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A Graph Theoretic Analysis of Leverage Eccentricity Centrality

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Abstract

In complex networks, the identification of key regions is determined through centrality measures. There are various centrality measures in which leverage centrality is specially designed for neural network. The concept of leverage centrality is the relationship between degree of a vertex relative to its neighbours and it operates under the principle that a vertex in a network is central if its immediate neighbours rely on it for information. In this paper, we built upon leverage centrality and introduce leverage eccentricity centrality, which is a measure of how eccentricity of a vertex rely on the eccentricity of all its neighbours. We investigate this property from a mathematical perspective. We first outline some of the basic properties and then compute leverage eccentricity centrality of vertices in different families of graphs, line graph of some special graphs, Mycielskian of some standard graphs, and molecular graph of cyclo-hydrocarbons.

Keywords: leverage centrality; leverage eccentricity centrality; neural network; cyclo-hydrocarbons.
2020 Mathematics Subject Classification: 05C07.

1. Introduction

In the analysis of complex networks, centrality measures play a crucial role in quantifying the importance or influence of individual vertices within a network structure. Classical centrality indices such as degree, betweenness, closeness, and eigenvector centrality have been extensively employed to identify key vertices in various contexts, including social networks, neuroscience, biological systems, and infrastructure networks. However, these traditional measures often fail to capture the subtle differences between vertices with similar local connectivity patterns or to account for the comparative advantage of a vertex relative to its immediate neighbours. To address these limitations, Leverage Centrality has emerged as a distinctive and insightful measure. Leverage Centrality, first introduced by Joyce [6], quantifies the relative degree difference between a vertex and its neighbouring vertices. It effectively evaluates how connectivity of a vertex contrasts with its local environment, providing an indicator of whether a vertex exerts influence over, or is dependent upon, its neighbours. In essence,

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the measure captures local dominance or vulnerability within a network, making it particularly valuable for identifying vertices that serve as local hubs in complex systems. Leverage Centrality is computationally efficient and locally sensitive, making it suitable for large-scale networks and dynamic systems where local structures significantly influence global behaviour.

Chemical graph theory is one of the well-investigated application areas with several researchers working on applications of graph theory to discover some chemical phenomena by means of mathematical modelling. If we represent the atoms of a molecule by vertices and the chemical covalent bonds between the atoms by edges, the graph obtained is a mathematical model of the given molecule. Such graphs are usually referred as molecular graphs or chemical graphs. They clearly give a graph-theoretical representation of the molecule under consideration and provide valuable information on the required chemical phenomena [7].

Let G be a simple, connected, finite and undirected nontrivial graph with vertex set $V(G)$ and edge set $E(G)$. Further, we denote the number of vertices and edges of G by $n = |V|$ and $m = |E|$, respectively. If $u, v \in V(G)$ are adjacent then we denote it by $u \sim v$[1], the degree of v denoted by $d(v)$ or $d_G(v)$ is defined as the number of edges incident with v, the distance between u and v is denoted by $d(u,v)$ or $d_G(u,v)$ and is defined as length of the shortest path connecting u and v in G. The vertices of line graph $L(G)$ are the edges of G with two vertices of line graph adjacent whenever the corresponding edges of G are adjacent [2,9]. The eccentricity of a vertex $v \in V(G)$ is $e_v = e(v) = max\{d(u,v) : u \in V(G)\}$. A graph G is said to be self centered if $e(u) = e(v)$ for all $u, v \in V(G)$[3]. For graph theoretic terminology, we refer to [5].

Leverage centrality is a measure of the relationship between degree of a given vertex v, and the degree of each of its neighbours, averaged over all neighbours, and is defined as shown below:

$$l(v) = \frac{1}{d(v)} \sum_{u \in N(v)} \frac{d(v) - d(u)}{d(v) + d(u)}$$

where, $N(v)$ stands for the set of vertices adjacent to the vertex v[11]. In this paper, we define and compute leverage eccentricity centrality for some families of graphs, line graph of some special graphs, Mycielskian of some standard graphs and molecular graph of cyclo-hydrocarbons.

2. Main Results

Definition 2.1. *Leverage eccentricity centrality is a measure of the relationship between eccentricity of a given vertex v, and the eccentricity of each of its neighbours, averaged over all neighbours, and is defined as shown below:*

$$le(v) = \frac{1}{e(v)} \sum_{u \in N(v)} \frac{e(v) - e(u)}{e(v) + e(u)}$$

where, $N(v)$ stands for the set of vertices adjacent to the vertex v.

We now proceed to compute $le(v)$ for some standard graphs.

Proposition 2.2.

1. *For any complete graph $K_n, n \geq 2$, $le(v) = 0$, for all $v \in V(G)$.*
2. *For any cycle graph C_n, $le(v) = 0$, for all $v \in V(G)$.*
3. *For any complete bipartite graph $K_{m,n}(m, n \geq 2)$, $le(v) = 0$.*
4. *For any wheel graph $W_{n+1}, (n \geq 4)$,* $le(v) = \begin{cases} \frac{-n}{3}, & \text{if d(v)=n;} \\ \frac{1}{6}, & \text{if d(v)=3.} \end{cases}$

Theorem 2.3. *For any connected graph G, $le(v) = 0$ if and only if the vertex v and all its neighbour vertex eccentricity are equal.*

Theorem 2.4. *For any graph G, $\sum_{v \in V(G)} le(v) \leq 0$.*

Proof. If all vertices of G have the same eccentricity then, $le(v) = 0$ and $\sum_{v \in V(G)} le(v) = 0$. Suppose there exists a graph in which an edge e with end vertices u and v where $e(u) > e(v)$. We note that the contribution of each edge uv to the sum of the leverage eccentricity centralities is $\frac{1}{e(u)}\left(\frac{e(u)-e(v)}{e(u)+e(v)}\right) - \frac{1}{e(v)}\left(\frac{e(u)-e(v)}{e(u)+e(v)}\right) < 0$. Hence for such graph, the sum of leverage eccentricity centralities is $\sum_{v \in V(G)} le(v) = \sum_{(uv) \in E(G)} \frac{1}{e(u)}\left(\frac{e(u)-e(v)}{e(u)+e(v)}\right) - \frac{1}{e(v)}\left(\frac{e(u)-e(v)}{e(u)+e(v)}\right) < 0$. □

Corollary 2.5. *For any star graph $K_{1,n}, (n \geq 2)$,* $le(v) = \begin{cases} \frac{-n}{3}, & \text{if d(v)=n;} \\ \frac{1}{6}, & \text{if d(v)=1.} \end{cases}$

Theorem 2.6. *For any self centered graph $le(v) = 0$.*

Proof. Suppose G is a self centered graph, then $e(u) = e(v)$ for all $u, v \in V(G)$. Thus, $le(v) = 0$ □

