International Journal of
Mathematics And its Applications
ISSN: 2347-1557

Int. J. Math. And Appl., **13(4)**(2025), 73–100
Available Online: http://ijmaa.in

Multifactor Balanced Asymmetrical Factorial Designs of Type III

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Abstract

This manuscript gives a method of constructing multifactor BAFDS of type III. Multi-factor BAFDS of type III are constructed from two factor BAFDS. The method was given by [21] and it generates a BAFD from two given BAFD's. The method can provide efficient BAFD's if efficient two factor BAFD's are used. The designs constructed are balanced with orthogonal factorial structure.

Keywords: Efficient BAFD's; Designs; Balance; Orthogonal Factorial Structure; BAFD's; multifactor BAFDS.

2020 Mathematics Subject Classification: 62K10, 62K15, 05B05.

1. Introduction

In many situations there arise scenarios when an experimenter has to use factors at different levels. The problem of obtaining confounded plans for such cases has received a good deal of attention. To this extent, [30], by trial and hit methods obtained confounded plans of the type $3^m \times 2^n$, where m and n are any positive integers. Using orthogonal arrays of strength 2 [18] gave methods for constructing Extended Group Divisible Designs $\{EGD\}$ for $s_1 \times s_2$ experiments in blocks of size $s_1 < s_2$. [26] starting from a basic $s_1 \times s_2$ design in blocks of size s_2 ($s_2 < s_1$, s_1 being a prime number or power of prime) obtained three factor designs. [20] constructed some series of designs from orthogonal latin squares for $s_1 \times s_2$ experiments in block of size s_1 and $s_2 - 1$ replications. [25] gave a class of balanced designs with OFS. [16] considered the use of balanced incomplete block designs for the construction of $s_1 \times s_2$ balanced factorial designs with OFS when $s_1 > s_2$. Informative accounts and subsequent developments have been done by [10,14,24]. [22] proposed a general method of obtaining block designs for asymmetrical confounded factorial experiments using block designs for symmetrical factorial experiments.

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[9] describes a general method of construction of supersaturated designs for asymmetrical factorial experiments obtained by exploiting the concept of resolvable orthogonal arrays and Hadamard matrices. [7] considered three forms of a general null hypothesis Ho on the factorial parameters of a general asymmetrical factorial paired comparison experiment in order to determine optimal or efficient designs. [13] constructed designs by using confounding through equation methods. Construction of confounded asymmetrical factorial experiments in row-column settings and efficiency factors of confounded effects was worked out. [27] identified a Kronecker product structure for a particular class of asymmetric factorial designs in blocks, including the classes of designs generated by several of the generalizations of the classical methods in literature. [4] focuses on the construction and analysis of an extra ordinary type of asymmetrical factorial experiment which corresponds to a fraction of asymmetrical factorial experiment as indicated by [5]. [2] establishes a lower bound to measure optimality with respect to a main effects model in a general asymmetric factorial experiment. [1] developed a method for the construction of $p \times 3 \times 2$ asymmetrical factorial experiments with $(p-1)$ replications. [23] proposed A general method of obtaining block designs for asymmetrical confounded factorial experiments using the block designs for symmetrical factorial experiments. [31] Constructed asymmetrical factorial designs containing clear effects. [29] explained how to choose an optimal $(s^2)s^n$ design for the practical need, where s is any prime or prime power and accordingly considered the clear effects criterion for selecting good designs. [17] dealt with situations where there was a need for designing an asymmetrical factorial experiment involving interactions. Failing to get a satisfactory answer to this problem from literature, the authors developed an adhoc method of constructing the design. It is transparent that the design provides efficient estimates for all the required main effects and interactions. The later part of this paper deals with the issues of how this method is extended to more general situations and how this adhoc method is translated into a systematic approach. [19] developed The R package DoE.base which can be used for creating full factorial designs and general factorial experiments based on orthogonal arrays. Besides design creation, some analysis functionality is also available, particularly (augmented) half-normal effects plots.

[12] Published a monograph that is an outcome of the research works on the construction of factorial experiments (symmetrical and asymmetrical). In this booklet, construction frameworks have been described for factorial experiments. The construction frameworks include general construction method of p^n factorial experiments, construction methods with confounded effects and detection method of confounded effects in a confounded plan. The concepts of combinatorial, matrix operations and linear equation technique have been deployed to develop the methods. [4] discussed an Alternative Method of Construction and Analysis of Asymmetrical Factorial Experiment of the type 6×22 in Blocks of Size 12. [4] focuses on the construction and analysis of an extra ordinary type of asymmetrical factorial designs which corresponds to fraction of a symmetrical factorial design as indicated by [5]. To construct this design, they used 3 choices and for each choice they used 5 different cases. Finding the block contents for each case showed that there are mainly two different cases for each choice. In the cases of

analysis of variance, is seen that, for the case where the highest order interaction effect is confounded in 4 replications, the loss of information is same for all the choices.

[11] in his book chapter discusses different methods of constructing systems of confounding for asymmetrical factorial designs, including: Combining symmetrical systems of confounding via the Kronecker product method, use of pseudo-factors, the method of generalized cyclic designs, method of finite rings (this method is also used to extend the Kempthorne parameterization from symmetrical to asymmetrical factorials), and the method of balanced factorial designs. He showed the equivalence of balanced factorial designs and extended group divisible partially balanced incomplete block designs, establishing again a close link between incomplete block designs and confounding in factorial designs.

[6] in her book chapter discusses confounding in single replicate experiments in which at least one factor has more than two levels. First, the case of three-levelled factors is considered and the techniques are then adapted to handle m-levelled factors, where m is a prime number. Next, pseudofactors are introduced to facilitate confounding for factors with non-prime numbers of levels. Asymmetrical experiments involving factors or pseudofactors at both two and three levels are also considered, as well as more complicated situations where the treatment factors have a mixture of $2, 3, 4$, and 6 levels. Analysis of an experiment with partial confounding is illustrated using the SAS and R software packages.

[8] shows that Asymmetrical single replicate factorial designs in blocks are constructed using the deletion technique. Results are given that are useful in simplifying expressions for calculating loss of information on main effects and interactions, due to confounding with blocks. Designs for estimating main effects and low order interactions are also given.

[15] in his work presents the results of a systematic literature review (SLR) and a taxonomical classification of studies about run orders for factorial designs published between 1952 and 2021. The objective here is to describe the findings, main and future research directions in this field. The main components considered in each study and the methodologies they used to obtain run sequences are also highlighted, allowing professionals to select an appropriate ordering for their problem. This review shows that obtaining orderings with good properties for an experimental design with any number of factors and levels is still an unresolved issue.

[28] in his present book gives, for the first time, a comprehensive and up-to-date account of the modern theory of factorial designs. Many major classes of designs are covered in the book. While maintaining a high level of mathematical rigor, it also provides extensive design tables for research and practical purposes.

[3] In his book, provides a rigorous, systematic, and up-to-date treatment of the theoretical aspects of factorial designs. To prepare readers for a general theory, the author first presents a unified treatment of several simple designs, including completely randomized designs, block designs, and row-column designs. As such, the book is accessible to readers with minimal exposure to experimental designs.

In this manuscript, we are going to discuss methods of constructing multifactor BAFDS by using known two factor BAFDS or other multifactor BAFDS already constructed.In particular we are going to construct multifactor balanced asymmetrical factorial designs of Type III. We are especially interested in BAFDS of which the main effects and lower order interactions can be estimated with high efficiencies.

The following combinatorial structures are used in the construction of multifactor BAFDS of type III.

Definition 1.1. *A $k \times b$ array with entries from a set of v symbols is called an orthogonal array of strength t if each $t \times b$ subarray of A contains all possible v^t column vectors with the same frequency $\lambda = \frac{b}{v^t}$. It is denoted $OA(b, k, v, t; \lambda)$; the number λ is called the index of the array. The numbers b and k are known as the number of assemblies and constraints of the orthogonal array respectively.*

Example 1.2.

$$\begin{matrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{matrix}$$

OA(8,4,2,3,1)

Definition 1.3. *Let A be a $k \times b$ array with entries from a set of v symbols. Consider the v^t ordered t-tuples $(x_1, \ldots, x_t)$ that can be formed from a t-rowed subarray of A, and let there be associated a non-negative integer $\lambda(x_1, \ldots, x_t)$ that is invariant under permutations of $x_1, \ldots, x_t$. If for any t-rowed subarray of A the v^t ordered t-tuples $(x_1, \ldots, x_t)$, each occur $\lambda(x_1, \ldots, x_t)$ times as a column, then A is said to ba a balanced array of strength t. It is denoted by $BA(b, k, v, t)$ and the numbers $\lambda(x_1, \ldots, x_t)$ are called the index parameters of the array.*

Example 1.4.

$$\begin{matrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{matrix}$$

BA(10,5,2,2)

$\lambda(0, 0) = \lambda(1, 1) = 2$ *and* $\lambda(0, 1) = \lambda(1, 0) = 3$.

In particular we are interested in the $BA[(ks - 1)s\lambda, ks, s, 2]$ with parameters $\lambda(x, y) = (k - 1)\lambda$ or $(k\lambda)$ according as $x = y$ or Not. For brevity we shall call it the balanced array of type T with index λ and denote it by $BA[T][k, s, \lambda]$.

Definition 1.5. *A transitive array $TA(b, k, v, t; \lambda)$ is a $k \times b$ array of v symbols such that for any choice of t rows, the $\frac{v!}{(v-t)!}$ ordered t-tuples of distinct symbols each occur λ times as a column.*

Example 1.6.

0	1	2	3	0	1	2	3	0	1	2	3
1	0	3	2	2	3	0	1	3	2	1	0
2	3	0	1	3	2	1	0	1	0	3	2
3	2	1	0	1	0	3	2	2	3	0	1

TA(12,4,4,2;1)

Definition 1.7. *A hadamard matrix of order n is an $n \times n$ matrix H_n of $+1's$ and $-1's$ whose rows are orthogonal, that is, which satisfies*

$$H_n H_n^T = nI_n \tag{1}$$

For example, here are hadamard matrices of order $1, 2$ and 4.

$$H_1 = [1], H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \tag{2}$$

These matrices are named after a French mathematician Jacques hadamard. He showed that if $A = (a_{ij})$ is an $n \times n$ matrix with $|a_{ij}| \leq 1$ then

$$|detA| \leq n^{\frac{n}{2}} \tag{3}$$

Hadamard matrices may be regarded as the special class of difference schemes $D(r, c, s)$ with $s = 2$, $r = c$ and index $\lambda = \frac{c}{2}$.

