International Journal of
Mathematics And its Applications
ISSN: 2347-1557

Int. J. Math. And Appl., **14(2)**(2026), 123–132
Available Online: http://ijmaa.in

Deformation in Orthotropic Elastic Media under the Effect of Triangular Irregularity

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Abstract

Closed-form analytical expressions for displacements have been obtained using Eigen value approach and Fourier transform method in orthotropic media, considering the triangular irregularity in one medium. To study the effect of irregularity, two orthotropic materials-Olivine and Topaz have been used. The variation in displacements with respect to horizontal distance has been analyzed by considering the irregularities of different size.

Keywords: Orthotropic; Triangular; Irregularity; Fourier Transform.

1. Introduction

To study and understand the propagation of seismic waves at continental margins, mountain fronts and similar regions, the problem of static deformation in an elastic medium with irregular thickness becomes important. This is because irregular thickness closely represents the natural geological environment. An elastic medium with irregular boundaries introduces considerable mathematical and physical complexity in the analysis of displacement and stress fields. The presence of orthotropy, where the material exhibits different mechanical properties along different physical directions, further complicates the behavior but also provides a more realistic model of crustal materials. In particular, orthotropic materials like Olivine and Topaz are often used in modeling due to their anisotropy elasticity. A number of researchers have studied irregular boundaries such as Noyer [7], Sato [12], Mal [3], Kar et al. [4], Chattopadhyay [1], Acharya and Roy [6], Selim [8, 9] and others. Garg et al [10] studied the deformation problem in an orthotropic medium using the eigen value approach.

In the present paper, we consider a homogenous orthotropic elastic medium subjected to triangular irregularity along the interface and a line load acting vertically downward on the normal line. The mathematical formulation is carried out using the Eigen value approach and Fourier transforms method to obtain the explicit solutions for the displacement field. The objective is to investigate how

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such geometric irregularity affects the deformation characteristics of the medium. The same type of model in orthotropic medium is studied by Madan and Gaba [5] but in their case, the irregularities at the interface were rectangular and parabolic. To evaluate the practical implications, two different types of orthotropic materials are considered. The variation of displacements with respect to horizontal and vertical directions has been derived and displacements for different value of irregularity size with respect to horizontal distance are obtained graphically and compared with uniform slip. The results provide insights into how surface irregularities and material anisotropy influence deformation, which is critical for applications in seismology and material science.

2. Formulation of Problem

We consider an infinite orthotropic elastic medium composed of two regions: Medium I and Medium II, with x_1-axis oriented vertically downwards. Suppose a normal line load S_0 per unit length acting parallel to x_3-axis. A triangular irregularity is introduced at the interface between Medium I and Medium II. The equation for triangular irregularity is described by the piecewise function:

$$x_1 = \varepsilon f(x_2) = \begin{cases} h\left(1 + \frac{x_2}{a}\right), & -a \le x_2 \le 0, \\ h\left(1 - \frac{x_2}{a}\right), & 0 \le x_2 \le a, \\ 0, \quad |x_2| \ge a \end{cases} \tag{1}$$

where h is the height and $2a$ is the base width of this triangular irregularity.

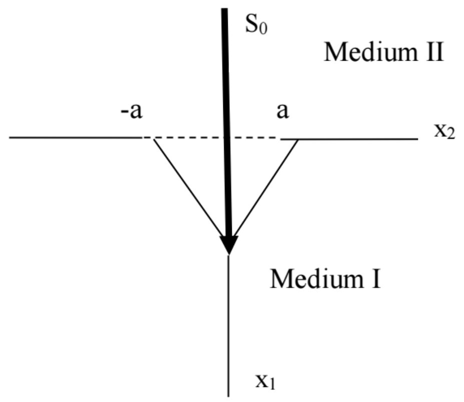

Figure 1: Geometry of the Problem

3. Basic Equations and Theoretical Framework

The equations of equilibrium in Cartesian Co-ordinate system (x_1, x_2, x_3) in absence of body forces, are

