq \frac{\rho_{(\alpha_1,\beta,\gamma)}[f\circ g]}{\lambda_{(\alpha,\beta,\gamma)}[f]}.$$

Proof. From Definition 1.1 and Definition 1.2, we have for arbitrary positive ε and for all sufficiently large values of r that

$$\alpha_1(\log(T_{f\circ g}(r))) \geqslant \left(\lambda_{(\alpha_1,\beta,\gamma)}[f\circ g]-\varepsilon\right)\beta(\log(\gamma(r))), \tag{3}$$
$$\alpha_1(\log(T_{f\circ g}(r))) \leq \left(\rho_{(\alpha_1,\beta,\gamma)}[f\circ g]+\varepsilon\right)\beta(\log(\gamma(r))). \tag{4}$$

Using Lemma 2.2, we have from Definition 1.1 and Definition 1.2, for arbitrary positive ε and for all sufficiently large values of r that

$$\alpha(\log(T_{W(f)}(r))) \geqslant \left(\lambda_{(\alpha,\beta,\gamma)}[f]-\varepsilon\right)\beta(\log(\gamma(r))), \tag{5}$$
$$\text{and } \alpha(\log(T_{W(f)}(r))) \leq \left(\rho_{(\alpha,\beta,\gamma)}[f]+\varepsilon\right)\beta(\log(\gamma(r))). \tag{6}$$

Again from Definition 1.1 and Definition 1.2, we have for arbitrary positive ε and for a sequence of values of r tending to infinity,

$$\alpha_1(\log(T_{f\circ g}(r))) \leq \left(\lambda_{(\alpha_1,\beta,\gamma)}[f\circ g]+\varepsilon\right)\beta(\log(\gamma(r))), \tag{7}$$
$$\alpha_1(\log(T_{f\circ g}(r))) \geqslant \left(\rho_{(\alpha_1,\beta,\gamma)}[f\circ g]-\varepsilon\right)\beta(\log(\gamma(r))). \tag{8}$$

Also, using Lemma 2.2, we get from Definition 1.1 and Definition 1.2, for arbitrary positive ε and for a sequence of values of r tending to infinity,

$$\alpha(\log(T_{W(f)}(r))) \leq \left(\lambda_{(\alpha,\beta,\gamma)}[f]+\varepsilon\right)\beta(\log(\gamma(r))), \tag{9}$$

and

$$\text{and } \alpha(\log(T_{W(f)}(r))) \geqslant \left(\rho_{(\alpha,\beta,\gamma)}[f]-\varepsilon\right)\beta(\log(\gamma(r))). \tag{10}$$

Now from (3) and (6) it follows for all sufficiently large values of r that

$$\frac{\alpha_1(\log(T_{f\circ g}(r)))}{\alpha(\log(T_{W(f)}(r)))} \geqslant \frac{\lambda_{(\alpha_1,\beta,\gamma)}[f\circ g]-\varepsilon}{\rho_{(\alpha,\beta,\gamma)}[f]+\varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\liminf_{r\to+\infty}\frac{\alpha_1(\log(T_{f\circ g}(r)))}{\alpha(\log(T_{W(f)}(r)))} \geqslant \frac{\lambda_{(\alpha_1,\beta,\gamma)}[f\circ g]}{\rho_{(\alpha,\beta,\gamma)}[f]}. \tag{11}$$

Combining (5) and (7), we have for a sequence of values of r tending to infinity that

$$\frac{\alpha_1(\log(T_{f\circ g}(r)))}{\alpha(\log(T_{W(f)}(r)))} \leq \frac{\lambda_{(\alpha_1,\beta,\gamma)}[f\circ g]+\varepsilon}{\lambda_{(\alpha,\beta,\gamma)}[f]-\varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\liminf_{r\to+\infty}\frac{\alpha_1(\log(T_{f\circ g}(r)))}{\alpha(\log(T_{W(f)}(r)))} \leq \frac{\lambda_{(\alpha_1,\beta,\gamma)}[f\circ g]}{\lambda_{(\alpha,\beta,\gamma)}[f]}. \tag{12}$$

Again from (3) and (9), for a sequence of values of r tending to infinity, we get

$$\frac{\alpha_1(\log(T_{f\circ g}(r)))}{\alpha(\log(T_{W(f)}(r)))} \geq \frac{\lambda_{(\alpha_1,\beta,\gamma)}[f\circ g]-\varepsilon}{\lambda_{(\alpha,\beta,\gamma)}[f]+\varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\limsup_{r\to+\infty}\frac{\alpha_1(\log(T_{f\circ g}(r)))}{\alpha(\log(T_{W(f)}(r)))} \geq \frac{\lambda_{(\alpha_1,\beta,\gamma)}[f\circ g]}{\lambda_{(\alpha,\beta,\gamma)}[f]}. \tag{13}$$