An experiment involving $2 \leq m$ factors $F_1, F_2, \ldots, F_m$ that appear at $s_1, \ldots, s_m$ (≥ 2) levels is called an $s_1 \times \cdots \times s_m$ factorial experiment (or an $s_1 \times \cdots \times s_m$ factorial for brevity). If $s_1 = \cdots = s_m = s$, we have s^m symmetrical BFD with $r =$ no of replications, $\lambda_i =$no of blocks in which any two treatments are i^{th} associates, $k =$block size, $b =$no of blocks.

Example 1.8. *Consider a 3×4 factorial arranged in twelve blocks as shown below*

$\longleftarrow$ *Blocks* $\longrightarrow$

00	00	00	01	01	01	02	02	02	03	03	03
11	12	13	10	12	13	10	11	13	10	11	12
22	23	21	23	20	22	21	23	20	22	20	21

The design is connected, proper with constant block size 3, and equireplicate with common replication number $r = 3$. It may be seen, by explicit computation, that for the above design by using Lemma 2.28

$$NN^{'} = \sum_{y \in \Omega^*} \lambda_y (J_1 - I_1)^{y_1} \otimes (J_2 - I_2)^{y_2} \otimes \ldots \otimes (J_m - I_m)^{y_m}$$

$$
\begin{aligned}
&= \lambda_{00}(J_1 - I_1)^0 \otimes (J_2 - I_2)^0 + \lambda_{01}(J_1 - I_1)^0 \otimes (J_2 - I_2)^1 \\
&+ \lambda_{10}(J_1 - I_1)^1 \otimes (J_2 - I_2)^0 + \lambda_{11}(J_1 - I_1)^1 \otimes (J_2 - I_2)^1 \\
&= 3I_1 \otimes I_2 + \lambda_{01} I_1 \otimes (J_2 - I_2) + \lambda_{10}(J_1 - I_1) \otimes I_2 + \lambda_{11}(J_1 - I_1) \otimes (J_2 - I_2)
\end{aligned}
$$

from the design it can be verified that $\lambda_{01} = \lambda_{10} = 0$ *while* $\lambda_{11} = 1$ *hence*

$$
\begin{aligned}
NN' &= 3I_1 \otimes I_2 + 1(J_1 - I_1) \otimes (J_2 - I_2) \\
&= 3I_1 \otimes I_2 + J_1 \otimes J_2 - J_1 \otimes I_2 - I_1 \otimes J_2 + I_1 \otimes I_2 \\
&= 4I_1 \otimes I_2 - I_1 \otimes J_2 - J_1 \otimes I_2 + J_1 \otimes J_2
\end{aligned}
$$

where as usual I_1 *and* I_2 *are* 3×3 *and* 4×4 *identity matrices and* J_1 *and* J_2 *are* 3×3 *and* 4×4 *matrices of all* $1'$*s. It follows that by* (29) *and* (30)

$$
NN' = 4Z^{11} - Z^{10} - Z^{01} + Z^{00}
$$

By Definition 2.29 that NN' *has property A. Hence the design is balanced and has OFS. Furthermore by Definition 2.27 it follows that:*

$$
\begin{aligned}
C &= 3(I_1 \otimes I_2) - \frac{1}{3}NN' \\
&= r(\otimes_{i=1}^{m} I_i) - k^{-1}NN' \\
&= \frac{5}{3}Z^{11} + \frac{1}{3}Z^{10} + \frac{1}{3}Z^{01} - \frac{1}{3}Z^{00}
\end{aligned}
$$

which also shows that C has the property A. Suppose

$$
P_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} \end{bmatrix}
$$

and P_2 *well chosen then*

$$
P^{10} = \begin{bmatrix} \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & 0 & 0 & 0 & 0 \\ \frac{1}{2\sqrt{6}} & \frac{1}{2\sqrt{6}} & \frac{1}{2\sqrt{6}} & \frac{1}{2\sqrt{6}} & \frac{1}{2\sqrt{6}} & \frac{1}{2\sqrt{6}} & \frac{1}{2\sqrt{6}} & \frac{1}{2\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix}
$$

and $Z^{11} = I_v$ *an identity matrix of order 12 hence* $P^{10}Z^{11}P^{10'} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I^{(10)}$ *where* $I^{(y)}$ *is identity matrix* *of order*

$$\begin{aligned}\prod(s_i-1)^{y_i} &= (s_1-1)^1(s_2-1)^0\\ &= (3-1)^1(4-1)^0\\ &= 2\end{aligned}$$

Similarly $P^{10}Z^{10}P^{10'} = 4I^{(10)}$, $P^{10}Z^{01}P^{10'} = P^{10}Z^{00}P^{10'} = 0$. *It also follows that*

$$Z^{10} = \begin{bmatrix} 1&1&1&1&0&0&0&0&0&0&0&0\\ 1&1&1&1&0&0&0&0&0&0&0&0\\ 1&1&1&1&0&0&0&0&0&0&0&0\\ 1&1&1&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&1&1&1&0&0&0&0\\ 0&0&0&0&1&1&1&1&0&0&0&0\\ 0&0&0&0&1&1&1&1&0&0&0&0\\ 0&0&0&0&1&1&1&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&1&1&1\\ 0&0&0&0&0&0&0&0&1&1&1&1\\ 0&0&0&0&0&0&0&0&1&1&1&1\\ 0&0&0&0&0&0&0&0&1&1&1&1 \end{bmatrix}; Z^{01} = \begin{bmatrix} 1&0&0&0&1&0&0&0&1&0&0&0\\ 0&1&0&0&0&1&0&0&0&1&0&0\\ 0&0&1&0&0&0&1&0&0&0&1&0\\ 0&0&0&1&0&0&0&1&0&0&0&1\\ 1&0&0&0&1&0&0&0&1&0&0&0\\ 0&1&0&0&0&1&0&0&0&1&0&0\\ 0&0&1&0&0&0&1&0&0&0&1&0\\ 0&0&0&1&0&0&0&1&0&0&0&1\\ 1&0&0&0&1&0&0&0&1&0&0&0\\ 0&1&0&0&0&1&0&0&0&1&0&0\\ 0&0&1&0&0&0&1&0&0&0&1&0\\ 0&0&0&1&0&0&0&1&0&0&0&1 \end{bmatrix}$$

$$\text{and } Z^{00} = \begin{bmatrix} 1&1&1&1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1&1&1&1 \end{bmatrix}$$

hence $C = \frac{5}{3}Z^{11} + \frac{1}{3}Z^{10} + \frac{1}{3}Z^{01} - \frac{1}{3}Z^{00}$ *that is,*

$$C = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 2 & 0 & 0 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 2 & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} \\ 0 & 0 & 0 & 2 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 2 & 0 & 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 2 & 0 & 0 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & 0 & 0 & 2 & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 & 0 & 2 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 2 & 0 & 0 & 0 \\ -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 2 & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & 0 & 0 & 2 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

hence $P^{10}CP^{10'} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 3I^{(10)}$ *where* $I^{(y)}$ *is the Identity matrix of order*

$$\begin{aligned} \prod (s_i - 1)^{y_i} &= (s_1 - 1)^1 (s_2 - 1)^0 \\ &= (3-1)^1(4-1)^0 \\ &= 2. \end{aligned}$$

Similarly it may be seen that $P^{01}CP^{01'} = \frac{8}{3}I^{(01)}, P^{11}CP^{11'} = \frac{5}{3}I^{(11)}$. *By equations* (25) *and* (26) *shows that* $\rho(1,0) = 3, \rho(0,1) = \frac{8}{3}$ *and* $\rho(1,1) = \frac{5}{3}$. *Hence by Corollary 2.26 the interaction efficiencies in the design under consideration are given by*

$$\begin{aligned} E[1,0] &= \frac{\rho(1,0)}{r} = \frac{3}{3} = 1.0 \\ E[0,1] &= \frac{\rho(0,1)}{r} = \frac{8}{3(3)} = \frac{8}{9} \\ E[1,1] &= \frac{\rho(1,1)}{r} = \frac{5}{3(3)} = \frac{5}{9} \end{aligned}$$

2. Type III Designs

Let there exist a $BA(T)(n_1, s, 1)$, by Corollary 2.5. This corresponds to $n_1 s \times s$ BAFD with $k = n_1 s$, $b = (n_1 s - 1)s$, and

$$\lambda(0,0) = n_1 s - 1; \;\; \lambda(0,1) = 0; \;\; \lambda(1,0) = n_1 - 1; \;\; \lambda(1,1) = n_1 \tag{4}$$

by equations (15), (16) and (17), the eigenvalues of NN^T are

$$g(1,0) = 0; \;\; g(0,1) = 0; \;\; g(1,1) = n_1 s \tag{5}$$

If there exists a resolvable $BA(T)(n_2, s, 1)$,then this corresponds to a resolvable $n_2 s \times s$ BAFD. By Theorem 2.8, if we replace the levels of the second factor of the $n_1 s \times s$ BAFD by the blocks of the $n_2 s \times s$ BAFD, we get an $n_1 s \times n_2 s \times s$ BAFD with $k = n_1 n_2 s^2, b = (n_1 s - 1)(n_2 s - 1)s$, and $\lambda(0,0,0) = \lambda(0,0)(n_2 s - 1) = (n_1 s - 1)(n_2 s - 1)$