$$\tau_{ij,j} = 0, \quad (i = 1, 2, 3), \tag{2}$$

where summation convention has been used on repeated indices and comma stands for differentiation. We know that

$$E_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \tag{3}$$

where E_{ij} is strain tensor of order 2 and u_i $(i = 1, 2, 3)$ are the displacements component along x_i $(i = 1, 2, 3)$. From generalized Hooke Law, the stress-strain relation is

$$\tau_{ij} = C_{ijks}E_{ks} = C_{ijks}u_{k,s}, \tag{4}$$

where C_{ijks} is elastic stiffness coefficient satisfying the relation

$$C_{ijks} = C_{jiks} = C_{ksij}, \tag{5}$$

From equation (1) and (3), we have obtained

$$C_{ijks}u_{k,s} = 0, \tag{6}$$

In this study, plane strain deformation is assumed in the x_1x_2 plane, implying that all displacement components are independent of x_3 coordinate. Hence

$$u_1 = u_1(x_1, x_2), \quad u_2 = u_2(x_1, x_2), \quad u_3 = 0, \tag{7}$$

Using equation (7) in (6), we obtained

$$C_{i1k1}u_{k,11} + C_{i2k2}u_{k,22} + (C_{i1k2} + C_{i2k1})u_{k,12} = 0, \tag{8}$$

In equation (8), we have used the contracted Voigt Notation for C_{ijks} according to rule

$$11 \to 1,\ 22 \to 2,\ 33 \to 3,\ 23 \to 4,\ 13 \to 5,\ 12 \to 6, \tag{9}$$

For Orthotropic material,

$$C_{14} = C_{15} = C_{16} = C_{24} = C_{25} = C_{26} = C_{36} = C_{45} = C_{46} = C_{56} = 0, \tag{10}$$

Using (9), (10) in (8), we obtained

$$C_{11}\frac{\partial^2 u_1}{\partial x_1^2} + C_{66}\frac{\partial^2 u_1}{\partial x_2^2} + (C_{66} + C_{12})\frac{\partial^2 u_1}{\partial x_1 \partial x_2} = 0, \tag{11}$$

$$(C_{66} + C_{12})\frac{\partial^2 u_1}{\partial x_1^2} + C_{66}\frac{\partial^2 u_2}{\partial x_1^2} + C_{22}\frac{\partial^2 u_2}{\partial x_2^2} = 0. \tag{12}$$

Let us define Fourier Transform $\overline{f}(s)$ of $f(x_2)$ by the relation

$$\overline{f}(s) = F[f(x_2)] = \int_{-\infty}^{\infty} f(x_2) e^{isx_2} dx_2 \tag{13}$$

so that the Fourier Inverse Transform is

$$f(x_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{f}(s) e^{-isx_2} ds, \tag{14}$$

where s is Fourier Transform Parameter. By using Fourier Transform Garg et al [10] solved equation (11) and (12) and obtained

$$u_1(x_1, x_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(AP_1 e^{\gamma_1 |s| x_1} + BP_2 e^{\gamma_2 |s| x_1} - CP_1 e^{-\gamma_1 |s| x_1} - DP_2 e^{\gamma_2 |s| x_1} \right) e^{-isx_2} ds, \tag{15}$$

$$u_2(x_1, x_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(Ae^{\gamma_1 |s| x_1} + Be^{\gamma_2 |s| x_1} + Ce^{-\gamma_1 |s| x_1} + De^{\gamma_2 |s| x_1} \right) e^{-isx_2} ds, \tag{16}$$

where

$$P_l = -i \frac{\gamma_l (C_{66} + C_{12})}{\gamma_l^2 C_{11} - C_{66}} = -i \frac{\gamma_l^2 C_{66} - C_{22}}{\gamma_l (C_{66} + C_{12})} \quad \text{for } l = 1, 2. \tag{17}$$

where

$$\gamma_1^2 = \frac{A_0 + \sqrt{A_0^2 - 4B_0}}{2}, \quad \gamma_2^2 = \frac{A_0 - \sqrt{A_0^2 - 4B_0}}{2} \tag{18}$$

where

$$\left. \begin{aligned} A_0 &= \frac{C_{11} C_{22} - 2C_{66} C_{12} - C_{12}^2}{C_{11} C_{66}}, \\ B_0 &= \frac{C_{22}}{C_{11}} \end{aligned} \right\} \tag{19}$$