Also, it follows from (4) and (5), for all sufficiently large values of r that

$$\frac{\alpha_1(\log(T_{f\circ g}(r)))}{\alpha(\log(T_{W(f)}(r)))} \leq \frac{\rho_{(\alpha_1,\beta,\gamma)}[f\circ g]+\varepsilon}{\lambda_{(\alpha,\beta,\gamma)}[f]-\varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\limsup_{r\to+\infty}\frac{\alpha_1(\log(T_{f\circ g}(r)))}{\alpha(\log(T_{W(f)}(r)))} \leq \frac{\rho_{(\alpha_1,\beta,\gamma)}[f\circ g]}{\lambda_{(\alpha,\beta,\gamma)}[f]}. \tag{14}$$

Now from (4) and (10), it follows for a sequence of values of r tending to infinity that

$$\frac{\alpha_1(\log(T_{f\circ g}(r)))}{\alpha(\log(T_{W(f)}(r)))} \leq \frac{\rho_{(\alpha_1,\beta,\gamma)}[f\circ g]+\varepsilon}{\rho_{(\alpha,\beta,\gamma)}[f]-\varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\liminf_{r\to+\infty}\frac{\alpha_1(\log(T_{f\circ g}(r)))}{\alpha(\log(T_{W(f)}(r)))} \leq \frac{\rho_{(\alpha_1,\beta,\gamma)}[f\circ g]}{\rho_{(\alpha,\beta,\gamma)}[f]}. \tag{15}$$

Combining (6) and (8), we get for a sequence of values of r tending to infinity that

$$\frac{\alpha_1(\log(T_{f\circ g}(r)))}{\alpha(\log(T_{W(f)}(r)))} \geqslant \frac{\rho_{(\alpha_1,\beta,\gamma)}[f\circ g]-\varepsilon}{\rho_{(\alpha,\beta,\gamma)}[f]+\varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\limsup_{r\to+\infty}\frac{\alpha_1(\log(T_{f\circ g}(r)))}{\alpha(\log(T_{W(f)}(r)))} \geqslant \frac{\rho_{(\alpha_1,\beta,\gamma)}[f\circ g]}{\rho_{(\alpha,\beta,\gamma)}[f]}. \tag{16}$$

Thus the theorem follows from $(11), (12), (13), (14), (15)$ and (16). □

Theorem 3.4. *Let f be a transcendental meromorphic function and g be a transcendental entire function such that $\sum\limits_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$, $0<\lambda_{(\alpha,\beta,\gamma)}[g]\leq\rho_{(\alpha,\beta,\gamma)}[g]<+\infty$ and $0<\lambda_{(\alpha_1,\beta,\gamma)}[f\circ g]\leq\rho_{(\alpha_1,\beta,\gamma)}[f\circ g]<+\infty$, then*

$$\begin{aligned}\frac{\lambda_{(\alpha_1,\beta,\gamma)}[f\circ g]}{\rho_{(\alpha,\beta,\gamma)}[g]} &\leq \liminf_{r\to+\infty}\frac{\alpha_1(\log(T_{f\circ g}(r)))}{\alpha(\log(T_{W(g)}(r)))}\\ &\leq \min\left\{\frac{\lambda_{(\alpha_1,\beta,\gamma)}[f\circ g]}{\lambda_{(\alpha,\beta,\gamma)}[g]},\frac{\rho_{(\alpha_1,\beta,\gamma)}[f\circ g]}{\rho_{(\alpha,\beta,\gamma)}[g]}\right\}\\ &\leq \max\left\{\frac{\lambda_{(\alpha_1,\beta,\gamma)}[f\circ g]}{\lambda_{(\alpha,\beta,\gamma)}[g]},\frac{\rho_{(\alpha_1,\beta,\gamma)}[f\circ g]}{\rho_{(\alpha,\beta,\gamma)}[g]}\right\}\\ &\leq \limsup_{r\to+\infty}\frac{\alpha_1(\log(T_{f\circ g}(r)))}{\alpha(\log(T_{W(g)}(r)))}\leq\frac{\rho_{(\alpha_1,\beta,\gamma)}[f\circ g]}{\lambda_{(\alpha,\beta,\gamma)}[g]}.\end{aligned}$$

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