$$\begin{aligned}
\lambda(0,0,1) &= \lambda(0,1)(n_2 s - 1) = 0 \\
\lambda(0,1,0) &= \lambda(0,0)(n_2 - 1) + \lambda(0,1)(n_2 s - n_2) = (n_1 s - 1)(n_2 - 1) \\
\lambda(0,1,1) &= \lambda(0,0)n_2 + \lambda(0,1)(n_2 s - n_2 - 1) = (n_1 s - 1)n_2 \\
\lambda(1,0,0) &= \lambda(1,0)(n_2 s - 1) = (n_1 - 1)(n_2 s - 1) \\
\lambda(1,0,1) &= \lambda(1,1)(n_2 s - 1) = n_1(n_2 s - 1) \\
\lambda(1,1,0) &= \lambda(1,0)(n_2 - 1) + \lambda(1,1)(n_2 s - 1) \\
&= (n_1 - 1)(n_2 - 1) + n_1(n_2 s - n_2) \\
\lambda(1,1,1) &= \lambda(1,0)n_2 + \lambda(1,1)(n_2 s - n_2 - 1) \\
&= (n_1 - 1)n_2 + n_1(n_2 s - n_2 - 1)
\end{aligned} \tag{6}$$

where $\lambda(0,0), \lambda(0,1), \lambda(1,0), \lambda(1,1)$ are given by equation (4). The eigenvalues of NN^T are

$$g[y_1, y_2, y_3] = \begin{cases} n_1 n_2 s^2, & \text{if } y_1 = y_2 = y_3 = 1; \\ n_1 n_2 s^2 (n_1 s - 1)(n_2 s - 1), & \text{if } y_1 = y_2 = y_3 = 0; \\ 0, & \text{Otherwise.} \end{cases} \tag{7}$$

hence $E[1,1,1] = 1 - \frac{1}{(n_1 s-1)(n_2 s-1)}$, and all the main effects and first order interactions are estimated with full efficiency. If further there exists a resolvable $BA(T)(n_3, s, 1)$, we can replace the levels of the third factor of the $n_1 s \times n_2 s \times s$ BAFD by the blocks of the $n_3 s \times s$ BAFD to obtain $n_1 s \times n_2 s \times n_3 s \times s$ BAFD with $k = n_1 n_2 n_3 s^3$ such that all the main effects and interactions are estimated with full efficiency except the third order interactions, which are estimated with efficiency, $1 - \frac{1}{(n_1 s-1)(n_2 s-1)(n_3 s-1)}$. Continuing this procedure, we can get an $n_1 s \times n_2 s \times \ldots \times n_L s \times s$ BAFD with $k = s^L n_1 n_2 \ldots n_L, b = s(n_1 s - 1)(n_2 s - 1) \ldots (n_L s - 1)$. The λ's can be calculated recursively by the following formulae:

Note: Replace y_i with x_i.

$$\begin{aligned}
\lambda(y_1,y_2,\ldots,y_{L-2},0,0) &= \lambda(y_1,y_2,\ldots,y_{L-2},0)(n_Ls-1)\\
\lambda(y_1,y_2,\ldots,y_{L-2},0,1) &= \lambda(y_1,y_2,\ldots,y_{L-2},1)(n_Ls-1)\\
\lambda(y_1,y_2,\ldots,y_{L-2},1,0) &= \lambda(y_1,y_2,\ldots,y_{L-2},0)(n_L-1)+\lambda(y_1,y_2,\ldots,y_{L-2},1)(n_Ls-n_L)\\
\lambda(y_1,y_2,\ldots,y_{L-2},1,1) &= \lambda(y_1,y_2,\ldots,y_{L-2},0)(n_L)+\lambda(y_1,y_2,\ldots,y_{L-2},1)(n_Ls-n_L-1)
\end{aligned} \tag{8}$$

we now prove that

$$E[1,1,\ldots,1] = 1-\frac{1}{(n_1s-1)(n_2s-1)\ldots(n_Ls-1)}$$

and all other efficiencies are 1. The proof is given by induction. Equation (19) can be written as

$$\begin{aligned}
g[y_1,y_2,\ldots,y_{L+1}] = \sum_{x_1,x_2,\ldots,x_{L-1}} & \left\{\prod_{i=1}^{L-1}\left\{(1-y_i)n_is-1\right\}^{x_i}\right\}\times\\
& \Big[\lambda(x_1,x_2,\ldots,x_{L-1},0,0)\times\left\{(1-y_L)n_Ls-1\right\}^{x_L}\left\{(1-y_{L+1})s-1\right\}^{x_{L+1}}\\
& +\lambda(x_1,x_2,\ldots,x_{L-1},0,1)\left\{(1-y_L)n_Ls-1\right\}^{x_L}\left\{(1-y_{L+1})s-1\right\}^{x_{L+1}}\\
& +\lambda(x_1,x_2,\ldots,x_{L-1},1,0)\left\{(1-y_L)n_Ls-1\right\}^{x_L}\left\{(1-y_{L+1})s-1\right\}^{x_{L+1}}\\
& +\lambda(x_1,x_2,\ldots,x_{L-1},1,1)\left\{(1-y_L)n_Ls-1\right\}^{x_L}\left\{(1-y_{L+1})s-1\right\}^{x_{L+1}}\Big]
\end{aligned}$$

using equation (7) we have

$$\begin{aligned}
g[y_1,y_2,\ldots,y_{L-1},0,0] = \sum_{x_1,x_2,\ldots,x_{L-1}} & \left\{\prod_{i=1}^{L-1}\left\{(1-y_i)n_is-1\right\}^{x_i}\right\}\Big[\lambda(x_1,x_2,\ldots,x_{L-1},0)(n_Ls-1)\\
& +\lambda(x_1,x_2,\ldots,x_{L-1},1)(n_Ls-1)\left\{(1-0)s-1\right\}^{1}\\
& +\left\{\begin{matrix}\lambda(x_1,x_2,\ldots,x_{L-1},0)(n_L-1)+\lambda(x_1,x_2,\ldots,x_{L-1},1)\\ (n_Ls-n_L)\end{matrix}\right\}\left\{n_Ls-1\right\}\\
& +\left\{\begin{matrix}\lambda(x_1,x_2,\ldots,x_{L-1},0)(n_L)+\lambda(x_1,x_2,\ldots,x_{L-1},1)\\ (n_Ls-n_L-1)\end{matrix}\right\}(n_Ls-1)(s-1)\Big]\\
& = n_Ls(n_Ls-1)\sum_{x_1,x_2,\ldots,x_{L-1}}\left\{\prod_{i=1}^{L-1}\left\{(1-y_i)n_is-1\right\}^{x_i}\right\}\Big[\lambda(x_1,x_2,\ldots,x_{L-1},0)\\
& +\lambda(x_1,x_2,\ldots,x_{L-1},1)(s-1)\Big]\\
& = n_Ls(n_Ls-1)g[y_1,y_2,\ldots,y_{L-1},0]
\end{aligned} \tag{9}$$

$$\begin{aligned}
g[y_1,y_2,\ldots,y_{L-1},0,1] = \sum_{x_1,x_2,\ldots,x_{L-1}} & \left\{\prod_{i=1}^{L-1}\left\{(1-y_i)n_is-1\right\}^{x_i}\right\}\Big[\lambda(x_1,x_2,\ldots,x_{L-1},0)(n_Ls-1)\\
& +\lambda(x_1,x_2,\ldots,x_{L-1},1)(n_Ls-1)\left\{(1-1)s-1\right\}^{1}\\
& +\Big[\lambda(x_1,x_2,\ldots,x_{L-1},0)(n_L-1)+\lambda(x_1,x_2,\ldots,x_{L-1},1)(n_Ls-n_L)\Big]
\end{aligned}$$

$$
\begin{aligned}
&\times \left\{(1-0)n_Ls-1\right\}^1 \\
&+\Big[\lambda(x_1,x_2,\ldots,x_{L-1},0)(n_L)+\lambda(x_1,x_2,\ldots,x_{L-1},1)(n_Ls-n_L-1)\Big] \\
&\times \left\{(1-0)n_Ls-1\right\}^1\left\{(1-1)s-1\right\}\Bigg] \\
&=(n_Ls-1)\sum_{x_1,x_2,\ldots,x_{L-1}}\left\{\textstyle\prod_{i=1}^{L-1}\left\{(1-y_i)n_is-1\right\}^{x_i}\right\}\Bigg[\lambda(x_1,x_2,\ldots,x_{L-1},0) \\
&-\lambda(x_1,x_2,\ldots,x_{L-1},1)+\lambda(x_1,x_2,\ldots,x_{L-1},0)(n_L-1) \\
&+\lambda(x_1,x_2,\ldots,x_{L-1},1)(n_Ls-n_L) \\
&-\lambda(x_1,x_2,\ldots,x_{L-1},0)(n_L)-\lambda(x_1,x_2,\ldots,x_{L-1},1)(n_Ls-n_L-1)\Bigg] \\
&=(n_Ls-1)(0)=0
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
g[y_1,y_2,\ldots,y_{L-1},1,0]=&\sum_{x_1,x_2,\ldots,x_{L-1}}\left\{\textstyle\prod_{i=1}^{L-1}\left\{(1-y_i)n_is-1\right\}^{x_i}\right\}\Bigg[\lambda(x_1,x_2,\ldots,x_{L-1},0)(n_Ls-1) \\
&+\lambda(x_1,x_2,\ldots,x_{L-1},1)(n_Ls-1)\left\{(1-0)s-1\right\}^1 \\
&+\Big[\lambda(x_1,x_2,\ldots,x_{L-1},0)(n_L-1)+\lambda(x_1,x_2,\ldots,x_{L-1},1)(n_Ls-n_L)\Big] \\
&\times\left\{(1-1)s-1\right\}^1 \\
&+\Big[\lambda(x_1,x_2,\ldots,x_{L-1},0)(n_L)+\lambda(x_1,x_2,\ldots,x_{L-1},1)(n_Ls-n_L-1)\Big] \\
&\times\left\{(1-1)n_Ls-1\right\}^1\left\{(1-0)s-1\right\}^1\Bigg] \\
=&\sum_{x_1,x_2,\ldots,x_{L-1}}\left\{\textstyle\prod_{i=1}^{L-1}\left\{(1-y_i)n_is-1\right\}^{x_i}\right\}\Bigg[\lambda(x_1,x_2,\ldots,x_{L-1},0)(n_Ls-1) \\
&+\lambda(x_1,x_2,\ldots,x_{L-1},1)(n_Ls-1)(s-1) \\
&-\lambda(x_1,x_2,\ldots,x_{L-1},0)(n_L-1)-\lambda(x_1,x_2,\ldots,x_{L-1},1)(n_Ls-n_L) \\
&-\lambda(x_1,x_2,\ldots,x_{L-1},0)(n_L)(s-1)-\lambda(x_1,x_2,\ldots,x_{L-1},1)(n_Ls-n_L-1)(s-1)\Bigg] \\
=&\sum_{x_1,x_2,\ldots,x_{L-1}}\left\{\textstyle\prod_{i=1}^{L-1}\left\{(1-y_i)n_is-1\right\}^{x_i}\right\} \\
&\times\Big[k_0n_Ls-k_0+k_1n_Ls^2-k_1n_Ls-k_1s+k_1-k_0n_L+k_0-k_1n_Ls+k_1n_L\Big] \\
&+\Big[-k_0n_Ls+k_0n_L-k_1n_Ls^2+k_1n_Ls+k_1n_Ls-k_1n_L+K_1s-k_1\Big] \\
=&\,0
\end{aligned} \tag{11}
$$