Using equation (4), (15) and (16), we obtained

$$\tau_{11}(x_1, x_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(AQ_1 e^{\gamma_1 |s| x_1} + BQ_2 e^{\gamma_2 |s| x_1} + CQ_1 e^{-\gamma_1 |s| x_1} + DQ_2 e^{\gamma_2 |s| x_1} \right) e^{-isx_2} ds, \tag{20}$$

$$\tau_{12}(x_1, x_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} C_{66} \left(AR_1 e^{\gamma_1 |s| x_1} + BR_2 e^{\gamma_2 |s| x_1} - CR_1 e^{-\gamma_1 |s| x_1} - DR_2 e^{\gamma_2 |s| x_1} \right) e^{-isx_2} ds, \tag{21}$$

where

$$\left. \begin{aligned} Q_l &= C_{11} P_l \gamma_l |s| - iC_{12} s, \\ R_l &= \gamma_l |s| - iP_l s, \qquad l = 1, 2. \end{aligned} \right\} \tag{22}$$

4. Solution of the Problem

Here in the infinite medium, irregularity is present at $x_1 = \varepsilon f(x_2)$. For $x_1 \succ \varepsilon f(x_2)$, represents medium I and $x_1 \prec \varepsilon f(x_2)$ represents medium II.

For Medium I

The displacements and stress components are expressed as follows:

$$u_1^I(x_1,x_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(-CP_1e^{-\gamma_1|s|x_1} - DP_2e^{\gamma_2|s|x_1}\right)e^{-isx_2}ds, \tag{23}$$

$$u_2^I(x_1,x_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(Ce^{-\gamma_1|s|x_1} + De^{\gamma_2|s|x_1}\right)e^{-isx_2}ds, \tag{24}$$

$$\tau_{11}^I(x_1,x_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(CQ_1e^{-\gamma_1|s|x_1} + DQ_2e^{\gamma_2|s|x_1}\right)e^{-isx_2}ds, \tag{25}$$

$$\tau_{12}^I(x_1,x_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty}C_{66}\left(-CR_1e^{-\gamma_1|s|x_1} - DR_2e^{\gamma_2|s|x_1}\right)e^{-isx_2}ds. \tag{26}$$

For Medium II

$$u_1^{II}(x_1,x_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(AP_1e^{\gamma_1|s|x_1} + BP_2e^{\gamma_2|s|x_1}\right)e^{-isx_2}ds, \tag{27}$$

$$u_2^{II}(x_1,x_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(Ae^{\gamma_1|s|x_1} + Be^{\gamma_2|s|x_1}\right)e^{-isx_2}ds, \tag{28}$$

$$\tau_{11}^{II}(x_1,x_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(AQ_1e^{\gamma_1|s|x_1} + BQ_2e^{\gamma_2|s|x_1}\right)e^{-isx_2}ds, \tag{29}$$

$$\tau_{12}^{II}(x_1,x_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty}C_{66}\left(AR_1e^{\gamma_1|s|x_1} + BR_2e^{\gamma_2|s|x_1}\right)e^{-isx_2}ds. \tag{30}$$

The boundary conditions are (at $x_1 = \varepsilon f(x_2)$)

$$\left.\begin{aligned} &u_1^I(\varepsilon f(x_2),x_2) = u_1^{II}(\varepsilon f(x_2),x_2)\\ &u_2^I(\varepsilon f(x_2),x_2) = u_2^{II}(\varepsilon f(x_2),x_2)\\ &\tau_{11}^I(\varepsilon f(x_2),x_2) - \tau_{11}^{II}(\varepsilon f(x_2),x_2) = -S_0\delta x_2\\ &\tau_{12}^I(\varepsilon f(x_2),x_2) = \tau_{12}^{II}(\varepsilon f(x_2),x_2)\end{aligned}\right\}, \tag{31}$$

where δx_2 is Dirac Delta Function.