Definition 2.7 ([8])**.** *The double star graph which is denoted by $S_{n,\,m},\ m, n \geq 2$ is a graph obtained from two star graphs $K_{1,\,n-1}$ and $K_{1,\,m-1}$ by joining their centers v_0 and u_0. So, $V(S_{n,\,m}) = V(K_{1,\,n-1}) \cup V(K_{1,\,m-1}) = \{v_0, v_1, \cdots, v_{n-1}\} \cup \{u_0, u_1, \cdots, u_{m-1}\}$ and $E(S_{n,\,m}) = \{v_0u_0, v_0v_i, u_0u_i : 1 \leq i \leq n-1; 1 \leq j \leq m-1\}$.*

Proposition 2.8. *The levereage eccentricity centrality of $S_{n,m}, m, n \geq 2$ is*

$$le(v) = \begin{cases} \frac{1-n}{10}, & \text{if d(v)=n;} \\ \frac{1-m}{10}, & \text{if d(v)=m;} \\ \frac{1}{15}, & \text{if d(v)=1.} \end{cases}$$

Proof. Clearly $e_{v_0} = e_{u_0} = 2$ and $e_{v_i} = e_{u_j} = 3$, for $i = 1, 2, \cdots, n-1$ and $j = 1, 2, \cdots, m-1$. Then

$$le(v_0) = \frac{1}{2}\left[\left(\frac{2-3}{2+3}\right)(n-1)\right] = \frac{1-n}{10}$$

$$le(u_0) = \frac{1}{2}\left[\left(\frac{2-3}{2+3}\right)(m-1)\right] = \frac{1-m}{10}$$
$$le(v_i) = le(u_j) = \frac{1}{3}\left[\left(\frac{3-2}{3+2}\right)\right] = \frac{1}{15}.$$

□

Definition 2.9 ([8]). *The graph Pl_n, $(n \geq 3)$, is obtained as the join of P_{n-2} and P_2.*

Proposition 2.10. *For any Pl_n, $n \geq 4$ graph,* $le(v) = \begin{cases} \frac{2-n}{3}, & \text{if } d(v)=n-1; \\ \frac{1}{3}, & \text{otherwise;} \end{cases}$

Proof. Here $e_{u_1} = e_{u_2} = 1$ and $e_{u_i} = 2$, where $i = 3, 4, \cdots, n$. We have

$$le(u_1) = le(u_2) = \frac{1}{1}\left[\left(\frac{1-2}{1+2}\right)(n-2)\right] = \frac{2-n}{3}$$
$$le(u_i) = \frac{1}{2}\left[\left(\frac{2-1}{2+1}\right)2\right] = \frac{1}{3}, i = 3, 4, \cdots, n.$$

□

Theorem 2.11. *Let $K_{1,p-1}$ be a star with $p \geq 3$, and let G_s be a spider graph which is constructed by subdividing each edge once in $K_{1,p-1}$. Then*

$$le(v) = \begin{cases} \frac{1-p}{10}, & \text{if } d(v)=p-1; \\ \frac{2}{105}, & \text{if } d(v)=2; \\ \frac{1}{28}, & \text{if } d(v)=1. \end{cases}$$

Proof. The spider graph G_s has $2p-1$ vertices and in accordance with the degree and eccentricity of vertices as displayed in Table 1. The vertex u of degree $p-1$ and eccentricity 2 is adjacent to $p-1$ vertices, each of degree 2 and eccentricity 3, then $le(u) = \frac{1}{2}\left[\left(\frac{2-3}{2+3}\right)(p-1)\right] = \frac{1-p}{10}$. The vertex $u_i, 1 \leq i \leq p-1$ of degree 2 and eccentricity 3 is adjacent to a vertex of degree 1 and eccentricity 4 and also adjacent to a vertex of degree $p-1$ and eccentricity 2, then $le(u_i) = \frac{1}{3}\left[\left(\frac{3-2}{3+2}\right) + \left(\frac{3-4}{3+4}\right)\right] = \frac{2}{105}$. The vertex $v_i, 1 \leq i \leq p-1$ of degree 1 and eccentricity 4 is adjacent to a vertex of degree 2 and eccentricity 3, then $le(v_i) = \frac{1}{4}\left[\left(\frac{4-3}{4+3}\right)\right] = \frac{1}{28}$.

Number of vertices	$d(v)$	$e(v)$
1	$p-1$	2
$p-1$	2	3
$p-1$	1	4

Table 1: Vertex partition of G_s.

□

Definition 2.12 ([8]). *The multi-star graph $K_{1,n,n,\cdots,n}$ is constructed as follows: Consider the star graph $K_{1,n}$ with vertex set $\{v_0, v_{11}, v_{12}, \cdots, v_{1n}\}$. Add an edge to each of the pendant vertices $v_{11}, v_{12}, \cdots, v_{1n}$ to get the* *resulting graph $K_{1,n,n}$ with vertices $\{v_0, v_{11}, \cdots, v_{1n}, v_{21}, \cdots, v_{2n}\}$. Again add an edge to each of the pendant vertices $v_{21}, \cdots, v_{2n}$ to get the graph $K_{1,n,n,n}$. Repeating this $(m-1)$ times, we get a graph $K_{1,\underbrace{n,\ n,\ldots,n}_{m-times}}$ called the multi-star graph with $mn+1$ vertices $v_0, v_{11}, v_{12}, \cdots, v_{1n}, v_{21}, \cdots, v_{2n}, v_{31}, \cdots, v_{3n}, \cdots, v_{m1}, \cdots, v_{mn}$ and mn edges as shown in Figure 1.*

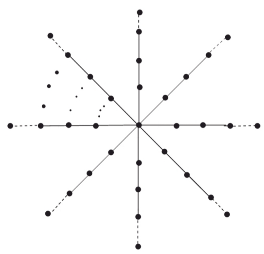

Figure 1: Multi-star graph $K_{1,\underbrace{n,n,\ldots,n}_{m-times}}$

Proposition 2.13. *For any multi-star graph $K_{1,\underbrace{n,n,\ldots,n}_{m-times}}$,*

$$le(v) = \begin{cases} \dfrac{n}{(n-1)(2n-1)}, & \text{if } e(v_0) = m; \\ \dfrac{2}{(m+1)(2m+1)(2m+3)}, & \text{if } e(v_{1j}) = m+1; \\ \dfrac{2}{(m+i)(4(m+i)^2-1)}, & \text{if } e(v_{ij}) = m+i,\ i \notin \{1,m\}; \\ \dfrac{2}{(2m-1)(4m-1)(4m-3)}, & \text{if } e(v_{(m-1)j}) = 2m-1; \\ \dfrac{1}{2m(4m-1)}, & \text{if } e(v_{mj}) = 2m. \end{cases}$$

Proof. Let $G = K_{1,\underbrace{n,n,\ldots,n}_{m-times}}$, we have $e_{v_0} = m$ and $e_{v_{ij}} = m+i$, for $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$.
For the central vertex v_0, we have

$$le(v_0) = \frac{1}{m}\left[\left(\frac{m-m+1}{m+m+1}\right)n\right] = \frac{n}{(n-1)(2n-1)}.$$

