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Lie Symmetry Solution of Third Order Nonlinear Ordinary Differential Equation

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Abstract: In this paper we used the method of Lie symmetry to solve and get a mathematical solution to a third order first degree nonlinear ordinary differential equation (ODE) of fourth degree in second derivative, which is common in waves of systems like water in shallow oceans because it yields exact solutions without depending on initial boundary values.

Keywords: Lie groups of transformations, Lie algebras, infinitesimal transformations, extended transformations, invariance under transformations, variation symmetries, Lie theory of differential equations, reduction of order and integrating factors.

1. Introduction

Norwegian mathematician Sophus Lie introduced the notion of Lie group to study the solutions of ODEs. Lie recognized the transformation properties of a nonlinear ODE under certain groups of continuous transformations as being fundamental in analyzing its solution. The nonlinear ODE is:

$$y''' - y'\left(\frac{y''}{y}\right)^4 = 0 \tag{1}$$

It's commonly used in many physical applications especially in engineering field and its very complex to be solved analytically.

2. Review of Literature

In related researches Oyombe, (2019) applied Lie symmetry analysis to solve (1) which yielded a general solution of the form:

$$V = \frac{1}{(U')^4}\int (U'')^4 (U')^2 (U)^{-4}\, dU \tag{2}$$

We have used this same (1) to sequentially get a mathematical solution as opposed to a general solution. We have also shown materials and methodology used in details. Opiyo [2] solved a third order nonlinear ODE and got a solution. The equation was of the form:

$$y''' = y\left(\frac{y''}{y'}\right)^3 \tag{3}$$

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and whose solution is:

$$V = \frac{1}{A\left(u'\right)^4} \int u\left(u''\right)^3\left(u'\right)^{-2} du \tag{4}$$

He applied Lie groups of transformations, Lie algebras, infinitesimal transformations, invariance under transformation, symmetry, Lie's integrating factor, method of canonical variables, Lie point symmetries and reduction of order. In our work, we used similar concepts except adjoint symmetries and method of canonical variables. Aminer [3] worked out a fourth order nonlinear wave equation. The form of the equation was:

$$\left(yy'\left(y\left(y'\right)^{-1}\right)''\right)' = 0 \tag{5}$$

and its solution is:

$$V = \frac{1}{u^3 + e^{2u'-1}} \int \left(u^3 + e^{2u'-1}\right)\left(4u^{-1}u''^2u'^{-4} - 4u''^3u'^{-6} - u^{-2}u'^{-2}u''\right) du \tag{6}$$

He applied Lie groups of transformations, Lie algebras, infinitesimal transformations, invariance under transformation, symmetry, Lie's integrating factor, method of canonical variables, Lie point symmetries, increasing of order and reduction of order. We used similar method during our research work. Yulia [4] solved a third order ODE which was quadratic in the second order derivative without a higher degree. He employed Lie groups, infinitesimal generators, transformation maps and group invariants. We applied the same to our problem. Kweyama [5] researched on Lie symmetries in generating solutions to differential equations that arise in particular physical systems. It was of the form:

$$2HH'' + 6H^2H' - H'^2 + aH^2 = b \tag{7}$$

where $H = H(t)$ and got a quadratic equation whose form is:

$$p^2 + 2pq - 1 = 0 \tag{8}$$

where p and q are invariants. He used Lie groups, Lie algebras, infinitesimal transformation, invariance under transformation, symmetry, Lie point symmetries, reduction of order, increasing the order, nonlocal symmetries and transformation of symmetries. We borrowed similar concepts to our work. Mehmet [6] solved a fourth order generalized Burger's equation by confining himself to the application of Lie point symmetries. Bluman and Anco [7] worked on how to find all the integrating factors and the corresponding first integrals for any system of ODEs. These integrating factors were shown to be all the solutions of both the adjoint system of the linearised system and a system which presents an extra adjoint invariance condition of the ODEs. Abraham-Shrauner [8] determined second order nonlinear ODE and obtained a solution by applying Lie groups, group invariants and order reduction to get his solution. Schwarz [9] solved a second order differential equation and obtained a solution successfully by using Lie groups, infinitesimal generators, Lie algebras, prolongation and differential invariants which yielded the solution.