By following the Madan and Gaba (2016), the values of coefficients are:

$$\left.\begin{aligned} A &= \frac{-S_0}{2C_{11}(P_2\gamma_2 - P_1\gamma_1)|s|} - \varepsilon\left(\frac{S_0}{2C_{11}(P_2\gamma_2 - P_1\gamma_1)|s|} + \frac{S_0\gamma_1 f(x_2)}{2C_{11}(P_2\gamma_2 - P_1\gamma_1)}\right),\\ B &= \frac{S_0}{2C_{11}(P_2\gamma_2 - P_1\gamma_1)|s|} + \varepsilon\left(\frac{S_0}{2C_{11}(P_2\gamma_2 - P_1\gamma_1)|s|} + \frac{S_0\gamma_2 f(x_2)}{2C_{11}(P_2\gamma_2 - P_1\gamma_1)}\right),\\ C &= \frac{S_0}{2C_{11}(P_2\gamma_2 - P_1\gamma_1)|s|} + \varepsilon\left(\frac{S_0}{2C_{11}(P_2\gamma_2 - P_1\gamma_1)|s|} + \frac{S_0\gamma_1 f(x_2)}{2C_{11}(P_2\gamma_2 - P_1\gamma_1)}\right),\\ D &= \frac{-S_0}{2C_{11}(P_2\gamma_2 - P_1\gamma_1)|s|} - \varepsilon\left(\frac{S_0}{2C_{11}(P_2\gamma_2 - P_1\gamma_1)|s|} + \frac{S_0\gamma_2 f(x_2)}{2C_{11}(P_2\gamma_2 - P_1\gamma_1)}\right)\end{aligned}\right] \tag{32}$$

5. Triangular Irregularity

By applying Fourier Transformation on equation (1), we obtained

$$\begin{aligned}
\overline{F}(s) &= [f(x_2)] = \frac{1}{\varepsilon}\int_{-\infty}^{\infty} f(x_2)e^{isx_2}dx_2 \\
&= \frac{1}{\varepsilon}\int_{-a}^{0} h\left(1+\frac{x_2}{a}\right)e^{isx_2}dx_2 + \frac{1}{\varepsilon}\int_{0}^{a} h\left(1-\frac{x_2}{a}\right)e^{isx_2}dx_2 \\
&= \frac{1}{\varepsilon}\left(\frac{2h}{as^2}\left(1-\cos(sa)\right)\right) = \frac{1}{\varepsilon}\left(\frac{2h}{as^2}\left(2\sin^2\left(\frac{as}{2}\right)\right)\right) \\
&= \frac{1}{\varepsilon}\frac{4h}{as^2}\sin^2\left(\frac{as}{2}\right).
\end{aligned} \tag{33}$$

where $\varepsilon = \frac{h}{2a} \ll 1$ is the perturbation factor and sgn is the signum function. By applying Fourier Inverse Transform of above equation (33), we obtained

$$f(x_2) = \frac{1}{\varepsilon}\frac{h}{2a}\left[(x_2-a)\,\mathrm{sgn}(x_2-a) + (x_2+a)\,\mathrm{sgn}(x_2+a) - 2x_2\,\mathrm{sgn}(x_2)\right] \tag{34}$$