For $v_{1j} \in V(G)$, $1 \le j \le n$, we have

$$le(v_{1j}) = \frac{1}{m+1}\left[\frac{m+1-m}{m+1+m} + \frac{m+1-(m+2)}{m+1+(m+2)}\right] = \frac{2}{(m+1)(2m+1)(2m+3)}.$$

For $v_{ij} \in V(G), i \notin \{1,m\}$

$$le(v_{ij}) = \frac{1}{m+i}\left[\frac{m+i-(m+i-1)}{m+i+(m+i-1)} + \frac{m+i-(m+i+1)}{m+i+(m+i+1)}\right] = \frac{2}{(m+i)(4(m+i)^2-1)}.$$

For $v_{(m-1)j} \in V(G)$, $1 \le j \le n$, we have

$$le(v_{(m-1)}j) = \frac{1}{2m-1}\left[\frac{2m-1-2m}{2m-1+2m} + \frac{2m-1-(2m-2)}{2m-1+(2m-2)}\right] = \frac{2}{(2m-1)(4m-1)(4m-3)}.$$

For $v_{mj} \in V(G)$, $1 \leq j \leq n$, we have

$$le(v_{mj}) = \frac{1}{2m}\left[\frac{2m-(2m-1)}{2m+(2m-1)}\right] = \frac{1}{2m(4m-1)}.$$

□

Definition 2.14 ([8]). *The Dutch windmill graph D_m^n $(m \geq 3,\ n \geq 2)$, is defined as the graph having n copies of C_m with a common vertex v_0. Note that $|V(D_m^n)| = n(m-1)+1$ and $|E(D_m^n)| = mn$. If $m = 3$, then D_m^n is called friendship graph [10].*

The vertices of graph D_m^n $(m \geq 6,\ n \geq 2)$, when m is even are classified as:
Type I: vertex with eccentricity $\frac{m}{2}$ all of whose neighbours have eccentricity $\frac{m}{2}+1$.
Type II: vertices with eccentricity $\frac{m}{2}+1$ which have one neighbour with eccentricity $\frac{m}{2}$ and other neighbour with eccentricity $\frac{m}{2}+2$.
Type III: vertices with eccentricity $\frac{m}{2}+i, 2 \leq i \leq \frac{m}{2}-1$ which have one neighbour with eccentricity $\frac{m}{2}+i-1$ and other neighbour with eccentricity $\frac{m}{2}+i+1$.
Type IV: vertices with eccentricity m which have two neighbours each with eccentricity $m-1$.
The vertices of graph D_m^n $(m \geq 7,\ n \geq 2)$, when m is odd are classified as:
Type I: vertex with eccentricity $\frac{m-1}{2}$ all of whose neighbours have eccentricity $\frac{m-1}{2}+1$.
Type II: vertices with eccentricity $\frac{m-1}{2}+1$ which have one neighbour with eccentricity $\frac{m-1}{2}$ and other neighbour with eccentricity $\frac{m-1}{2}+2$.
Type III: vertices with eccentricity $\frac{m-1}{2}+i, 2 \leq i \leq \frac{m-1}{2}-1$ which have one neighbour with eccentricity $\frac{m-1}{2}+i-1$ and other neighbour with eccentricity $\frac{m-1}{2}+i+1$.
Type IV: vertices with eccentricity $m-1$ which have one neighbour with eccentricity $m-2$ and one neighbour with eccentricity $m-1$.

Proposition 2.15. *For a D_m^n $(m \geq 6,\ n \geq 2)$ graph, when m is even,*

$$le(v) = \begin{cases} \dfrac{-4n}{m^2+m}, & \text{if } v \text{ is a vertex of Type I;} \\ \dfrac{4}{(m+1)(m+2)(m+3)}, & \text{if } v \text{ is a vertex of Type II;} \\ \dfrac{4}{(m+2i)(m+2i+1)(m+2i-1)}, & \text{if } v \text{ is a vertex of Type III;} \\ \dfrac{2}{2m^2-m}, & \text{if } v \text{ is a vertex of Type IV.} \end{cases}$$

Proof. In D_m^n, $|V(G)| = n(m-1)+1$. Let m be even and v_0 be the central vertex of D_m^n. Then v_0 has eccentricity $\frac{m}{2}$, the n vertices at distance $\frac{m}{2}$ from v_0 have eccentricity m and the remaining $(m-2)n$

vertices have eccentricity $\frac{m}{2}+i$, where $1 \leq i \leq \frac{m}{2}-1$. If v is a vertex of Type I, then

$$le(v) = \frac{1}{\left(\frac{m}{2}\right)}\left[\left(\frac{\frac{m}{2}-\left(\frac{m}{2}+1\right)}{\frac{m}{2}+\left(\frac{m}{2}+1\right)}\right)2n\right] = \frac{-4n}{m^2+m}.$$

If v is a vertex of Type II, then

$$le(v) = \frac{1}{\left(\frac{m}{2}+1\right)}\left[\left(\frac{\frac{m}{2}+1-\frac{m}{2}}{\frac{m}{2}+1+\frac{m}{2}}\right)+\frac{\left(\frac{m}{2}+1\right)-\left(\frac{m}{2}+2\right)}{\left(\frac{m}{2}+1\right)+\left(\frac{m}{2}+2\right)}\right] = \frac{4}{(m+1)(m+2)(m+3)}.$$

If v is a vertex of Type III, then

$$\begin{aligned} le(v) &= \frac{1}{\left(\frac{m}{2}+i\right)}\left[\frac{\left(\frac{m}{2}+i\right)-\left(\frac{m}{2}+i+1\right)}{\left(\frac{m}{2}+i\right)+\left(\frac{m}{2}+i+1\right)}+\frac{\left(\frac{m}{2}+i\right)-\left(\frac{m}{2}+i-1\right)}{\left(\frac{m}{2}+i\right)+\left(\frac{m}{2}+i-1\right)}\right] \\ &= \frac{4}{(m+2i)(m+2i+1)(m+2i-1)}. \end{aligned}$$

where $2 \leq i \leq \frac{m}{2}-1$. If v is a vertex of Type IV, then

$$le(v) = \frac{1}{m}\left[\left(\frac{m-(m-1)}{m+(m-1)}\right)2\right] = \frac{2}{2m^2-m}.$$

□

Proposition 2.16. *For a D_m^n $(m \geq 7,\ n \geq 2)$ graph, when m is odd.*

$$le(v) = \begin{cases} \dfrac{-4n}{m^2-m}, & \textit{if } v \textit{ is a vertex of Type I;} \\ \dfrac{4}{m(m+1)(m+2)}, & \textit{if } v \textit{ is a vertex of Type II;} \\ \dfrac{4}{(m+2i-1)(m+2i)(m+2i-2)}, & \textit{if } v \textit{ is a vertex of Type III;} \\ \dfrac{1}{2m^2-5m+3}, & \textit{if } v \textit{ is a vertex of Type IV.} \end{cases}$$