3. Materials and Methodology

3.1. Lie Groups of Transformations

A Group:

A group K is a non-empty set of elements with a law of composition Ω defined between the elements satisfying the following conditions:

(1). **Closure Property:** If x and y are elements of K, then $\Omega(x,y)$, $\forall\ x,y \in K$; then $\Omega(x,y) \in K$.

(2). **Associative Property:** For any elements x, y and z of K, $\forall\ x,y,z \in K$; then $\Omega(x, \Omega(y,z)) = \Omega(\Omega(x,y), z)$.

(3). **Identity Property:** K contains a unique element called identity element I such that for any element x of K, there exist an identity element $I \in K$ such that: then $\Omega(I,x) = \Omega(x,I) = x$.

(4). **Inverse Property:** For any element x of K there is a unique element in K called inverse element x^{-1} such that $\forall\ x \in K$, $\exists$ inverse element $x^{-1} \in K$, then $\Omega\left(x^{-1}, x\right) = \Omega\left(x, x^{-1}\right) = I$.

A Group of Transformations:

Consider a transformations set: $x = (x_1, x_2, \cdot, x_m)$ lie in a region $D \subset R^m$. Consider another transformations set: $x^* = X(x,\varepsilon)$ defined for each x in $D \subset R$ depending on real parameter ε lying in $S \subset R$. Suppose $\Omega(\varepsilon,\delta)$ defines a composition parameter law ε, δ and forms a transformation group on D [7]. $x^* = X(x,\varepsilon)$, $x^{**} = X(x, \Omega(\varepsilon,\delta))$. Hence, it is a transformations Lie group.

3.2. Lie Algebras

Commutator:

If G_1 and G_2 are vector fields then their commutator (also known as a Lie bracket) is defined as follows:

$$[G_1, G_2] = G_1 G_2 - G_2 G_1 \tag{9}$$

Lie Algebra:

L, Lie algebra is a vector space over some field F, on which commutation is defined satisfying the following Sophus Lie conditions:

1. Closure:

$$G_1, G_2 \in L \Rightarrow [G_1 G_2] \in L$$

2. Skew-symmetry:

$$[G_1, G_2] = -[G_2, G_1] \tag{10}$$

3. Bi-linearity:

$$[k_1 G_1 + k_2 G_2, G_3] = k_1 [G_1, G_3] + k_2 [G_2, G_3] \tag{11}$$

$$[G_1, k_1 G_2 + k_2 G_3] = k_1 [G_1, G_2] + k_2 [G_1, G_3] \tag{12}$$

where k_1 and k_2 are constants.

4. Jacobi's Identity:

$$[G_1, [G_2, G_3]] + [G_2, [G_3, G_1]] + [G_3, [G_1, G_2]] = 0 \tag{13}$$

for all G_1, G_2 and G_3 in L. If $[G_1, G_2] = 0$ then we say G_1 and G_2 commute and if all the elements of L commute then L is called Abelian Lie algebra.

3.3. Infinitesimal Transformations

Let us consider a transformation of one-parameter: $x^* = \mathrm{X}(x, y, \lambda)$ and $y^* = Y(x, y, \lambda)$ where λ is a continuous parameter. By taking the Taylor series expansion of this transformation about the point $\lambda = \lambda_0$ we generate:

$$x^* = x + \left(\frac{\partial \mathrm{X}}{\partial \lambda}\right)_{\lambda=\lambda_0} (\lambda - \lambda_0) + \dots \tag{14}$$

$$y^* = y + \left(\frac{\partial Y}{\partial \lambda}\right)_{\lambda=\lambda_0} (\lambda - \lambda_0) + \dots \tag{15}$$

The partial derivatives evaluated at $\lambda = \lambda_0$ with respect to group parameter λ are known as infinitesimals and are functions of x and$y$. Let us denote them by:

$$\left(\frac{\partial X}{\partial \lambda}\right)_{\lambda=\lambda_0} = \xi(x, y) \tag{16}$$

$$\left(\frac{\partial Y}{\partial \lambda}\right)_{\lambda=\lambda_0} = \eta(x, y) \tag{17}$$

Considering the values of λ sufficiently close to λ_0 by writing the coordinates of the transformation as follows: $x^* = x + \xi(x, y)(\lambda - \lambda_0)$ and $y^* = y + \eta(x, y)(\lambda - \lambda_0)$ where terms of second and higher degree in $(\lambda - \lambda_0)$ have been neglected. This transformation is known as an infinitesimal transformation.