By substituting the values of A, B, C, D in (23), (24), (27), (28) and value of $f(x_2)$ from equation (34), the corresponding displacement and stress components for infinite medium are derived as follows:

$$u_1(x_1,x_2) = \frac{s_0}{4\pi c_{11}(P_2\gamma_2 - P_1\gamma_1)}\left[\begin{array}{l} P_1\log {T_1}^2 - P_2\log {T_2}^2 \\ +E\left\{\begin{array}{l} P_1x_1\log {T_1}^2 - \frac{P_2}{2a}x_1\log {T_2}^2 \\ \mp\frac{{x_1}^2}{2a}\left[\begin{array}{l}(x_2-a)\,\mathrm{sgn}(x_2-a) \\ +(x_2+a)\,\mathrm{sgn}(x_2+a) - 2x_2\,\mathrm{sgn}(x_2)\end{array}\right] \\ \left(\frac{P_1{\gamma_1}^2}{{T_1}^2} - \frac{P_2{\gamma_2}^2}{{T_2}^2}\right)\end{array}\right\}\end{array}\right] \tag{35}$$

$$u_2(x_1,x_2) = \mp\frac{s_0}{4\pi c_{11}(P_2\gamma_2 - P_1\gamma_1)}\left[\begin{array}{l} \log {T_1}^2 - \log {T_2}^2 \\ +E\left\{\begin{array}{l} -\frac{x_1}{a}\log {T_1}^2 - \frac{x_1}{a}\log {T_2}^2 \\ \pm{x_1}^2\left[\begin{array}{l}(x_2-a)\,\mathrm{sgn}(x_2-a) \\ +(x_2+a)\,\mathrm{sgn}(x_2+a) - 2x_2\,\mathrm{sgn}(x_2)\end{array}\right] \\ \left(\frac{{\gamma_1}^2}{{T_1}^2} - \frac{{\gamma_2}^2}{{T_2}^2}\right)\end{array}\right\}\end{array}\right] \tag{36}$$

where E is the parameter irregularity and

$$\left.\begin{aligned} T_1^2 &= x_2^2 + \gamma_1^2 x_1^2, \\ T_2^2 &= x_2^2 + \gamma_2^2 x_1^2. \end{aligned}\right\} \tag{37}$$

6. Particular Case

When we take $E = 0$, the corresponding results (35) and (36) represents displacements for an orthotropic infinite elastic medium without surface irregularity.

7. Graphical Interpretation

To study the effect of irregularity on the displacement field due to a normal line load, two orthotropic materials-Olivine and Topaz as reported by Verma [11] and Love [1] respectively have been considered. The elastic stiffnesses coefficients of these materials are as follows:

Olivine

$$C_{11} = 324, C_{22} = 198, C_{33} = 59, C_{12} = 78, C_{23} = 79, C_{31} = 79, C_{66} = 79.3, C_{44} = 66.7, C_{55} = 81,$$

in terms of unit stress of 10^{11} dyne cm^{-2}.

Topaz

$$C_{11} = 2870, C_{22} = 3560, C_{33} = 3000, C_{12} = 1280, C_{23} = 900, C_{31} = 860, C_{66} = 1330, C_{44} = 1100, C_{55} = 1350,$$

in terms of unit stress of 10^{11} dyne cm^{-2}. The influence of interface irregularity on the displacement field within an orthotropic medium has been demonstrated well with help of figure (2) and (3) for the fixed value of $x_1 = 1$ and for different size of irregularities.

Horizontal Distance (y)	Displacement for L=0 in terms of 10^4	Displacement for L=0.2 in terms of 10^5	Displacement for L=0.4 in terms of 10^5	Displacement for L=0.6 in terms of 10^5
0.0	6.5303	1.2350	1.8171	2.3991
0.2	5.6069	0.8667	1.1726	1.4786
0.4	3.7124	0.4730	0.5748	0.6766
0.6	1.7932	0.2080	0.2368	0.2655
0.8	0.0988	0.0069	0.0039	0.0010
1.0	-1.3681	-0.1539	-0.1710	-0.1880
1.2	-2.6474	-0.2957	-0.3266	-0.3576
1.4	-3.7770	-0.4234	-0.4691	-0.5148
1.6	-4.7864	-0.5366	-0.5946	-0.6526
1.8	-5.6979	-0.6384	-0.7071	-0.7757
2.0	-6.5282	-0.7310	-0.8092	-0.8873