where $k_0=\lambda(x_1,x_2,\ldots,x_{L-1},0)$ and $k_1=\lambda(x_1,x_2,\ldots,x_{L-1},1)$

$$
g[y_1,y_2,\ldots,y_{L-1},1,1]=\sum_{x_1,x_2,\ldots,x_{L-1}}\left\{\textstyle\prod_{i=1}^{L-1}\left\{(1-y_i)n_is-1\right\}^{x_i}\right\}\Bigg[\lambda(x_1,x_2,\ldots,x_{L-1},0,0)
$$

$$
\begin{aligned}
&+\lambda(x_1,x_2,\ldots,x_{L-1},0,1)\left\{(1-y_{L+1})s-1\right\}^{x_{L+1}}\\
&+\lambda(x_1,x_2,\ldots,x_{L-1},1,0)\left\{(1-y_L)n_Ls-1\right\}^{x_L}\\
&+\lambda(x_1,x_2,\ldots,x_{L-1},1,1)\left\{(1-y_L)n_Ls-1\right\}^{x_L}\left\{(1-y_{L+1})s-1\right\}^{x_{L+1}}\Bigg]\\
&=\sum_{x_1,x_2,\ldots,x_{L-1}}\left\{\prod_{i=1}^{L-1}\left\{(1-y_i)n_is-1\right\}^{x_i}\right\}\Bigg[\lambda(x_1,x_2,\ldots,x_{L-1},0)(n_Ls-1)\\
&+\lambda(x_1,x_2,\ldots,x_{L-1},1)(n_Ls-1)\left\{(1-1)s-1\right\}^{1}\\
&+\left[\lambda(x_1,x_2,\ldots,x_{L-1},0)(n_L-1)+\lambda(x_1,x_2,\ldots,x_{L-1},1)(n_Ls-n_L)\right]\times\\
&\left\{(1-1)n_Ls-1\right\}^{1}\\
&+\left[\lambda(x_1,x_2,\ldots,x_{L-1},0)(n_L)+\lambda(x_1,x_2,\ldots,x_{L-1},1)(n_Ls-n_L-1)\right]\\
&\times\left\{(1-1)n_Ls-1\right\}^{1}\left\{(1-1)s-1\right\}^{1}\Bigg]\\
&=\sum_{x_1,x_2,\ldots,x_{L-1}}\left\{\prod_{i=1}^{L-1}\left\{(1-y_i)n_is-1\right\}^{x_i}\right\}\Bigg[\lambda(x_1,x_2,\ldots,x_{L-1},0)(n_Ls-1)\\
&-\lambda(x_1,x_2,\ldots,x_{L-1},1)(n_Ls-1)\\
&-\lambda(x_1,x_2,\ldots,x_{L-1},0)(n_L-1)-\lambda(x_1,x_2,\ldots,x_{L-1},1)(n_Ls-n_L)\\
&+\left[\lambda(x_1,x_2,\ldots,x_{L-1},0)(n_L)+\lambda(x_1,x_2,\ldots,x_{L-1},1)(n_Ls-n_L-1)\right]\Bigg]\\
&=\sum_{x_1,x_2,\ldots,x_{L-1}}\left\{\prod_{i=1}^{L-1}\left\{(1-y_i)n_is-1\right\}^{x_i}\right\}\\
&\times\begin{bmatrix}k_0n_Ls-k_0-k_1n_Ls+k_1-k_0n_L+k_0-k_1n_Ls\\ +k_1n_L+k_0n_L+k_1n_Ls-k_1n_L-k_1\end{bmatrix}\\
&=\sum_{x_1,x_2,\ldots,x_{L-1}}\left\{\prod_{i=1}^{L-1}\left\{(1-y_i)n_is-1\right\}^{x_i}\right\}\left[k_0n_Ls-k_1n_Ls\right]\\
&=n_Ls\sum_{x_1,x_2,\ldots,x_{L-1}}\left\{\prod_{i=1}^{L-1}\left\{(1-y_i)n_is-1\right\}^{x_i}\right\}\left[k_0-k_1\right]\\
&=n_Ls\sum_{x_1,x_2,\ldots,x_{L-1}}\left\{\prod_{i=1}^{L-1}\left\{(1-y_i)n_is-1\right\}^{x_i}\right\}\begin{bmatrix}\lambda(x_1,x_2,\ldots,x_{L-1},0)\\ -\lambda(x_1,x_2,\ldots,x_{L-1},1)\end{bmatrix}\\
&=n_Lsg[y_1,y_2,\ldots,y_{L-1},1] \qquad (12)
\end{aligned}
$$

where $k_0=\lambda(x_1,x_2,\ldots,x_{L-1},0)$ and $k_1=\lambda(x_1,x_2,\ldots,x_{L-1},1)$. By the recursive formulae (9), (10), (11), (12) and the initial values (7) we have

$$\begin{aligned} g(y_1, y_2, \ldots, y_{L+1}) &= s^L \prod_{i=1}^{L} n_i \quad if \quad y_1 = y_2 = \ldots = y_{L+1} = 1 \\ &= s^L \prod_{i=1}^{L} n_i(n_i s - 1) \quad if \quad y_1 = y_2 = \ldots = y_{L+1} = 0 \\ &= 0 \qquad\qquad\qquad otherwise \end{aligned} \tag{13}$$

hence the efficiencies are

$$E[y_1, y_2, \ldots, y_{L+1}] = 1 - \frac{1}{\prod_{i=1}^{L}(n_i s - 1)} \tag{14}$$

If $y_1 = y_2 = \ldots = y_{L+1} = 1$ and 0 otherwise. From the discussion of the type III designs, we state the following theorem;

Theorem 2.1. *If there exists a $BA(T)(n_1, s, 1)$, and a resolvable $BA(T)(n_i, s, 1)$ for $i = 2, 3, \ldots, l$ then we can always construct an $n_1 s \times n_2 s \times \ldots n_L s$ BAFD with*

$$k = s^L \prod_{i=1}^{L} ni, b = s \prod_{i=1}^{L}(n_i s - 1)$$

and

$$r = \prod_{i=1}^{L}(n_i s - 1)$$

such that $E[1, 1, \ldots, 1] = \frac{-1}{r} + 1$ and all other efficiencies are 1.0.

Proof. If there exist a $BA(T)(n_1, s, 1)$ by Corollary 2.5 this corresponds to $n_1 s \times s$ BAFD with $k = n_1 s, b = (n_1 s - 1)s$ If there exist a $BA(T)(n_2, s, 1)$ by Theorem 2.8 we can replace levels of the second factor in $n_1 s \times s$ by blocks of $n_2 s \times s$ to obtain $k = (n_1 s)(n_2 s) = s^2 n_1 n_2, b = (n_1 s - 1)(n_2 s - 1)s$. Continuing with this procedure if there exist a $BA(T)(n_L, s, 1)$ this corresponds to $n_L s \times s$ BAFD. If we replace the levels of L^{th} factor in $n_1 s \times n_2 s \times \cdots \times n_{L-1} s \times s$ by using blocks of $n_L s$ we obtain $n_1 s \times n_2 s \times \cdots n_{L-1} s \times n_L s$ BAFD with $k = s^L \prod_{i=1}^{L} n_i$, $b = (n_1 - 1)(n_2 - 1) \cdots (n_L s - 1)s = s \prod_{i=1}^{L}(n_i s - 1)$ and by using equations (13) and (14) it follows that $E[1, 1, \cdots, 1] = -\frac{1}{r} + 1$ and all other efficiencies are 1.00. □

Example 2.2. *A $BA(T)(3, 2, 1)$ is given in example 2.10 and a resolvable $BA(T)(2, 2, 1)$ given in example 2.6 which is equivalent to the following 4×2 resolvable BAFD.*

x_0	x_1	y_0	y_1	z_0	z_1
00	01	00	01	00	01
10	11	11	10	11	10
21	20	20	21	21	20
31	30	31	30	30	31

Table 1: 4×2 Resolvable BAFD

where x_0, x_1, y_0, y_1, z_0, z_1 represent the blocks can be used to construct a $6 \times 4 \times 2$ BAFD with $k = 24$, $b = 30$, $r = 15$, $\lambda(0,0,1) = 0$, $\lambda(0,1,0) = 5$, $\lambda(0,1,1) = 10$, $\lambda(1,0,0) = 6$, $\lambda(1,0,1) = 9$, $\lambda(1,1,0) = 8$, $\lambda(1,1,1) = 7$. The efficiencies are $E(1,1,1) = \frac{14}{15}$, and all other efficiencies are 1.0. The design can be expressed as the same table in example 2.10 the differences are the rows representing the levels of the first factor and the x_0, x_1, y_0, y_1, z_0, z_1 representing the blocks as shown above.