Proof. Let m be odd and v_0 be the central vertex of D_m^n. Then v_0 has eccentricity $\frac{m-1}{2}$, the $2n$ vertices at a distance $\frac{m-1}{2}$ from v_0 have eccentricity $m-1$ and the remaining $(m-3)n$ vertices have eccentricity $\frac{m-1}{2}+i$, where $1 \leq i \leq \frac{m-1}{2}-1$. By replacing the value of m by $m-1$ in the above proposition we get the desired result. □

Theorem 2.17. *Let $G = C_3^n, n \geq 2$, be the friendship graph. Then for $v \in V(G)$,*

$$le(v) = \begin{cases} \frac{1}{6}, & \text{if } e(v)=2; \\ \frac{-2n}{3}, & \text{if } e(v)=1. \end{cases}$$

Proof. The order and size of G are $2n+1$ and $3n$, respectively. In reference to the degree and eccentricity of vertices we partition $V(G)$ as shown in Table 2. If v is a vertex in G of eccentricity 2, then $le(v) = \frac{1}{2}\left[\frac{2-1}{2+1}\right] = \frac{1}{6}$. If v is a vertex in G of eccentricity 1, then $le(v) = \frac{1}{1}\left[\left(\frac{1-2}{1+2}\right)2n\right] = \frac{-2n}{3}$. □

Number of vertices	$d(v)$	$e(v)$
$2n$	2	2
1	$2n$	1

Table 2: Vertex partition of C_3^n.

Theorem 2.18. *Let $H = L(C_3^n)$ be the line graph of friendship graph. Then for $v \in V(H)$,*

$$le(v) = \begin{cases} \frac{2}{15}, & \text{if } e(v)=3; \\ \frac{-1}{10}, & \text{if } e(v)=2. \end{cases}$$

Proof. The line graph H of friendship graph has $3n$ vertices and $n(2n+1)$ edges. In reference to the degree and eccentricity of vertices we partition $V(H)$ as shown in Table 3. If v is a vertex in H of eccentricity 3, then $le(v) = \frac{1}{3}\left[\left(\frac{3-2}{3+2}\right)2\right] = \frac{2}{15}$. If v is a vertex in H of eccentricity 2, then $le(v) = \frac{1}{2}\left[\left(\frac{2-3}{2+3}\right)\right] = \frac{-1}{10}$. □

Number of vertices	$d(v)$	$e(v)$
$2n$	$2n$	2
n	2	3

Table 3: Vertex partition of $L(C_3^n)$.

Definition 2.19 ([10])**.** *The graph $CW_n = C_n \circ K_1$ is called a crown graph.*

Theorem 2.20. *Let $G = CW_n$,*

1. *If n is even, then*

$$le(v) = \begin{cases} \frac{-2}{(n+2)(n+3)}, & \text{if } e(v) = \frac{n}{2}+1; \\ \frac{2}{(n+4)(n+3)}, & \text{if } e(v) = \frac{n}{2}+2. \end{cases}$$

2. *If n is odd, then*

$$le(v) = \begin{cases} \dfrac{-2}{(n+1)(n+2)}, & \text{if } e(v) = \dfrac{n-1}{2}+1; \\ \dfrac{2}{(n+2)(n+3)}, & \text{if } e(v) = \dfrac{n-1}{2}+2. \end{cases}$$

Proof. The crown graph has $2n$ vertices and $2n$ edges and in accordance with the degree and eccentricity of vertices as displayed in Tables 4 and 5. We get the following cases by considering the information in the mentioned tables.

Case (i): When n is even, for $v \in V(G)$ with $e(v) = \frac{n}{2}+1$, we have

$$le(v) = \frac{1}{\left(\frac{n}{2}+1\right)}\left[\frac{\left(\frac{n}{2}+1\right)-\left(\frac{n}{2}+2\right)}{\left(\frac{n}{2}+1\right)+\left(\frac{n}{2}+2\right)}\right] = \frac{-2}{(n+2)(n+3)}.$$

Similarly, for $v \in V(G)$ with $e(v) = \frac{n}{2}+2$, we have

$$le(v) = \frac{1}{\left(\frac{n}{2}+2\right)}\left[\frac{\left(\frac{n}{2}+2\right)-\left(\frac{n}{2}+1\right)}{\left(\frac{n}{2}+2\right)+\left(\frac{n}{2}+1\right)}\right] = \frac{2}{(n+3)(n+4)}.$$

Number of vertices	$d(v)$	$e(v)$
n	1	$\frac{n}{2}+2$
n	3	$\frac{n}{2}+1$

Table 4: Vertex partition of CW_n, when n is even.

Case (ii): When n is odd, for $v \in V(G)$ with $e(v) = \frac{n-1}{2}+1$, we have

$$le(v) = \frac{1}{\left(\frac{n-1}{2}+1\right)}\left[\frac{\left(\frac{n-1}{2}+1\right)-\left(\frac{n-1}{2}+2\right)}{\left(\frac{n-1}{2}+1\right)+\left(\frac{n-1}{2}+2\right)}\right] = \frac{-2}{(n+1)(n+2)}.$$

Similarly, for $v \in V(G)$ with $e(v) = \frac{n-1}{2}+2$, we have

$$le(v) = \frac{1}{\left(\frac{n-1}{2}+2\right)}\left[\frac{\left(\frac{n-1}{2}+2\right)-\left(\frac{n-1}{2}+1\right)}{\left(\frac{n-1}{2}+2\right)+\left(\frac{n-1}{2}+1\right)}\right] = \frac{2}{(n+2)(n+3)}.$$

□

Number of vertices	$d(v)$	$e(v)$
n	1	$\frac{n-1}{2}+2$
n	3	$\frac{n-1}{2}+1$

Table 5: Vertex partition of CW_n, when n is odd.

Theorem 2.21. *Let $H = L(CW_n)$,*

1. *If n is even, then*

$$le(v) = \begin{cases} \dfrac{4}{(n+1)(n+2)}, & \text{if } v \text{ corresponds to pendant edge of } CW_n; \\ \dfrac{-4}{n(n+1)}, & \text{otherwise.} \end{cases}$$

2. *If n is odd, then $le(v) = 0$.*

Proof. The line graph H of crown graph has $2n$ vertices and $3n$ edges. We get the following cases by considering the information of the mentioned in Tables 6 and 7.

Case (i): When n is even, for $v \in V(H)$ if v corresponds to pendant edge of CW_n with $e(v) = \frac{n}{2} + 1$ and

$$le(v) = \frac{1}{\left(\frac{n}{2}+1\right)} \left[\left(\frac{\frac{n}{2}+1-\left(\frac{n}{2}\right)}{\frac{n}{2}+1+\left(\frac{n}{2}\right)} \right) 2 \right] = \frac{4}{(n+1)(n+2)}.$$

Similarly, for $v \in V(H)$ if v corresponds to an edge on cycle C_n of CW_n with $e(v) = \frac{n}{2}$ and

$$le(v) = \frac{1}{\left(\frac{n}{2}\right)} \left[\left(\frac{\frac{n}{2}-\left(\frac{n}{2}+1\right)}{\frac{n}{2}+\left(\frac{n}{2}+1\right)} \right) 2 \right] = \frac{-4}{n(n+1)}.$$

Number of vertices	$d(v)$	$e(v)$
n	2	$\frac{n}{2}+1$
n	4	$\frac{n}{2}$

Table 6: Vertex partition of $L(CW_n)$, when n is even.