Table 1: Displacements for Olivine Material for Different Values of E

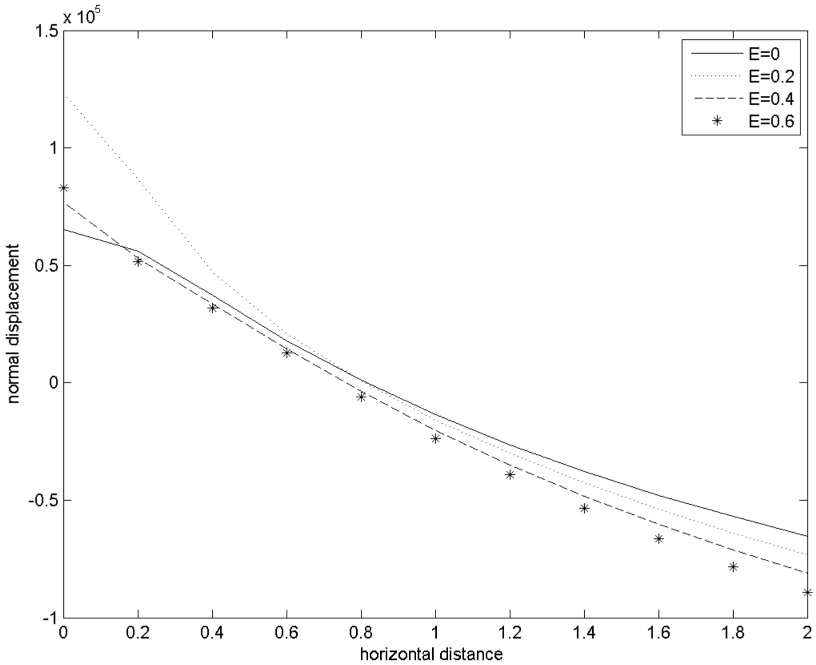

Figure 2: Displacement Variation (u) with Distance in Horizontal direction (x_2) on the plane at $x_1 = 1$ for Olivine Material

As shown in figure (2), in the interval $[0, 0.8]$, the magnitude of displacement decreases with increasing horizontal distance from the load application point and in the interval $[0.8, 2]$, the magnitude of displacement increases with increasing horizontal distance. It is observed that the presence and size of the surface irregularity significantly influence the magnitude of displacement. We observed that the magnitude of displacement is highest near the origin and gradually reduces till 0.8 and then starts increasing. In Table 2, we noticed that displacement values in the irregular orthotropic Topaz material are complex valued. This indicates that, in the physical sense, no real displacement or deformation occurs in the medium under the given conditions. In other words, the irregular orthotropic topaz material does not exhibit real deformation in response to the applied normal line load.

Horizontal Distance (x)	Displacement for E = 0 in terms of 10^5	Displacement for E = 0.2 in terms of 10^5
0.0	-4.0342	$-4.5636 + i0.3696$
0.2	-4.0526	$-4.5470 + i0.3796$
0.4	-4.1483	$-4.5607 + i0.4107$
0.6	2.1402	$2.7310 - i0.4802$
0.8	1.4614	$1.8651 - i0.4155$
1.0	0.2968	$0.7563 - i0.3499$
1.2	-1.2423	$-1.0238 - i0.2975$
1.4	-2.9192	$-3.2156 - i0.2634$
1.6	-4.5543	$-5.2009 - i0.2449$
1.8	-6.0700	$-6.9748 - i0.2372$
2.0	-7.4493	$-8.5667 - i0.2362$