Example 2.3. *A $BA(T)(2,2,2)$ given in example 2.11 and a resolvable $BA(T)(2,2,1)$ given in Example 2.6 which is equivalent to the following 4×2 resolvable BAFD.*

x_0	x_1	y_0	y_1	z_0	z_1
00	01	00	01	00	01
10	11	11	10	11	10
21	20	20	21	21	20
31	30	31	30	30	31

Table 2: 4×2 Resolvable BAFD

where $x_0, x_1, y_0, y_1, z_0, z_1$ represent the blocks can be used to construct a $4 \times 4 \times 2$ BAFD with different parameters as the ones given in Theorem 2.1 and hence with different values of λ as the ones given in equation (6). For this design $k = 16$, $b = 36$, $r = 18$, $\lambda(1,0) = 6$, $\lambda(1,1) = 12$, $\lambda(2,0) = 10$, $\lambda(2,1) = 8$ and the efficiencies are $E[1,0] = E[1,1] = E[2,0] = E[0,1] = 1.00$ and $E[2,1] = \frac{8}{9} \approx 1 - \frac{1}{r}$. The $4 \times 4 \times 2$ BAFD is given below

Blocks	1	2	3	4	5	6	7	8	9	10	11	12
Levels of F_1	Levels	of	F_2	and	F_3							
0	x_0	x_0	x_0	x_0	x_0	x_0	x_1	x_1	x_1	x_1	x_1	x_1
1	x_1	x_1	x_0	x_1	x_0	x_1	x_0	x_0	x_1	x_0	x_1	x_0
2	x_0	x_0	x_1	x_1	x_1	x_1	x_1	x_1	x_0	x_0	x_0	x_0
3	x_1	x_1	x_1	x_0	x_1	x_0	x_0	x_0	x_0	x_1	x_0	x_1

Block	13	14	15	16	17	18	19	20	21	22	23	24
Levels of F_1	Levels	of	F_2	and	F_3							
0	y_0	y_0	y_0	y_0	y_0	y_0	y_1	y_1	y_1	y_1	y_1	y_1
1	y_1	y_1	y_0	y_1	y_0	y_1	y_0	y_0	y_1	y_0	y_1	y_0
2	y_0	y_0	y_1	y_1	y_1	y_1	y_1	y_1	y_0	y_0	y_0	y_0
3	y_1	y_1	y_1	y_0	y_1	y_0	y_0	y_0	y_0	y_1	y_0	y_1

Block	25	26	27	28	29	30	31	32	33	34	35	36
Levels of F_1	Levels	of	F_2	and	F_3							
0	z_0	z_0	z_0	z_0	z_0	z_0	z_1	z_1	z_1	z_1	z_1	z_1
1	z_1	z_1	z_0	z_1	z_0	z_1	z_0	z_0	z_1	z_0	z_1	z_0
2	z_0	z_0	z_1	z_1	z_1	z_1	z_1	z_1	z_0	z_0	z_0	z_0
3	z_1	z_1	z_1	z_0	z_1	z_0	z_0	z_0	z_0	z_1	z_0	z_1

Table 3: $4 \times 4 \times 2$ BAFD

Other BAFDs that can be constructed by using Theorem 2.1 include $4 \times 4 \times 2, 6 \times 6 \times 3, 6 \times 3 \times 3, 6 \times 9 \times 3, 8 \times 4 \times 4 \ldots$ etc.

Corollary 2.4. *If s is a prime power,then there exists a $(2s)^L \times s^m (m \geq 1)$ BAFD with $k = 2^L s^{L+m-1}$, $r = (2s-1)^L (s-1)^{m-1}$, $b = (2s-1)^L (s-1)^{m-1}$, $E(L,m) = 1 - \frac{1}{r}$, and all other efficiencies are* 1.

Proof. This is a consequence of Theorem 2.1 since a resolvable $BA(T)(2,s,1)$ and a $BA(T)(1,s,1)$i.e a $TA[s(s-1),s,s,2]$ exists for s a prime power. If in addition to the conditions in Theorem 2.1, there exists a resolvable BIBD with $n_{L+1}s$ treatments and block size n_{L+1}, then we can replace the levels of the last factor of the $n_1 s \times n_2 s \times \ldots \times n_L s \times s$ BAFD by the blocks of the BIBD to get an $n_1 s \times n_2 s \times \ldots n_L s \times n_{L+1} s$ BAFD with block size $n_1 \ldots n_L n_{L+1} s^L$. All the main effects and interactions are estimated with full efficiency except the L^{th} order interactions. □

Corollary 2.5. *In an $s_1 \times s_2$ BAFD with block size $s_2 (> s_1)$ the main effects of F_1 and F_2 are estimated with full efficiency if and only if $s_2 = ms_1, \lambda_{10} = 0$ and $\frac{\lambda_{01}}{\lambda_{11}} = \frac{m-1}{m}$ for some m. The design is equivalent to a $BA[(ms_1-1)s_1\lambda, ms_1, s_1, 2]$ with parameters $\lambda(x,y) = (m-1)\lambda$ or $m\lambda$ according as $x = y$ or not, i.e. a $BA(T)(m, s_1, \lambda)$. By Theorem 2.12 for any given m and s_1 we can always construct a $BA(T)(m, s_1, \lambda)$ for some λ. Thus we can always construct an $ms_1 \times s_1$ BAFD such that all main effects are estimated with full efficiency, but a large replication may be needed. The construction of a $BA(T)(m, s_1, 1)$ for some m and s_1 are discussed in Corollary 2.14, 2.16, and 2.18. In Example 2.21 and 2.22 we also gave a $BA(T)[4,3,2]$ and a $BA(T)(3,4,2)$.*

The following examples use Corollary 2.5

Example 2.6. *A 2×4 BAFD with $b = 6, k = 4, r = 3, \lambda_{10} = 0$, can be constructed from a $BA(T)(2,2,1) = BA[6,4,2,2]$ with $\lambda(x,y) = 1$ or 2 according as $x = y$ or not*

Blocks	1	2	3	4	5	6
Levels of F_2	Levels	of	F_1			
0	1	0	1	0	1	0
1	0	1	1	0	0	1
2	1	0	0	1	0	1
3	0	1	0	1	1	0

Table 4: Table of a 2×4 BAFD

In this design, the efficiencies are: $E[0,1] = 1$, $E[1,0] = 1$ and $E[1,1] = \frac{2}{3}$.

Example 2.7. *A 7×42 BAFD with $b = 287, k = 42, r = 41, \lambda_{10} = 0, \lambda_{01} = 5, \lambda_{11} = 6$ can be constructed from a $BA(T)[6,7,1] = BA[287,42,7,2]$ with $\lambda(x,y) = 5$ or 6 according as $x = y$ or Not. The efficiencies of this designs are: $E[0,1] = 1.0$, $E[1,0] = 1.0$ and $E[1,1] = \frac{40}{41}$*

Let N be the incidence matrix of a $BAFD$ the eigenvalues of NN^T are given by

$$g(1,0) = r + (s_2 - 1)\lambda_{01} - \lambda_{10} - (s_2 - 1)\lambda_{11} \quad (15)$$

$$g(0,1) = r - \lambda_{01} + (s_1 - 1)\lambda_{10} - (s_1 - 1)\lambda_{11} \quad (16)$$

$$g(1,1) = r - \lambda_{01} - \lambda_{10} + \lambda_{11} \quad (17)$$

Theorem 2.8. *Let there be a BAFD with the incidence matrix N in $n+1$ factors $F_0, F_{m+1}, \ldots, F_{m+n}$ at $q, s_{m+1}, \ldots, s_{m+n}$ levels respectively in b blocks of k plots each.Also let there be two BAFDs with incidence matrices N^* and $N^*{}_{pq}$ as given by equations* (20) *and* (21) *respectively.If the level $j-1$ of the factor F_0 is replaced by the block $A_{iq+j}(j = 1, 2, \ldots, q)$ in each of the treatments of N ,then the design obtained by adjoining the p designs so formed (for $i = 0, 1, 2, \ldots, p-1$) is a BAFD in $m+n$ factors in bp blocks of kk^* plots each.*

This method generates an $m+n$ factor BAFD from an $n+1$ factor $BAFD$ and an m factor $BAFD$. Thus from the two-factor $BAFD$'s we can generate a three-factor $BAFD$. If the two-factor $BAFD$s are efficient, then three-factor $BAFD$ is also efficient. We can therefore construct efficient multi-factor $BAFD$'s step by step from efficient two-factor $BAFD$'s. While applying this method, the number of blocks does not increase so quickly as in the first method, but the block size does increase.

Example 2.9. *Let N be the incidence matrix of the* 3×6 *BAFD constructed by identifying rows,columns and symbols, with the levels of the second factor, the blocks ,and the levels of the first factors respectively in the $BA(T)(2,3,1)$ given in Example 2.19. Let N^* be the incidence matrix of the resolvable* 3^2 *symmetrical balanced factorial design given below*

x_0	x_1	x_2	y_0	y_1	y_2
00	01	02	00	01	02
11	12	10	12	10	11
22	20	21	21	22	20

Table 5: 3^2 Symmetrical BFD

where $x_0, x_1, x_2, y_0, y_1, y_2$ represents blocks.Then by Theorem 2.8 we can construct a $3^2 \times 6$ BAFD with $r = 10$, $b = 30$, $\lambda(2,0) = 5$, $\lambda(0,1) = 2$,$\lambda(2,1) = 3$, $\lambda(1,1) = 4$, $\lambda(1,0) = 0$, $E[2,1] = \frac{9}{10}$ and all main effects and first order interactions are estimated with full efficiency.The BAFD is given below.