Case (ii): When n is odd, the eccentricity of all the vertices is $\frac{n+1}{2}$, hence $le(v) = 0$.

Number of vertices	$d(v)$	$e(v)$
n	2	$\frac{n+1}{2}$
n	4	$\frac{n+1}{2}$

Table 7: Vertex partition of $L(CW_n)$, when n is odd.

□

Definition 2.22 ([10]). *The graph obtained by adding a vertex between every pair of adjacent vertices of the cycle C_n in the wheel W_{n+1} is called a gear graph G_n.*

Theorem 2.23. *Let $G = G_n, n \geq 4$, be the gear graph. Then for $v \in V(G)$,*

$$le(v) = \begin{cases} \frac{-1}{35}, & \text{if } e(v)=3; \\ \frac{1}{14}, & \text{if } e(v)=4; \\ \frac{-n}{10}, & \text{if } e(v)=2. \end{cases}$$

Proof. The gear graph has $2n+1$ vertices and $3n$ edges. Based on the degree and eccentricity we partition $V(G)$ as shown in Table 8. If v is a vertex in G of eccentricity 3, then $le(v) = \frac{1}{3}\left[\left(\frac{3-4}{3+4}\right)2 + \frac{3-2}{3+2}\right] = \frac{-1}{35}$. If v is a vertex in G of eccentricity 4, then $le(v) = \frac{1}{4}\left[\left(\frac{4-3}{4+3}\right)2\right] = \frac{1}{14}$. If v is a vertex in G of ecentricity 2, then $le(v) = \frac{1}{2}\left[\left(\frac{2-3}{2+3}\right)n\right] = \frac{-n}{10}$. □

Number of vertices	$d(v)$	$e(v)$
n	3	3
n	2	4
1	n	2

Table 8: Vertex partition of G_n.

Theorem 2.24. *Let $H = L(G_n), n \geq 4$ be the line graph of gear graph G_n. Then for $v \in V(H)$,*

$$le(v) = \begin{cases} \frac{-1}{5}, & \text{if } e(v)=2; \\ \frac{1}{15}, & \text{if } e(v)=3. \end{cases}$$

Proof. The line graph H has order $3n$ and size $\frac{n^2+7n}{2}$. In reference to the degree and eccentricity of vertices we partition $V(H)$ as shown in Table 9. If v is a vertex in H of eccentricity 2, then $le(v) = \frac{1}{2}\left[\left(\frac{2-3}{2+3}\right)2\right] = \frac{-1}{5}$. If v is a vertex in H of eccentricity 3, then $le(v) = \frac{1}{3}\left[\frac{3-2}{3+2}\right] = \frac{1}{15}$. □

Number of vertices	$d(v)$	$e(v)$
n	$n+1$	2
$2n$	3	3

Table 9: Vertex partition of $L(G_n)$.

Definition 2.25 ([10])**.** *The Helm Graph H_n is a graph constructed by adjoining a pendant edge at each vertex of the cycle C_n in a wheel graph W_{n+1}.*

Theorem 2.26. *Let $G = H_n, n \geq 4$, be the helm graph. Then for $v \in V(G)$,*

$$le(v) = \begin{cases} \frac{-n}{10}, & \text{if } d(v)=n; \\ \frac{2}{105}, & \text{if } d(v)=4; \\ \frac{1}{28}, & \text{if } d(v)=1. \end{cases}$$

Proof. In G, there are $2n+1$ vertices and $3n$ edges. In reference to the degree and eccentricity of vertices we partition $V(G)$ as shown in Table 10. If v is a vertex in G of eccentricity 2, then $le(v) = \frac{1}{2}\left[\left(\frac{2-3}{2+3}\right)n\right] = \frac{-n}{10}$. If v is a vertex in G of eccentricity 3, then $le(v) = \frac{1}{3}\left[\left(\frac{3-2}{3+2}\right)+\left(\frac{3-4}{3+4}\right)\right] = \frac{2}{105}$. If v is a vertex in G of eccentricity 4, then $le(v) = \frac{1}{4}\left[\frac{4-3}{4+3}\right] = \frac{1}{28}$. □

Number of vertices	$d(v)$	$e(v)$
n	4	3
n	1	4
1	n	2

Table 10: Vertex partition of H_n.

Theorem 2.27. *Let $H = L(H_n), n \geq 4$ be the line graph of helm graph. Then for $v \in V(H)$,*

$$le(v) = \begin{cases} \frac{1}{15}, & \text{if } d(v)=3; \\ \frac{2}{15}, & \text{if } d(v)=6; \\ \frac{-3}{10}, & \text{if } d(v)=n+2. \end{cases}$$

Proof. The line graph H of helm graph is of order $3n$ and size $\frac{n(n+11)}{2}$. In reference to the degree and eccentricity of vertices we partition $V(H)$ as shown in Table 11. If v is a vertex in H of degree 3, then $le(v) = \frac{1}{3}\left[\left(\frac{3-2}{3+2}\right)\right] = \frac{1}{15}$. If v is a vertex in H of degree 6, then $le(v) = \frac{1}{3}\left[\left(\frac{3-2}{3+2}\right)2\right] = \frac{2}{15}$. If v is a vertex in H of degree $n+2$, then $le(v) = \frac{1}{2}\left[\left(\frac{2-3}{2+3}\right)3\right] = \frac{-3}{10}$. □

Number of vertices	$d(v)$	$e(v)$
n	$n+2$	2
n	6	3
n	3	3

Table 11: Vertex partition of $L(H_n)$.

Definition 2.28 ([10])**.** *A flower graph F_n is obtained from helm graph by joining each pendant vertex to the central vertex.*

Theorem 2.29. *Let $G = F_n, n \geq 3$, be the flower graph. Then for $v \in V(G)$*

$$le(v) = \begin{cases} \frac{-2n}{3}, & \text{if } d(v)=2n; \\ \frac{1}{6}, & \text{otherwise.} \end{cases}$$

Proof. The flower graph G is of order $2n+1$ and size $4n$. In reference to the degree and eccentricity of vertices we partition $V(G)$ as shown in Table 12. If v is a vertex in G with degree $2n$ and of eccentricity 1, then $le(v) = \frac{1}{1}\left[\left(\frac{1-2}{1+2}\right)2n\right] = \frac{-2n}{3}$, and for the remaining vertices in G, $le(v) = \frac{1}{2}\left[\left(\frac{2-1}{2+1}\right)\right] = \frac{1}{6}$. □

Number of vertices	$d(v)$	$e(v)$
n	4	2
n	2	2
1	$2n$	1

Table 12: Vertex partition of F_n.