Infinitesimal Generator:

The one-parameter Lie group of transformations of infinitesimal generator is an operator:

$$X = X(x) = \gamma(x).\nabla = \sum_{i=1}^{n} \gamma_i(x) \frac{\partial}{\partial x_i} \tag{18}$$

where the gradient operator ∇ is given as; $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}\right)$ for any function that is differentiable: $F(x) = F(x_1, x_2, \dots, x_n)$ and

$$XF(x) = \gamma(x).\nabla F(x) = \sum_{i=1}^{n} \gamma_i(x) \frac{\partial F(x)}{\partial x_i} \tag{19}$$

Thus a one-parameter transformations of Lie group is "equivalent" to its infinitesimal generator in the same way it is "equivalent" to its infinitesimal transformation.

3.4. Prolongations (Extended Transformations)

When we want to apply a transformations point: $x^* = X(x, y, \omega)$ and $y^* = Y(x, y, \omega)$ to the differential equation:

$$H\left(x, y, y', y'', y''', ..., y^{(m)}\right) = 0 \tag{20}$$

$y' = \frac{dy}{dx}$. We want to transform the derivatives $y^{(m)}$ that is to extend (prolong) the point transformation to the derivatives. The task here is extending on the transformation (20) acting on (x, y) to the $(x, y, y_1, y_2, y_3, ..., y_m)$ space with the property of preserving the contact of differentials conditions: $dx, dy, dy_1, dy_2, ..., dy_m$

$$\begin{aligned} dy &= y_1 dx \\ dy_1 &= y_2 dx \\ dy_2 &= y_3 dx \\ &\vdots \\ dy_m &= y_{m+1} dx \end{aligned} \tag{21}$$

3.5. Invariance under Transformations

Invariant

An invariant is that which remains unchanged when its constituents change. The concept of invariance has a physical basis in the conservation laws of mechanics. A function$f$ under a Lie group is invariant if and only if

$$f\left(x^{*}, y^{*}\right)=f(\mathrm{X}(x, y, \lambda), Y(x, y, \lambda))=f(x, y) \tag{22}$$

The function must read the same when expressed in the new variables.

3.6. Variation Symmetries

Symmetry

Symmetry is an operation which leaves invariant that upon which it operates. Symmetry of a transformation geometrical object apparently leaves the object unchanged. Consider the transformation of infinitesimal form:

$$x_{i}^{*}=x_{i}+\varepsilon \omega_{i}, \quad i=1,2, \ldots, m \tag{23}$$

where ε is a parameter of smallness. Here equation (23) can be written as $x_i^*=(1+\varepsilon G)\,x_i$, where $G=\omega_i\frac{\partial}{\partial x_i}$ is a differential operator called the generator of the transformation (23). Consider:

$$G=\omega \frac{\partial}{\partial x}+\phi \frac{\partial}{\partial y} \tag{24}$$

Under the infinitesimal transformation generated by G, a function $f(x, y)$ becomes:

$$f^{*}\left(x^{*}, y^{*}\right)=(1+\varepsilon G) f(x, y)=f+\varepsilon\left(\omega \frac{\partial f}{\partial x}+\phi \frac{\partial f}{\partial y}\right) \tag{25}$$

If the form of f is unchanged then $f^*(x^*, y^*)=f(x, y)$ or

$$\omega \frac{\partial f}{\partial x}+\phi \frac{\partial f}{\partial y}=0 \tag{26}$$

then G is called a symmetry of f.

3.7. Lie Theory of Differential Equations

Lie Point Symmetries of ODEs

Point symmetry is a symmetry in which the infinitesimals depend only on coordinates [4]. We describe Lie point symmetry as a point symmetry that depends continuously on at least one-parameter, thus the parameters can vary continuously over a set of scalar non-zero measure. Lie point symmetries of ODEs are of the form:$G=\omega\frac{\partial}{\partial x}+\phi\frac{\partial}{\partial y}$ where ω and ϕ are coefficients functions of only x and y. To apply a point transformation to an m^{th} order ODE, $f\left(x, y, y', y'', \ldots, y^{(m)}\right)=0$, where $y'=\frac{dy}{dx}$, $y''=\frac{d^2y}{dx^2}, \ldots, y^{(m)}=\frac{d^my}{dx^m}$, we need to know how derivatives undergo infinitesimal transformation: $x^*=x+\in\omega(x,y)$, $y^*=y+\in\phi(x,y)$ which has a generator given by

$$G=\omega(x, y) \frac{\partial}{\partial x}+\phi(x, y) \frac{\partial}{\partial y} \tag{27}$$