Blocks	1	2	3	4	5	6	7	8	9	10
Levels of F_3	Levels	of	F_1	and	F_2					
0	x_0	x_0	x_0	x_0	x_0	x_1	x_1	x_1	x_1	x_1
1	x_1	x_2	x_1	x_2	x_0	x_2	x_0	x_2	x_0	x_1
2	x_2	x_1	x_1	x_0	x_2	x_0	x_2	x_2	x_1	x_0
3	x_2	x_2	x_0	x_1	x_1	x_0	x_0	x_1	x_2	x_2
4	x_0	x_1	x_2	x_2	x_1	x_1	x_2	x_0	x_0	x_2
5	x_1	x_0	x_2	x_1	x_2	x_2	x_1	x_0	x_2	x_0

Blocks	11	12	13	14	15	16	17	18	19	20
Levels of F_3	Levels	of	F_1	and	F_2					
0	x_2	x_2	x_2	x_2	x_2	y_0	y_0	y_0	y_0	y_0
1	x_0	x_1	x_0	x_1	x_2	y_1	y_2	y_1	y_2	y_0
2	x_1	x_0	x_0	x_2	x_1	y_2	y_1	y_1	y_0	y_2
3	x_1	x_1	x_2	x_0	x_0	y_2	y_2	y_0	y_1	y_1
4	x_2	x_0	x_1	x_1	x_0	y_0	y_1	y_2	y_2	y_1
5	x_0	x_2	x_1	x_0	x_1	y_1	y_0	y_2	y_1	y_2

Blocks	21	22	23	24	25	26	27	28	29	30
Levels of F_3	Levels	of	F_1	and	F_2					
0	y_1	y_1	y_1	y_1	y_1	y_2	y_2	y_2	y_2	y_2
1	y_2	y_0	y_2	y_0	y_1	y_0	y_1	y_0	y_1	y_2
2	y_0	y_2	y_2	y_1	y_0	y_1	y_0	y_0	y_2	y_1
3	y_0	y_0	y_1	y_2	y_2	y_1	y_1	y_2	y_0	y_0
4	y_1	y_2	y_0	y_0	y_2	y_2	y_0	y_1	y_1	y_0
5	y_2	y_1	y_0	y_2	y_0	y_0	y_2	y_1	y_0	y_1

Table 6: $3^2 \times 6$ BAFD

If there exists $TA[s_i(s_i-1), s_m, s_i, 2]$ for $i = 1, 2, \ldots, m-1$ then we can construct an $s_1 \times s_2 \times \ldots s_m$ BAFD with $k = s_m$, $b = \prod_{i=1}^{m-1} s_i(s_i-1)$, $r = \prod_{i=1}^{m-1}(s_i-1)$, $\lambda(1,1,\ldots,1) = 1$ and other λ's being 0. By Theorem 2.25 the eigenvalues of NN^T of a BAFD are given by

$$\begin{aligned} g(y_1, y_2, \ldots, y_m) &= rk - k\rho(y_1, y_2, \ldots, y_m) \\ &= rk - \left\{ r(k-1) - \textstyle\sum_{x\in\Omega} \lambda(x) \left\{ \prod_{i=1}^{m} \left[(1-y_i)s_i - 1 \right]^{x_i} \right\} \right\} \quad (18) \\ &= r + \sum_{x\in\Omega} \lambda(x) \left\{ \textstyle\prod_{i=1}^{m} \left[(1-y_i)s_i - 1 \right]^{x_i} \right\} \quad (19) \end{aligned}$$

By Theorem 2.23 we can construct a 4×6 BAFD with $k = 12, r = \lambda_{00} = 15, b = 30, \lambda_{10} = 5, \lambda_{01} = 6, \lambda_{11} = 8$ with $E[1,0] = 1, E[0,1] = 1, E[1,1] = \frac{14}{15}$.

Example 2.10. *A* 4×6 *BAFD with block size* 12 *can be constructed using* $BA[10, 6, 2, 2]$ *and a resolvable BIBD with 4 treatments and block size* 2 *as shown below. Consider the following BIBD with* 4 *treatments and block size* 2 *where* $X_0, X_1, Y_0, Y_1, Z_0, Z_1$ *represents the blocks.*

X_0	X_1	Y_0	Y_1	Z_0	Z_1
0	2	0	1	0	1
1	3	2	3	3	2

Table 7: Table of BIBD[4,6,2]

Also consider the $BA(T)(3, 2, 1)$ given below

0	0	0	0	0	1	1	1	1	1
0	0	1	1	1	0	0	0	1	1
1	0	0	1	1	1	1	0	0	0
0	1	1	0	1	0	1	1	0	0
1	1	1	0	0	1	0	0	0	1
1	1	0	1	0	0	0	1	1	0

Table 8: Table of BA(T)[3,2,1]

Blocks	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
levels of F_2	Levels	of	F_1																											
0	X_0	X_0	X_0	X_0	X_0	X_1	X_1	X_1	X_1	X_1	Y_0	Y_0	Y_0	Y_0	Y_0	Y_1	Y_1	Y_1	Y_1	Y_1	Z_0	Z_0	Z_0	Z_0	Z_0	Z_1	Z_1	Z_1	Z_1	Z_1
1	X_0	X_0	X_1	X_1	X_1	X_0	X_0	X_0	X_1	X_1	Y_0	Y_0	Y_1	Y_1	Y_1	Y_0	Y_0	Y_0	Y_1	Y_1	Z_0	Z_0	Z_1	Z_1	Z_1	Z_0	Z_0	Z_0	Z_1	Z_1
2	X_1	X_0	X_0	X_1	X_1	X_1	X_1	X_0	X_0	X_0	Y_1	Y_0	Y_0	Y_1	Y_1	Y_1	Y_1	Y_0	Y_0	Y_0	Z_1	Z_0	Z_0	Z_1	Z_1	Z_1	Z_1	Z_0	Z_0	Z_0
3	X_0	X_1	X_1	X_0	X_1	X_0	X_1	X_1	X_0	X_0	Y_0	Y_1	Y_1	Y_0	Y_1	Y_0	Y_1	Y_1	Y_0	Y_0	Z_0	Z_1	Z_1	Z_0	Z_1	Z_0	Z_1	Z_1	Z_0	Z_0
4	X_1	X_1	X_1	X_0	X_0	X_1	X_0	X_0	X_0	X_1	Y_1	Y_1	Y_1	Y_0	Y_0	Y_1	Y_0	Y_0	Y_0	Y_1	Z_1	Z_1	Z_1	Z_0	Z_0	Z_1	Z_0	Z_0	Z_0	Z_1
5	X_1	X_1	X_0	X_1	X_0	X_0	X_0	X_1	X_1	X_0	Y_1	Y_1	Y_0	Y_1	Y_0	Y_0	Y_0	Y_1	Y_1	Y_0	Z_1	Z_1	Z_0	Z_1	Z_0	Z_0	Z_0	Z_1	Z_1	Z_0

Table of a 4×6 BAFD

The example below uses Theorem 2.20.

Example 2.11. *Let* $M = [0, 1]$. *Among the differences of the corresponding elements of any two rows of the following array* 0 *occurs twice wheres* 1 *occurs four times*

0	0	0	0	0	0
1	1	0	1	0	1
0	0	1	1	1	1
1	1	1	0	1	0

hence we can construct a $BA[12, 4, 2, 2]$ *shown in table 9 below*

0	0	0	0	0	0	1	1	1	1	1	1
1	1	0	1	0	1	0	0	1	0	1	0
0	0	1	1	1	1	1	1	0	0	0	0
1	1	1	0	1	0	0	0	0	1	0	1

Table 9: Table $BA[12, 4, 2, 2]$ = BA(T)[2,2,2]

Parameters of this Balanced array:

$$\lambda(0,0) = \lambda(1,1) = 2; \;\; \lambda(0,1) = \lambda(1,0) = 4$$

Theorem 2.12. *For all k and s,there always exists a* $BA[T][k, s, \lambda]$ *for some* λ.

Proof. For all k and s, there exists a $TA[(ks-1)ksn, ks, ks, 2]$ for some n. Let the symbols of the transitive array be denoted by $[0, 1, ..., ks-1]$. If we replace each symbol in the transitive array by $x(mod k)$. Then the transitive array becomes a $BA[(ks-1)ksn, ks, s, 2]$ with parameters $\lambda(x, y) = (k-1)kn$ or k^2n according as $x = y$ or not, which is a $BA[T][ks, s, kn]$. The method of construction in Theorem 2.12 does not usually provide balanced arrays with a small number of assemblies as we desire. □

Example 2.13. *Suppose* $k = 2$,$s = 2$, *and* $n = 1$ *then we can construct a* $TA[(ks-1)ksn, ks, ks, 2] = TA[12, 4, 4, 2]$

3	1	0	2	1	3	2	0	2	0	1	3
1	3	2	0	2	0	1	3	3	1	0	2
2	0	1	3	3	1	0	2	1	3	2	0
0	2	3	1	0	2	3	1	0	2	3	1

Table 10: $TA[12, 4, 4, 2]$

replacing every symbol in $TA[12, 4, 4, 2]$ by $x(mod2)$, we have a $BA[12, 4, 2, 2] = BA(T)[2, 2, 2]$

1	1	0	0	1	1	0	0	0	0	1	1
1	1	0	0	0	0	1	1	1	1	0	0
0	0	1	1	1	1	0	0	1	1	0	0
0	0	1	1	0	0	1	1	0	0	1	1

Table 11: BA[12,4,2,2]

parameters of $BA[12,4,4,2]$ are $\lambda(0,0) = \lambda(1,1) = 2 = (k-1)kn$ and $\lambda(1,0) = \lambda(0,1) = 4 = k^2n$.