Theorem 2.30. *Let $H = L(F_n)$ be the line graph of flower graph. Then for $v \in V(H)$*

$$le(v) = \begin{cases} \frac{-3}{10}, & \text{if } d(v)=2n+2; \\ \frac{-1}{10}, & \text{if } d(v)=2n; \\ \frac{2}{15}, & \text{if } d(v)=4 \text{ or } d(v)=6. \end{cases}$$

Proof. The line graph H of flower graph has $4n$ vertices and $2n(n+3)$ edges. In reference to the degree and eccentricity of vertices we partition $V(H)$ as shown in Table 13. If v is a vertex in H of degree $2n+2$, then $le(v) = \frac{1}{2}\left[\left(\frac{2-3}{2+3}\right)3\right] = \frac{-3}{10}$. If v is a vertex in H of degree $2n$, then $le(v) = \frac{1}{2}\left[\left(\frac{2-3}{2+3}\right)\right] = \frac{-1}{10}$. If v is a vertex in H of degree 4 or degree 6, then $le(v) = \frac{1}{3}\left[\left(\frac{3-2}{3+2}\right)2\right] = \frac{2}{15}$. □

Number of vertices	$d(v)$	$e(v)$
n	$2n+2$	2
n	$2n$	2
n	4	3
n	6	3

Table 13: Vertex partition of $L(F_n)$.

3. Leverage Eccentricity Centrality of Mycielskian Graph

Definition 3.1 ([8]). *Let $G = (V,\ E)$ be a graph with $V(G) = \{v_i : 1 \leq i \leq n\}$. Then the Mycielskian graph $\mu(G)$ of G is obtained by adding $n+1$ new vertices $\{u_i : 1 \leq i \leq n\} \cup \{w_0\}$, making w adjacent to u_i, $1 \leq i \leq n$ and making u_i adjacent to $N(v_i)$ in G. The vertex w_0 is called root of $\mu(G)$.*

Theorem 3.2. *The leverage eccentricity centrality of vertex of Mycielskian graph of P_n, $n \geq 7$, is given by*

$$le(v) = \begin{cases} \frac{-n}{10}, & \text{if } d(v)=n; \\ \frac{-1}{35}, & \text{if } d(v)=3; \\ \frac{1}{14}, & \text{if } d(v)=4; \\ \frac{1}{28}, & \text{if } v \text{ is a pendant vertex of } G; \\ \frac{2}{105}, & \text{otherwise.} \end{cases}$$

Proof. Let $V = \{v_1, v_2, \cdots, v_n\}$, be the vertex set of P_n, and let $V(\mu(P_n)) = \{v_1, v_2, \cdots, v_n\} \cup \{u_1, u_2, \cdots, u_n\} \cup \{w_0\}$ and $|V(\mu(P_n))| = 2n+1$. Here $d(u_i) = 3$,

$e(u_i) = 3, 2 \leq i \leq n-1$ and each of these $u_i, 2 \leq i \leq n-1$ is adjacent to v_{i-1}, v_{i+1} and w_0. Then $e(w_0) = 2$, $e(v_i) = 4, 1 \leq i \leq n$, $e(u_i) = 3, 1 \leq i \leq n$. $d(w_0) = n$, $d(v_1) = d(v_n) = 2$, $d(v_i) = 4, 2 \leq i \leq n-1$, $d(u_1) = d(u_n) = 2$ and $d(u_i) = 3, 2 \leq i \leq n-1$. Also, $N(w_0) = \{u_1, u_2, \ldots, u_n\}$, $N(u_1) = \{w_0, v_2\}$, $N(u_n) = \{w_0, v_{n-1}\}$, and $N(u_i) = \{w_0, v_{i-1}, v_{i+1}\}$, $N(v_1) = \{v_2, u_2\}$, $N(v_n) = \{v_{n-1}, u_{n-1}\}$, $N(v_i) = \{v_{i-1}, v_{i+1}, u_{i-1}, u_{i+1}\}$. Therefore,

$$
\begin{aligned}
le(w_0) &= \frac{1}{2}\left[\left(\frac{2-3}{2+3}\right)n\right] = \frac{-n}{10} \\
le(u_1) = le(u_n) &= \frac{1}{3}\left[\left(\frac{3-4}{3+4}\right) + \left(\frac{3-2}{3+2}\right)\right] = \frac{2}{105} \\
le(u_i) &= \frac{1}{3}\left[\left(\frac{3-4}{3+4}\right)2 + \left(\frac{3-2}{3+2}\right)\right] = \frac{-1}{35}, 2 \leq i \leq n-1 \\
le(v_1) = le(v_n) &= \frac{1}{4}\left[\left(\frac{4-3}{4+3}\right)\right] = \frac{1}{28} \\
le(v_i) &= \frac{1}{4}\left[\left(\frac{4-3}{4+3}\right)2\right] = \frac{1}{14}, 2 \leq i \leq n-1
\end{aligned}
$$

□

Theorem 3.3. *The leverage eccentricity centrality of vertex of Mycielskian graph of C_n, $n \geq 8$, is given by*

$$
le(v) = \begin{cases} \frac{-n}{10}, & \text{if } d(v)=n; \\ \frac{-1}{35}, & \text{if } d(v)=3; \\ \frac{1}{14}, & \text{if } d(v)=4. \end{cases}
$$

Proof. Let $V = \{v_1, v_2, \cdots, v_n\}$, be the vertex set of C_n, and let $V(\mu(C_n)) = \{v_1, v_2, \cdots, v_n\} \cup \{u_1, u_2, \cdots, u_n\} \cup \{w_0\}$ and $|V(\mu(C_n))| = 2n+1$. Here, $d(w_0) = n, d(u_i) = 3$, $1 \leq i \leq n$, $d(v_i) = 4$, $1 \leq i \leq n$. Then $e(w_0) = 2, e(u_i) = 3$, $1 \leq i \leq n$, $e(v_i) = 4$, $1 \leq i \leq n$. Also $N(w_0) = \{u_1, u_2, \ldots, u_n\}$, $N(u_1) = \{w_0, v_2, v_n\}$, $N(u_n) = \{w_0, v_{n-1}, v_1\}$, $N(u_i) = \{w_0, v_{i-1}, v_{i+1}\}, 2 \leq i \leq n-1$, $N(v_1) = \{v_2, v_n, u_2, u_n\}$, $N(v_n) = \{v_{n-1}, v_1, u_1, u_{n-1}\}$, $N(v_i) = \{v_{i-1}, v_{i+1}, u_{i-1}, u_{i+1}\}, 2 \leq i \leq n-1$. Therefore

$$
\begin{aligned}
le(w_0) &= \frac{1}{2}\left[\left(\frac{2-3}{2+3}\right)n\right] = \frac{-n}{10} \\
le(u_i) &= \frac{1}{3}\left[\left(\frac{3-4}{3+4}\right)2 + \left(\frac{3-2}{3+2}\right)\right] = \frac{-1}{35} \\
le(v_i) &= \frac{1}{4}\left[\left(\frac{4-3}{4+3}\right)2\right] = \frac{1}{14}.
\end{aligned}
$$