Table 2: Displacements for Topaz Material for Different Values of E

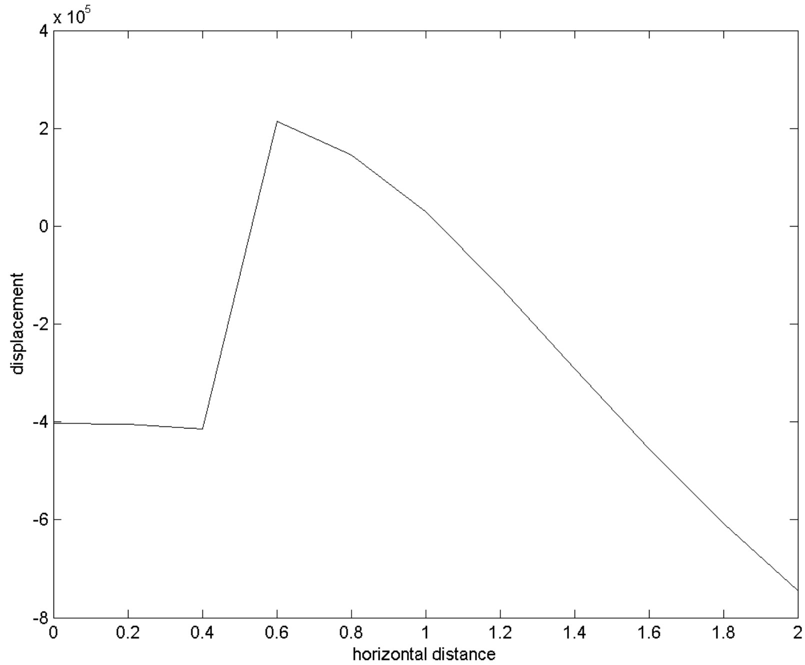

Figure 3: Displacement Variation (u) with Distance in Horizontal direction (x_2) on the plane at $x_1 = 1$ for Topaz Material

In figure 3, we have observed that from horizontal distance 0 to 0.4, distance is approximately constant, from 0.4 to 0.6, displacement increases and after that displacement decreases.

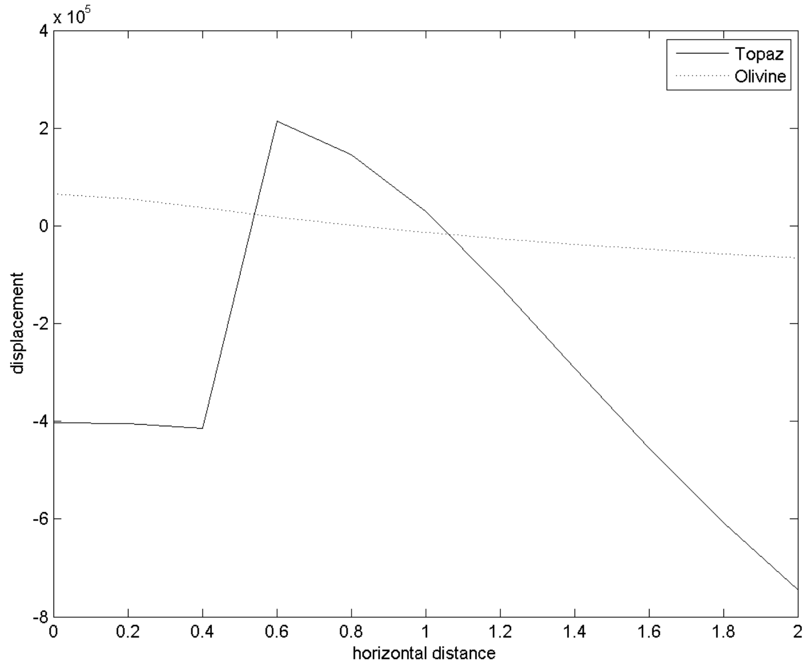

Figure 4: Displacement (u) Variation with Distance (y) in horizontal direction on the plane at $x = 1$ for Topaz and Olivine Material for $E = 0$

In figure 4, it is evident that the displacement behavior differs significantly between Topaz and Olivine under identical loading and geometric conditions. The displacements in the Olivine medium shows a smooth, monotonic trend, indicating stable deformation behavior. In contrast, the displacement in the Topaz medium fluctuates drastically and includes abrupt changes, including large positive and negative peaks.

Thus, from table 2 and figure 4, we observed that there is no real physical deformation in the irregular Topaz medium, such a behavior is likely due to the high stiffness and anisotropic nature which make it more resistant to the surface deformation under normal line loading. This result indicates the importance of material selection in deformation analysis and significant role played by surface irregularities in orthotropic media.

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