Corollary 2.14. *If a hadamard matrix of order* $4k$ *exists, then a* $BA(T)[k,2,1]$ *exists, and can always be constructed.*

Proof. If a hadamard matrix of order $4k$ exists, we can arrange its elements such that all the elements in the first column and the first row are $+1$. All other columns must then contain $2k(+1's)$ and $2k(-1's)$. Deleting $2k$ rows whose second column is 1. We obtain $OA[4k,2k,2,2]$ with all the elements equal to $+1$ in the first column and equal to -1 in the second column. We then construct a $BA(T)[k,2,1]$ since the $OA[4k,2k,2,2]$ is partly resolvable. □

Example 2.15. *Using the Sylvester type hadamard matrix of order* 8*,* $k = 4$*, that leads to an* $OA[16,8,2,2]$ *We then obtain* $BA[T][4,2,1]$*.*

0	0	0	0	0	0	0	1	1	1	1	1	1	1
1	0	1	0	1	0	1	0	1	0	1	0	1	0
0	1	1	0	0	1	1	1	0	0	1	1	0	0
1	1	0	0	1	1	0	0	0	1	1	0	0	1
0	0	0	1	1	1	1	1	1	1	0	0	0	0
1	0	1	1	0	1	0	0	1	0	0	1	0	1
0	1	1	1	1	0	0	1	0	0	0	0	1	1
1	1	0	1	0	0	1	0	0	1	0	1	1	0

Table 12: Table $BA[14,8,2,2] = BA[(T)[4,2,1]]$

Parameters of $BA(T)[4,2,1]$

- $\lambda(0,0) = \lambda(1,1) = 3$
- $\lambda(0,1) = \lambda(1,0) = 4$

Corollary 2.16. *If k and s are both powers of the same prime p a* $BA(T)[k,s,1]$ *can always be constructed.*

Proof. We can always construct a completely resolvable orthogonal array $OA[\lambda s^2, \lambda(s+1)+1, s, 2]$ by deleting any $\lambda + 1$ constraints(factors) we obtain $OA[\lambda s^2, \lambda s, s, 2]$. □

Example 2.17. *For* $k = 3$ *and* $s = 3$ *we can construct a* $BA(T)[3,3,1]$ *by first constructing a completely resolvable* $OA[27,9,3,2]$ *We eventually obtain* $BA(T)[3,3,1]$

0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2
0	1	2	0	1	2	0	1	2	1	2	0	1	2	0	1	2	0	2	0	1	2	0	1	2	0	1
0	2	1	0	2	1	0	2	1	1	0	2	1	0	2	1	0	2	2	1	0	2	1	0	2	1	0
0	0	0	1	1	1	2	2	2	1	1	1	2	2	2	0	0	0	2	2	2	0	0	0	1	1	1
0	1	2	1	2	0	2	0	1	1	2	0	2	0	1	0	1	2	2	0	1	0	1	2	1	2	0
0	2	1	1	0	2	2	1	0	1	0	2	2	1	0	0	2	1	2	1	0	0	2	1	1	0	2
0	0	0	2	2	2	1	1	1	1	1	1	0	0	0	2	2	2	2	2	2	1	1	1	0	0	0
0	1	2	2	0	1	1	2	0	1	2	0	0	1	2	2	0	1	2	0	1	1	2	0	0	1	2
0	2	1	2	1	0	1	0	2	1	0	2	0	2	1	2	1	0	2	1	0	1	0	2	0	2	1

Table 13: An $OA(27, 9, 3, 2)\lambda = 3$

0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2
1	2	0	1	2	0	1	2	2	0	1	2	0	1	2	0	0	1	2	0	1	2	0	1
2	1	0	2	1	0	2	1	0	2	1	0	2	1	0	2	1	0	2	1	0	2	1	0
0	0	1	1	1	2	2	2	1	1	2	2	2	0	0	0	2	2	0	0	0	1	1	1
1	2	1	2	0	2	0	1	2	0	2	0	1	0	1	2	0	1	0	1	2	1	2	0
2	1	1	0	2	2	1	0	0	2	2	1	0	0	2	1	1	0	0	2	1	1	0	2
0	0	2	2	2	1	1	1	1	1	0	0	0	2	2	2	2	2	1	1	1	0	0	0
1	2	2	0	1	1	2	0	2	0	0	1	2	2	0	1	0	1	1	2	0	0	1	2
2	1	2	1	0	1	0	2	0	2	0	2	1	2	1	0	1	0	1	0	2	0	2	1

Table 14: Table $BA(T)[3, 3, 1] = BA[24, 9, 3, 2]$

Parameters are; $\lambda(0,0) = \lambda(1,1) = \lambda(2,2) = 2$; $\lambda(0,1) = \lambda(1,0) = \lambda(0,2) = \lambda(2,0) = \lambda(1,2) = \lambda(2,1) = 3$.

Corollary 2.18. *If $s = p^n, k = 2s^l$ where p is an odd prime, $n \geq 1$ and $l \geq 0$, then a $BA(T)[k, s, 1]$ can always be constructed.*

Proof. We can always construct $OA[ks^2, ks, s, 2]$ by developing a difference scheme $D(2s, 2s, s)$. We then construct a $BA(T)(k, s, \lambda)$ □

Example 2.19. *For $s = 3$ and $k = 2$ implies $3 = 3^1, k = 2.3^0 \mapsto n = 1$ and $l = 0$ We can therefore construct*

$$OA[2.3^2, 2.3, 3, 2] = OA[18, 6, 3, 2]$$

by developing a difference scheme $D(2s, 2s, s) = D(6, 6, 3)$

Table 15 shows a difference scheme $D(6, 6, 3)$ constructed in a similar way from $GF(3)$

0	0	0	0	0	0
0	1	2	1	2	0
0	2	1	1	0	2
0	2	2	0	1	1
0	0	1	2	2	1
0	1	0	2	1	2

Table 15: A difference Scheme $D(6, 6, 3)$

0	0	0	0	0	0	1	1	1	1	1	1	2	2	2	2	2	2
0	1	2	1	2	0	1	2	0	2	0	1	2	0	1	0	1	2
0	2	1	1	0	2	1	0	2	2	1	0	2	1	0	0	2	1
0	2	2	0	1	1	1	0	0	1	2	2	2	1	1	2	0	0
0	0	1	2	2	1	1	1	2	0	0	2	2	2	0	1	1	0
0	1	0	2	1	2	1	2	1	0	2	0	2	0	2	1	0	1

Table 16: Table $OA[18,6,3,2]$

From this orthogonal array we obtain $BA(T)[2,3,1]$

0	0	0	0	0	1	1	1	1	1	2	2	2	2	2
1	2	1	2	0	2	0	2	0	1	0	1	0	1	2
2	1	1	0	2	0	2	2	1	0	1	0	0	2	1
2	2	0	1	1	0	0	1	2	2	1	1	2	0	0
0	1	2	2	1	1	2	0	0	2	2	0	1	1	0
1	0	2	1	2	2	1	0	2	0	0	2	1	0	1

Table 17: Table $BA(T)[2,3,1] = BA[15,6,3,2]$

Parameters of $BA(T)[2,3,1]$

- $\lambda(0,0) = \lambda(1,1) = \lambda(2,2) = 1$
- $\lambda(0,1) = \lambda(1,0) = \lambda(0,2) = \lambda(2,0) = \lambda(1,2) = \lambda(2,1) = 2$

Theorem 2.20. *Let M be a module of s elements. It is possible to choose k rows and N columns ($N = \lambda_1 + \lambda_2(s-1), \lambda_1$ and λ_2 integers)*

$$\begin{array}{cccccc} a_{11} & a_{12} & . & . & . & a_{1N} \\ a_{21} & a_{22} & . & . & . & a_{2N} \\ . & . & . & . & . & . \\ a_{k1} & a_{k2} & . & . & . & a_{kN} \end{array}$$

with elements belonging to M such that among the differences of the corresponding elements of any two rows, the element 0 occurs λ_1 times and the other non zero elements occur λ_2 times, then by adding the elements of the module to the elements in the above array and reducing mod s, we can generate Ns columns: this constitutes a $BA[N,k,s,2]$ with parameters $\lambda(x,y) = \lambda_1$ or λ_2 according as $x = y$ or $x \neq y$.

Example 2.21. *Let $M = [0,1,2]$. Among the Differences of corresponding elements of any two rows of the following array,*0 *occurs* 6 *times wheres* 1 *and* 2 *each occur* 8 *times.*

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	2	2	0	1	0	0	2	1	2	1	2	1	1	0	2	0	0	1	2	1	2
1	1	2	2	0	1	0	0	2	1	2	2	2	1	1	0	2	0	0	1	2	1
2	1	1	2	2	0	1	0	0	2	1	1	2	2	1	1	0	2	0	0	1	2
1	2	1	1	2	2	0	1	0	0	2	2	1	2	2	1	1	0	2	0	0	1
2	1	2	1	1	2	2	0	1	0	0	1	2	1	2	2	1	1	0	2	0	0
0	2	1	2	1	1	2	2	0	1	0	0	1	2	1	2	2	1	1	0	2	0
0	0	2	1	2	1	1	2	2	0	1	0	0	1	2	1	2	2	1	1	0	2
1	0	0	2	1	2	1	1	2	2	0	2	0	0	1	2	1	2	2	1	1	0
0	1	0	0	2	1	2	1	1	2	2	0	2	0	0	1	2	1	2	2	1	1
2	0	1	0	0	2	1	2	1	1	2	1	0	2	0	0	1	2	1	2	2	1
2	2	0	1	0	0	2	1	2	1	1	1	1	0	2	0	0	1	2	1	2	2

hence we can construct a $BA[66,12,3,2]$ with parameters $\lambda(x,y) = 6$ or 8 according as $x = y$ or not. i.e $BA(T)[4,3,2]$.

Example 2.22. *Let $M = [0,1,2,3]$. Among the differences of the corresponding elements of any two rows of the following array,0 occurs 4 times, wheres 1,2 and 3 occur 6 times each.*

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	1	2	0	2	1	1	3	2	3	1	0	3	2	0	2	3	3	1	2	1
3	3	0	1	2	0	2	1	1	3	2	1	1	0	3	2	0	2	3	3	1	2
2	3	3	0	1	2	0	2	1	1	3	2	1	1	0	3	2	0	2	3	3	1
3	2	3	3	0	1	2	0	2	1	1	1	2	1	1	0	3	2	0	2	3	3
1	3	2	3	3	0	1	2	0	2	1	3	1	2	1	1	0	3	2	0	2	3
1	1	3	2	3	3	0	1	2	0	2	3	3	1	2	1	1	0	3	2	0	2
2	1	1	3	2	3	3	0	1	2	0	2	3	3	1	2	1	1	0	3	2	0
0	2	1	1	3	2	3	3	0	1	2	0	2	3	3	1	2	1	1	0	3	2
2	0	2	1	1	3	2	3	3	0	1	2	0	2	3	3	1	2	1	1	0	3
1	2	0	2	1	1	3	2	3	3	0	3	2	0	2	3	3	1	2	1	1	0
0	1	2	0	2	1	1	3	2	3	3	0	3	2	0	2	3	3	1	2	1	1

hence we can construct a $BA[88,12,4,2]$ with parameters $\lambda(x,y) = 4$ or 6 according as $x = y$ or not,i.e $BA(T)[3,4,2]$.