In terms of the quantities x^* and y^* we have the derivative:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d\left(y+\varepsilon\phi\right)}{d\left(x+\varepsilon\omega\right)} \\ &= \frac{\frac{dy}{dx}+\varepsilon\frac{d\phi}{dx}}{1+\varepsilon\frac{d\omega}{dx}} \\ &= \left(y'+\varepsilon\phi'\right)\left(1-\varepsilon\omega'+\varepsilon^2\omega'^2-...\right) \\ &= y'+\varepsilon\left(\phi'-y'\omega'\right) \end{aligned} \tag{28}$$

which we have terminated at $O\left(\varepsilon^2\right)$. The primes here are for total differentiation with respect to x. Now our second derivative gives:

$$\begin{aligned}\frac{d^2y^*}{dx^{*2}} &= \frac{d}{dx^*}\left(\frac{dy^*}{dx^*}\right) \\ &= \frac{d\left[y'+\varepsilon\left(\phi'-y'\omega'\right)\right]}{d\left(x+\varepsilon\omega\right)} \\ &= \frac{\frac{dy'}{dx}+\varepsilon\frac{d}{dx}\left(\phi'-y'\omega'\right)}{1+\varepsilon\omega'} \\ &= y''+\varepsilon\left(\phi''-2y''\omega'-y'\omega''\right)\end{aligned} \tag{29}$$

Further, the third derivative is as follows:

$$\frac{d^3y^*}{dx^{*3}} = y'''+\varepsilon\left(\phi'''-3y'''\omega'-3y''\omega''-y'\omega'''\right)$$

In general, we generate the formula:

$$\frac{d^my^*}{dx^{*m}} = y^{(m)}+\varepsilon\left(\phi^{(m)}-\sum_{i=1}^{m}C_i^m y^{(i+1)}\omega^{(m-i)}\right) \tag{30}$$

where the superscript (i) denotes $\frac{d^i}{dx^i}$ and C_i^m is the number of combinations of m objects taken i at a time. We indicate that a generator G has been extended by writing

$$G^{[1]} = G+\left(\phi'-y'\omega'\right)\frac{\partial}{\partial y'} \tag{31}$$

$$G^{[2]} = G^{[1]}+\left(\phi''-2y''\omega'-y'\omega''\right)\frac{\partial}{\partial y''} \tag{32}$$

for the first and second extensions respectively. For an m^{th} order differential equation, the m^{th} extension is of the form:

$$G^{[m]} = G+\sum_{i=1}^{m}\left\{\phi^{(i)}-\sum_{j=1}^{i}\begin{bmatrix} i \\ j\end{bmatrix}y^{(i+1-j)}\omega^{(i)}\right\}\frac{\partial}{\partial y^{(i)}} \tag{33}$$

The generator $G=\omega\frac{\partial}{\partial x}+\phi\frac{\partial}{\partial y}$ is symmetry of the differential equation $E(x,y,y',y'',...,y^{(m)})=0$ if and only if $G^{[m]}E_{E=0}=0$ which means that the action of the m^{th} extension of G on E is zero when the original equation is satisfied.

3.8. Reduction of Order

If a differential equation: $E\left(x,y,y',...,y^{(m)}\right)=0$ has symmetry:

$$G=\omega\left(x,y\right)\frac{\partial}{\partial x}+\phi\left(x,y\right)\frac{\partial}{\partial y} \tag{34}$$

we obtain an equation of order $(m-1)$ in a systematic manner. This is achieved by using the $zero^{th}$ and first order differential invariants which are the two characteristics associated with $G^{[1]}$. The characteristics are obtained by solving the following system of ODEs:

$$\frac{dx}{\omega}=\frac{dy}{\phi}=\frac{dy'}{\phi'-y'\omega'} \tag{35}$$

3.9. Integrating Factors

3.9.1. Theorem on Integrating Factor

Consider a first order ODE: $M(x, y)dx + N(x, y)dy = 0$ which admits a one-parameter Lie group G with an infinitesimal generator: $X = \sigma(x, y)\frac{\partial}{\partial x} + \psi(x, y)\frac{\partial}{\partial y}$ if and only if the function:

$$\rho = \frac{1}{\sigma M + \psi N} \tag{36}$$

is the integrating factor provided that $\sigma M + \psi N \neq 0$

4. Results and Discussion

We solved a third order nonlinear ODE of fourth degree in second derivative which is a form of a wave equation: $F\left(x, y, y', y'', y'''\right) = 0$ or

$$y''' = f\left(x, y, y', y''\right) \tag{37}$$

Our objective was to solve the special case (37) using the method of Lie symmetry. By expressing (1) in other ways gives: $y''' - y'\left(\frac{y''^4}{y^4}\right) = 0$, when the power is brought into the bracket:

$$y''' - y'\left(y\right)^{-4}\left(y''\right)^4 = 0 \tag{38}$$

after applying the law of indices and removal of the fraction. By applying the m^{th} extension of G given as:

$$G^{[m]} = G + \sum_{i=1}^{m}\left(\phi^{(i)} - \sum_{j=1}^{i}\begin{bmatrix} i \\ j \end{bmatrix} y^{(i+1-j)}\omega^{(i)}\right)\frac{\partial}{\partial y^{(i)}} \tag{39}$$

where m is the order, i is the upper limit and j is the lower limit. Then the third extension of $G^{[3]}$ is:

$$G^{[3]} = \omega\frac{\partial}{\partial x} + \phi\frac{\partial}{\partial y} + \left(\phi' - \omega' y'\right)\frac{\partial}{\partial y'} + \left(\phi'' - 2y''\omega' - y'\omega''\right)\frac{\partial}{\partial y''} + \left(\phi''' - 3y'''\omega' - 3y''\omega'' - y'\omega'''\right)\frac{\partial}{\partial y'''} \tag{40}$$

Now, manipulating $G^{[3]}$ on (38) yields:

$$\begin{aligned}\Big(\omega\frac{\partial}{\partial x} + \phi\frac{\partial}{\partial y} + \left(\phi' - \omega' y'\right)\frac{\partial}{\partial y'} + \left(\phi'' - 2\omega' y'' - \omega'' y'\right)\frac{\partial}{\partial y''} \\ + \left(\phi''' - 3\omega' y''' - 3\omega'' y'' - \omega''' y'\right)\frac{\partial}{\partial y'''}\Big)\left(y''' - y'\left(y''\right)^4\left(y\right)^{-4}\right) = 0\end{aligned} \tag{41}$$

Hence, expansion of (41) gives:

$$\begin{aligned}\Rightarrow \omega\frac{\partial}{\partial x}\left[y''' - y'\left(y''\right)^4\left(y\right)^{-4}\right] + \phi\frac{\partial}{\partial y}\left[y''' - y'\left(y''\right)^4\left(y\right)^{-4}\right] \\ + \left(\phi' - \omega' y'\right)\frac{\partial}{\partial y'} + \left(\phi'' - 2y''\omega' - y'\omega''\right)\frac{\partial}{\partial y''}\left[y''' - y'\left(y''\right)^4\left(y\right)^{-4}\right] \\ + \left(\phi''' - 3y'''\omega' - 3y''\omega'' - y'\omega'''\right)\frac{\partial}{\partial y'''}\left[y''' - y'\left(y''\right)^4\left(y\right)^{-4}\right] = 0\end{aligned} \tag{42}$$

From (42) after simplifying and then combining gives:

$$\omega\left[\, y^{(iv)} - \left(y''\right)^5\left(y\right)^{-4} - 4y'\left(y''\right)^3\left(y\right)^{-4}y''' + 4\left(y'\right)^2\left(y''\right)^4\left(y\right)^{-5}\,\right]$$

$$+\phi\left[4y'\left(y''\right)^{4}(y)^{-5}\right]+\left(\phi'-\omega'y'\right)\left[-\left(y''\right)^{5}(y)^{-4}\right]$$
$$+\left(\phi''-2y''\omega'-y'\omega''\right)\left[-4y'\left(y''\right)^{3}(y)^{-4}y'''\right]+\left(\phi'''-3y'''\omega'-3y''\omega''-y'\omega'''\right)=0 \tag{43}$$