□

Theorem 3.4. *The leverage eccentricity centrality of vertex of Mycielskian graph of $K_n, n \geq 4$ is given by $le(v) = 0$, for all $v \in \mu(K_n)$.*

Theorem 3.5. *For Mycielskian graph of $K_{1,n}$ we have $le(v) = 0$, for all $v \in \mu(K_{1,n})$.*

3.1 Leverage eccentricity centrality of $C_n^{R_r}$

A chemical compound denoted by $C_n^{R_r}$ is obtained by joining an alkyl R_r in place of each hydrogen atom in Cycloalkene [7] as shown in Figure 3. We denote the group of alkyls by R_r, $r \in \mathbb{Z}^+$. For example R_1, R_2, R_3, ... denote methyl, ethyl, propyl, ..., respectively, as shown in Figure 2. When we put an alkyl instead of each hydrogen atom in the cycloalkene, we get a new class of cycloalkenes. We denote the chemical graphs that are formed by attaching an R_r instead of each hydrogen atom in a cycloalkene by $C_n^{R_r}$.

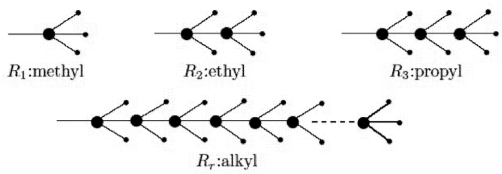

Figure 2: The first few alkyls

Figure 3 represents the molecular structure of $C_n^{R_r}$. The total number of vertices in the molecular structure and molecular graph are naturally equal. We calculate leverage eccentricity centrality of the graph shown in Figure 4.

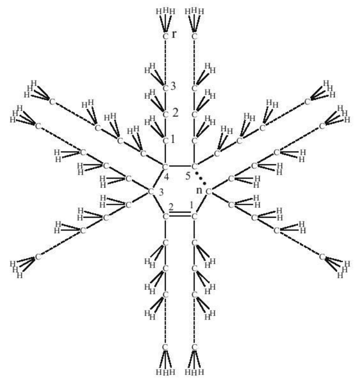

Figure 3: Molecular structure of $C_n^{R_r}$

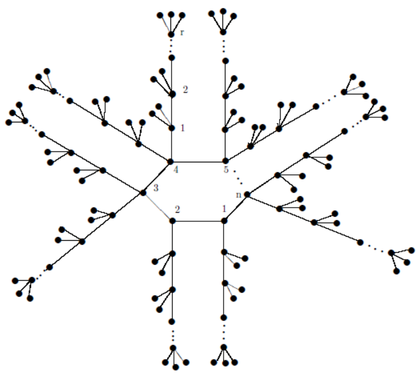

Figure 4: Molecular graph of $C_n^{R_r}$

Vertex corresponding to	Number of vertices	Eccentricity	
		for even n	for odd n
Carbon atom on cycle	n	$\frac{n}{2}+r+1$	$\frac{n-1}{2}+r+1$
Carbon atom not on cycle	$2nr-2r$	$\frac{n}{2}+r+1+i,\ 1\leq i\leq r$	$\frac{n-1}{2}+r+1+i,\ 1\leq i\leq r$
Hydrogen atom	$4nr+2n-4r-2$	$\frac{n}{2}+r+i+2,\ 1\leq i\leq r$	$\frac{n-1}{2}+r+i+2,\ 1\leq i\leq r$

Table 14: Eccentricity values for vertices of $C_n^{R_r}$

The vertices of $C_n^{R_r}, n$ is even are classified as:

Type I: v is a vertex corresponding to carbon atom on cycle C_n with degree 4 and eccentricity $\frac{n}{2}+r+1$, $(n-2)$ in number.

Type II: v is a vertex corresponding to carbon atom on cycle C_n with degree 3 and eccentricity $\frac{n}{2}+r+1$, 2 in number.

Type III: v is a vertex corresponding to carbon atom not on cycle C_n with eccentricity $\frac{n}{2}+r+2$ which has one neighbour with eccentricity $\frac{n}{2}+r+1$, and three neighbours with eccentricity $\frac{n}{2}+r+3$, $(2n-2)$ in number.

Type IV: For every i, $2\leq i\leq r-1$, v is a vertex corresponding to carbon atom not on cycle C_n with eccentricity $\frac{n}{2}+r+i+1$, which has one neighbour with eccentricity $\frac{n}{2}+r+i$, and three neighbours with eccentricity $\frac{n}{2}+r+i+2$, $(2nr-4n-2r+4)$ in number.

Type V: v is a vertex corresponding to carbon atom not on cycle C_n with eccentricity $\frac{n}{2}+2r+1$ which has three neighbours with eccentricity $\frac{n}{2}+2r+2$ and one neighbour with eccentricity $\frac{n}{2}+2r$, $(2n-2)$ in number.

Type VI: v is a vertex corresponding to hydrogen atom with eccentricity $\frac{n}{2}+r+i+2, 1\leq i\leq r$, which has one neighbour with eccentricity $\frac{n}{2}+r+i+1, 1\leq i\leq r$, $4r(n-1)+2n-2$ in number.

The vertices of $C_n^{R_r}, n$ is odd are classified as:

Type I: v is a vertex corresponding to carbon atom on cycle C_n with degree 4 and eccentricity $\frac{n-1}{2}+r+1$, $(n-2)$ in number.

Type II: v is a vertex corresponding to carbon atom on cycle C_n with degree 3 and eccentricity $\frac{n-1}{2}+r+1$, 2 in number.

Type III: v is a vertex corresponding to carbon atom not on cycle C_n with eccentricity $\frac{n-1}{2}+r+2$ which has one neighbour with eccentricity $\frac{n-1}{2}+r+1$, and three neighbours with eccentricity $\frac{n-1}{2}+r+3$, $(2n-2)$ in number.

Type IV: For every i, $2\leq i\leq r-1$, v is a vertex corresponding to carbon atom not on cycle C_n with eccentricity $\frac{n-1}{2}+r+i+1$, which has one neighbour with eccentricity $\frac{n-1}{2}+r+i$, and three neighbours with eccentricity $\frac{n-1}{2}+r+i+2$, $(2nr-4n-2r+4)$ in number.

Type V: v is a vertex corresponding to carbon atom not on cycle C_n with eccentricity $\frac{n-1}{2}+2r+1$

which has three neighbours with eccentricity $\frac{n-1}{2}+2r+2$ and one neighbour with eccentricity $\frac{n-1}{2}+2r$, $(2n-2)$ in number.

Type VI: v is a vertex corresponding to hydrogen atom with eccentricity $\frac{n-1}{2}+r+i+2, 1 \leq i \leq r$, which has one neighbour with eccentricity $\frac{n-1}{2}+r+i+1, 1 \leq i \leq r$, $4r(n-1)+2n-2$ in number.