Assume that there exists a BAFD with m factors $F_1, F_2, \ldots F_m$ at $s_1, s_2, \ldots, s_m$ levels respectively, each of the $v*$ treatments replicated r^* times in b^* blocks of k^* plots each, with the incidence matrix:

$$N^* = [A_1^*|A_2^*|\ldots|A_b^**] \tag{20}$$

Further assume that $b^* = pq$, and the pq bloacks can be divided into p groups of q blocks each, such that the design consisting of p blocks formed by adding together all the blocks of a group is a BAFD. The incidence matrix is

$$N_{pq}^* = \left[\sum_{j=1}^{q} A^*{}_j | \sum_{j=1}^{q} A^*{}_{j+q} | \ldots | \sum_{j=1}^{q} A^*{}_{pq-q+j}\right] \tag{21}$$

for a resolvable design N^*, the corresponding N_{pq}^* exists with $p = r^*$.

Theorem 2.23. *If there exists a resolvable BIBD with qs treatments and block size q, then there exists a $ps \times qs$ BAFD with block size pqs such that all main effects are estimated with full efficiency.*

Proof. Construct a $BA(T)(p,s,n)$ for some integer n by Theorem 2.12. In the resolvable BIBD, there being s blocks in each replication , we can number the block in each replication by $0, 1, \cdots, s-1$. Replacing each symbol in the balanced array by a group of symbols which represents blocks in the BIBD for each replication, we obtain a $pqs \times [ps-1]snr'$ matrix, where r' is the number of replications in the BIBD. Assign i^{th} level of F_1 to the rows from the $(i_q+1)^{th}$ to the $(i+1)^{th}$, where $i = 0, 1, \cdots, ps-1$.Identifying columns and symbols with blocks and the levels of F_2, we get a $ps \times qs$ design with block size pqs.
We shall show that all the main effects of the design constructed above are estimated with full efficiency.Let λ' be the number of blocks in which two treatments occur together in the BIBD, then $(qs-1)\lambda' = (q-1)r'$. Assume that $r' = (qs-1)m$ and $\lambda' = (q-1)m$, where m need not be an integer. Let $\lambda_{01}, \lambda_{10}, \lambda_{11}$ denote the parameters and r denote the number of replications in the $ps \times qs$ design, then through inspection we have □

$$\lambda(x,y) = (ps-1)^{x+1}(qs-1)^{y+1}(p-1)^{x}(q-1)^{y}mn + (xy)(pq)(s-1)^{xy}mn \tag{22}$$

$x, y = 0$ or 1 in mod 2

so

$$\left\{\begin{array}{c} \lambda_{01} = (ps-1)(q-1)mn \\ \lambda_{10} = (qs-1)(p-1)mn \\ \lambda_{11} = (p-1)(q-1)mn + pq(s-1)mn \\ \lambda_{00} = r = (ps-1)(qs-1)mn \end{array}\right\} \tag{23}$$

if we substitute the parameters of the equations (15), (16) and (17) in equations (23) and Corollary 2.26 we get

$$E[0,1] = E[1,0] = 1 \quad \text{and} \quad E[1,1] = -\frac{s-1}{(ps-1)(qs-1)} + 1$$

Given any q and s, there always exists a resolvable BIBD with qs treatments and block size q if the number of replications is allowed to be large.

Example 2.24. *The irreducible BIBD of qs treatments with block size q in which each of the $\binom{qs}{q}$ possible q− element combinations form a block is resolvable with parameters*

$$v = qs, \quad b = \binom{qs}{q}, \quad r = \binom{qs-1}{q-1}, \quad k = q, \quad \lambda = \binom{qs-2}{q-2} \tag{24}$$

Let $F_1, F_2, \ldots, F_m$ be m factors at $s_1, s_2, \ldots, s_m$ levels respectively and N be the incidence matrix of a BAFD

Theorem 2.25. *The eigenvalues of NN' of a BAFD are $g(y_1, y_2, \ldots, y_m)'s$ with corresponding eigenvectors given by the columns of $p^{y'}$ where $y = (y_1, y_2, \ldots, y_m) \in \Omega$.*

It should be noted that the multiplicity of $g(y_1, y_2, \ldots, y_m)$ is $\prod_{i=1}^{m}(s_i - 1)^{y_i}$. Since $C = r(\otimes_{i=1}^{m} I_i) - k^{-1}NN'$, The columns of $P^{y'}$ $y \in \Omega$ are also the eigenvectors of C with corresponding eigenvalues

$$\rho(y) = r - \frac{1}{k}g(y_1, y_2, \ldots, y_m) \tag{25}$$

$$= r - \frac{1}{k}g(y), \qquad y \in \Omega \tag{26}$$

Corollary 2.26. *Let $E(y)$ denote the interaction efficiencies of a BAFD, where $g(y)$ denotes the eigenvalues of NN^T then $E(y) = 1 - \frac{1}{rk}g(y)$ and $E(y) = 1$ if and only if $g(y) = 0$*

Definition 2.27. *Suppose we have a $C-$ matrix of the design in $v(= s_1 s_2 \ldots s_m)$ treatment combinations, then the design is said to possess property A if*

$$C = \sum_{y \in \Omega^*} g(y)(J_1 - I_1)^{y_1} \otimes (J_2 - I_2)^{y_2} \otimes \ldots \otimes (J_m - I_m)^{y_m} \tag{27}$$

where $g(y)'s$ are constants depending on y_i's and $y_i = 0$ or 1 and $(J_i - I_i)^{y_i} = J_i - I_i$ if $y_i = 1$ while $(J_i - I_i)^{y_i} = I_i$ if $y_i = 0$

The element which is in the $(x_1, x_2, \ldots, x_m)^{th}$ row and $(y_1, y_2, \ldots, y_m)^{th}$ column of the matrix (the treatments are in lexicographic order) is 1 if $(x_1, x_2, \ldots, x_m)$ and $(y_1, y_2, \ldots, y_m)$ are $(y_1, y_2, \ldots, y_m)^{th}$ associates, and 0 otherwise. Two treatments which are $(y_1, y_2, \ldots, y_m)^{th}$ associates occur together in $\lambda y_1 y_2 \ldots y_m$ blocks; hence we have the following lemma

Lemma 2.28. *Let N be the incidence matrix of a BAFD; then*

$$NN' = \sum_{y \in \Omega^*} \lambda y_1 y_2 \ldots y_m (J_1 - I_1)^{y_1} \otimes (J_2 - I_2)^{y_2} \otimes \ldots \otimes (J_m - I_m)^{y_m} \tag{28}$$

where $\lambda_{000\ldots0}$ is defined to be r.

Further let $J_i = s_i \otimes s_i$ to be a matrix with all elements equal to 1. Let Ω^* be the set of all $m-$ component binary vectors, that is $\Omega^* = \Omega \cup \{(0, 0, \ldots, 0)\}$ where Ω is the set of $2^m - 1$ none null binary $m-$ tuples. For $y = (y_1, y_2, \ldots, y_m) \in \Omega^*$ let

$$Z^y = \otimes_{i=1}^{m} Z_i^{y_i} \tag{29}$$

where for $1 \le i \le m$,

$$\begin{aligned} Z_i^{y_i} &= I_i \quad \text{if} \quad y_i = 1 \\ &= J_i \quad \text{if} \quad y_i = 0 \end{aligned} \tag{30}$$

Definition 2.29. *A $v \times v$ matrix G where $v = \Pi s_i$ will be said to have property A if it is of the form $G = \sum_{y \in \Omega^*} h(y) Z^y$ where $h(y), y \in \Omega^*$, are real numbers*

$$\text{Let} \quad M^{000\ldots0} = \otimes_{i=1}^{m} (s_i^{-1} J_i), \tag{31}$$

Which defines M^y for every $y \in \Omega^*$. Also let the $(v-1) \times v$ matrix P be defined as

$$P = (\ldots, P^{y'}, \ldots)', \tag{32}$$

where P^y is included in P for every $y \in \Omega$. For example if $m = 2$ then $P = (P^{01'}, P^{10'}, P^{11'})'$

Definition 2.30. *An $r \times c$ array D with entries from $\mathcal{A}$ is called a **difference scheme** based on $(\mathcal{A}, +)$ if it has the property that for all i and j with $1 \leq i, j \leq c$, the vector difference between the i^{th} and j^{th} columns contains every element of $\mathcal{A}$ equally often if $i \neq j$*

Example 2.31. *We illustrate the construction for the case $p = 3, m = 2, n = 1$, this will result to a difference scheme $D(9,9,3)$. In this special case the field $GF(p^n)$ in the construction is actually the subfield of $GF(p^m)$ and the multiplication of elements of $GF(p^n)$ is the same in both fields. Table 18 is a multiplication table for $GF(3^2)$, based on the irreducible polynomial $f(x) = x^2 + x + 2$, we represent the nine elements of $GF(3^2)$ in condensed notation writing* 0 *as* 00, 1 *as* 10, $1 + 2x$ *as* 01 *and so on.*

(*)	00	10	20	01	11	21	02	12	22
00	00	00	00	00	00	00	00	00	00
10	00	10	20	01	11	21	02	12	22
20	00	20	10	02	22	12	01	21	11
01	00	01	02	12	10	11	21	22	20
11	00	11	22	10	21	02	20	01	12
21	00	21	12	11	02	20	22	10	01
02	00	02	01	21	20	22	12	11	10
12	00	12	21	22	01	10	11	20	02
22	00	22	11	20	12	01	10	02	21

Table 18: Multiplication table for $GF(3^2)$

Upon applying the map: For $m = 2 : b_0 + b_1 x$ is the polynomial and for $n = 1 : b_0$ is the polynomial hence we apply the map. $b_0 + b_1 x \mapsto b_0$ to the entries of this table to obtain the Difference Scheme $D(9,9,3)$ based on $(GF(3), +)$ which is exhibited in Table 19

0	0	0	0	0	0	0	0	0
0	1	2	0	1	2	0	1	2
0	2	1	0	2	1	0	2	1
0	0	0	1	1	1	2	2	2
0	1	2	1	2	0	2	0	1
0	2	1	1	0	2	2	1	0
0	0	0	2	2	2	1	1	1
0	1	2	2	0	1	1	2	0
0	2	1	2	1	0	1	0	2

Table 19: A difference scheme based on $(GF(3), +)$

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