Thus, from (38) we have: $y'''-y'\left(y''\right)^{4}(y)^{-4}=0$

$$\Rightarrow y'''=y'\left(y''\right)^{4}(y)^{-4} \tag{44}$$

and $y^{(iv)}=\left(y'''\right)'$ hence $y^{(iv)}=\left(y'\left(y''\right)^{4}(y)^{-4}\right)'$

$$y^{(iv)}=\left(y''\right)^{5}(y)^{-4}+4y'\left(y''\right)^{3}y'''(y)^{-4}-4\left(y'\right)^{2}\left(y''\right)^{4}(y)^{-5} \tag{45}$$

By substituting gives:

$$\omega\left[\ \left(y''\right)^{5}(y)^{-4}+4y'\left(y''\right)^{3}y'''(y)^{-4}-4\left(y'\right)^{2}\left(y''\right)^{4}(y)^{-5}-\left(y''\right)^{5}(y)^{-4}\right.$$
$$\left.-4y'\left(y''\right)^{3}(y)^{-4}y'''+4\left(y'\right)^{2}\left(y''\right)^{4}(y)^{-5}\ \right]+\left[\ 4y'\left(y''\right)^{4}(y)^{-5}\ \right]\phi$$
$$-\left[\left(y''\right)^{5}(y)^{-4}\right]\left(\phi'-\omega'y'\right)-\left[4y'\left(y''\right)^{3}(y)^{-4}y'''\right]\left(\phi''-2y''\omega'-y'\omega''\right)$$
$$+\left(\phi'''-3y'''\omega'-3y''\omega''-y'\omega'''\right)=0 \tag{46}$$

Further simplification gives:

$$\omega\left(y''\right)^{5}(y)^{-4}+4\omega y'\left(y''\right)^{3}(y)^{-4}y'''-4\omega\left(y'\right)^{2}\left(y''\right)^{4}(y)^{-5}-\omega\left(y''\right)^{5}(y)^{-4}$$
$$-4\omega y'\left(y''\right)^{3}(y)^{-4}y'''+4\omega\left(y'\right)^{2}\left(y''\right)^{4}(y)^{-5}+4\phi y'\left(y''\right)^{4}(y)^{-4}-\phi'\left(y''\right)^{5}(y)^{-4}$$
$$+\omega'y'\left(y''\right)^{5}(y)^{-4}-4\phi''y'\left(y''\right)^{3}(y)^{-4}y'''+8\omega'y'\left(y''\right)^{4}(y)^{-4}y'''+4\omega''\left(y'\right)^{2}\left(y''\right)^{3}(y)^{-4}y'''$$
$$+\phi'''-3\omega'y'''-3\omega''y''-y'\omega'''=0 \tag{47}$$

Again when simplified, it gives:

$$4\phi y'(y)^{-4}\left(y''\right)^{4}-\phi'(y)^{-4}\left(y''\right)^{5}-4\phi''y'(y)^{-4}\left(y''\right)^{3}y'''+\phi'''$$
$$+8\omega'y'(y)^{-4}\left(y''\right)^{4}y'''+\omega'y'(y)^{-4}\left(y''\right)^{5}-3\omega'y'''-3\omega''y''$$
$$4\omega''(y)^{-4}\left(y'\right)^{2}\left(y''\right)^{3}y'''-y'\omega'''=0 \tag{48}$$

By expressing first, second and third derivatives of ω and ϕ in terms of partial derivatives given that: $\omega=\omega(x,y)$ then $d(x)=\left(\frac{\partial\omega}{\partial x}\right)dx+\left(\frac{\partial\omega}{\partial y}\right)dy$. Therefore

$$\omega'=\frac{\partial\omega}{\partial x}+y'\frac{\partial\omega}{\partial y} \tag{49}$$
$$\omega''=\frac{\partial^{2}\omega}{\partial x^{2}}+2y'\frac{\partial^{2}\omega}{\partial x\partial y}+y'^{2}\frac{\partial^{2}\omega}{\partial y^{2}}+y''\frac{\partial\omega}{\partial y} \tag{50}$$
$$\omega'''=\frac{\partial^{3}\omega}{\partial x^{3}}+3y'\frac{\partial^{3}\omega}{\partial x^{2}\partial y}+3y''\frac{\partial^{2}\omega}{\partial x\partial y}+y'''\frac{\partial\omega}{\partial y}+3y'y''\frac{\partial^{2}\omega}{\partial y^{2}}+y'^{3}\frac{\partial^{3}\omega}{\partial y^{3}}+3y'^{2}\frac{\partial^{3}\omega}{\partial x\partial y^{2}} \tag{51}$$