Proposition 3.6. *If $G = C_n^{R_r}$ with $n \geq 3$ and n is even, then for $v \in V(G)$*

$$le(v) = \begin{cases} \frac{-4}{(n+2r+2)(n+2r+3)}, & \text{if } v \text{ is a vertex of Type I;} \\ \frac{-2}{(n+2r+2)(n+2r+3)}, & \text{if } v \text{ is a vertex of Type II;} \\ \frac{-4(n+2r+2)}{(n+2r+3)(n+2r+5)}, & \text{if } v \text{ is a vertex of Type III.} \\ \frac{-4(n+2r+2i)}{(n+2r+2i+1)(n+2r+2i+2)(n+2r+2i+3)}, & \text{if } v \text{ is a vertex of Type IV;} \\ \frac{-4(n-4r)}{(n+4r+1)(n+4r+2)(n+4r+3)}, & \text{if } v \text{ is a vertex of Type V;} \\ \frac{2}{(n+2r+2i+3)(n+2r+2i+4)}, & \text{if } v \text{ is a vertex of Type VI.} \end{cases}$$

Proof. In $C_n^{R_r}$, $|V(C_n^{R_r})| = 6nr+3n-6r-2$ in which $n+2nr-2r$ vertices correspond to carbon atoms and $4nr+2n-4r-2$ pendant vertices correspond to hydrogen atoms. Let n be even. We consider the following cases by using Table 14.

Case 1: If v is a vertex of Type I, then

$$\begin{aligned} le(v) &= \frac{1}{\left(\frac{n}{2}+r+1\right)} \left[\left(\frac{\left(\frac{n}{2}+r+1\right) - \left(\frac{n}{2}+r+2\right)}{\left(\frac{n}{2}+r+1\right) + \left(\frac{n}{2}+r+2\right)} \right) 2 \right] \\ &= \frac{2}{(n+2r+2)} \left[\frac{-2}{n+2r+3} \right] \\ &= \frac{-4}{(n+2r+2)(n+2r+3)}. \end{aligned}$$

Case 2: If v is a vertex of Type II, then

$$\begin{aligned} le(v) &= \frac{1}{\left(\frac{n}{2}+r+1\right)} \left[\frac{\left(\frac{n}{2}+r+1\right) - \left(\frac{n}{2}+r+2\right)}{\left(\frac{n}{2}+r+1\right) + \left(\frac{n}{2}+r+2\right)} \right] \\ &= \frac{-2}{(n+2r+2)(n+2r+3)}. \end{aligned}$$

Case 3: If v is a vertex of Type III, then

$$le(v) = \frac{1}{\left(\frac{n}{2}+r+2\right)} \left[\frac{\left(\frac{n}{2}+r+2\right) - \left(\frac{n}{2}+r+1\right)}{\left(\frac{n}{2}+r+2\right) + \left(\frac{n}{2}+r+1\right)} \right] + \left[\left(\frac{\left(\frac{n}{2}+r+2\right) - \left(\frac{n}{2}+r+3\right)}{\left(\frac{n}{2}+r+2\right) + \left(\frac{n}{2}+r+3\right)} \right) 3 \right]$$

$$= \frac{-4(n+2r+2)}{(n+2r+3)(n+2r+5)}.$$

Case 4: If v is a vertex of Type IV, then

$$le(v) = \frac{1}{\left(\frac{n}{2}+r+i+1\right)} \left[\frac{\left(\frac{n}{2}+r+i+1\right) - \left(\frac{n}{2}+r+1+i-1\right)}{\left(\frac{n}{2}+r+i+1\right) + \left(\frac{n}{2}+r+1+i-1\right)} \right.$$
$$\left. + \frac{\left(\frac{n}{2}+r+i+1\right) - \left(\frac{n}{2}+r+1+i+1\right)}{\left(\frac{n}{2}+r+i+1\right) + \left(\frac{n}{2}+r+1+i+1\right)} + \left[\left(\frac{\left(\frac{n}{2}+r+i+1\right) - \left(\frac{n}{2}+r+i+2\right)}{\left(\frac{n}{2}+r+i+1\right) + \left(\frac{n}{2}+r+1+i+1\right)} \right) 2 \right] \right]$$
$$= \frac{-4(n+2r+2i)}{(n+2r+2i+1)(n+2r+2i+2)(n+2r+2i+3)}.$$

Case 5: If v is a vertex of Type V, then

$$le(v) = \frac{1}{\left(\frac{n}{2}+2r+1\right)} \left[\left(\frac{\left(\frac{n}{2}+2r+1\right) - \left(\frac{n}{2}+2r+2\right)}{\left(\frac{n}{2}+2r+1\right) + \left(\frac{n}{2}+2r+2\right)} \right) 3 \right] + \left[\left(\frac{\left(\frac{n}{2}+2r+1\right) - \left(\frac{n}{2}+2r\right)}{\left(\frac{n}{2}+2r+1\right) + \left(\frac{n}{2}+2r\right)} \right) \right]$$
$$= \frac{-4(n-4r)}{(n+4r+2)(n+4r+3)(n+4r+1)}.$$

Case 6: If v is a vertex of Type VI, then

$$le(v) = \frac{1}{\left(\frac{n}{2}+r+i+2\right)} \left[\frac{\left(\frac{n}{2}+r+i+2\right) - \left(\frac{n}{2}+r+1+i\right)}{\left(\frac{n}{2}+r+i+2\right) + \left(\frac{n}{2}+r+1+i\right)} \right]$$
$$= \frac{2}{(n+2r+2i+4)(n+2r+2i+3)}, 1 \le i \le r.$$

□

Proposition 3.7. *If $G = C_n^{R_r}$ with $n \geq 3$ and n is odd, then for $v \in V(G)$*

$$le(v) = \begin{cases} \frac{-4}{(n+2r+1)(n+2r+2)}, & \text{if } v \text{ is a vertex of Type I;} \\ \frac{-2}{(n+2r+1)(n+2r+2)}, & \text{if } v \text{ is a vertex of Type II;} \\ \frac{-4(n+2r+1)}{(n+2r+2)(n+2r+4)}, & \text{if } v \text{ is a vertex of Type III.} \\ \frac{-4(n+2r+2i-1)}{(n+2r+2i)(n+2r+2i+1)(n+2r+2i+2)}, & \text{if } v \text{ is a vertex of Type IV;} \\ \frac{-4(n-1-4r)}{(n+4r)(n+4r+1)(n+4r+2)}, & \text{if } v \text{ is a vertex of Type V;} \\ \frac{2}{(n+2r+2i+2)(n+2r+2i+3)}, & \text{if } v \text{ is a vertex of Type VI.} \end{cases}$$

Proof. By replacing the value of n by $n-1$ in the above proposition we get the desired results. □

4. Conclusion

In this paper, we compute the precise values of leverage eccentricity centrality of vertices in different families of graphs, line graph of some special graphs, Mycielskian of some standard graphs and molecular graph of cyclo-hydrocarbons.

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