and also $\phi=\phi(x,y)$ then $d(\phi)=\left(\frac{\partial\phi}{\partial x}\right)dx+\left(\frac{\partial\phi}{\partial y}\right)dy$. Therefore

$$\phi'=\frac{\partial\phi}{\partial x}+y'\frac{\partial\phi}{\partial y} \tag{52}$$

$$\phi'' = \frac{\partial^2 \phi}{\partial x^2} + 2y'\frac{\partial^2 \phi}{\partial x \partial y} + y'^2 \frac{\partial^2 \phi}{\partial y^2} + y''\frac{\partial \phi}{\partial y} \tag{53}$$

$$\phi''' = \frac{\partial^3 \phi}{\partial x^3} + 3y'\frac{\partial^3 \phi}{\partial x^2 \partial y} + 3y''\frac{\partial^2 \phi}{\partial x \partial y} + y'''\frac{\partial \phi}{\partial y} + 3y'^2\frac{\partial^3 \phi}{\partial x \partial y^2} + 3y'y''\frac{\partial^2 \phi}{\partial y^2} + y'^3\frac{\partial^3 \phi}{\partial y^3} \tag{54}$$

When we substitute (49), (50), (51), (52), (53) into (48) we get:

$$\Rightarrow 4\phi y'(y)^{-4}(y'')^4 - \phi'(y)^{-4}(y'')^5 - 4\phi'' y'(y)^{-4}(y'')^3 y''' + \phi''' + 8\omega' y'(y)^{-4}(y'')^4 y''' + \omega' y'(y)^{-4}(y'')^5$$
$$- 3\omega' y''' - 3\omega'' y'' + 4\omega''(y)^{-4}(y')^2(y'')^3 y''' - y'\omega''' = 0 \tag{55}$$

When we expand, it yields:

$$\begin{aligned}
&4\phi y'(y)^{-4}(y'')^4 - (y)^{-4}(y'')^5\frac{\partial \phi}{\partial x}(y)^{-4}(y'')^5 y'\frac{\partial \phi}{\partial y} - 4y'(y)^{-4}(y'')^3 y'''\frac{\partial^2 \phi}{\partial x^2}\\
&- 8(y')^2(y)^{-4}(y'')^3 y'''\frac{\partial^2 \phi}{\partial x \partial y} - 4(y')^3(y)^{-4}(y'')^3 y'''\frac{\partial^2 \phi}{\partial y^2} - 4y'(y)^{-4}(y'')^4 y'''\frac{\partial \phi}{\partial y} + \frac{\partial^3 \phi}{\partial x^3}\\
&+ 3y'\frac{\partial^3 \phi}{\partial x^2 \partial y} + 3y''\frac{\partial^2 \phi}{\partial x \partial y} + y'''\frac{\partial \phi}{\partial y} + 3(y')^2\frac{\partial^3 \phi}{\partial x \partial y^2} + 3y''y'\frac{\partial^2 \phi}{\partial y^2} + (y')^3\frac{\partial^3 \phi}{\partial y^3} + y'''\frac{\partial \phi}{\partial y}\\
&+ 8y'(y)^{-4}(y'')^4 y'''\frac{\partial \omega}{\partial x} + 8(y')^2(y)^{-4}(y'')^4 y'''\frac{\partial \omega}{\partial y} + y'(y)^{-4}(y'')^5\frac{\partial \omega}{\partial x} + (y')^2(y)^{-4}(y'')^5\frac{\partial \omega}{\partial y}\\
&- 3y'''\frac{\partial \omega}{\partial x} - 3y'''y'\frac{\partial \omega}{\partial y} - 3y''\frac{\partial^2 \omega}{\partial x^2} - 6y''y'\frac{\partial^2 \omega}{\partial x \partial y} - 3y''(y')^2\frac{\partial^2 \omega}{\partial y^2} - 3(y'')^2\frac{\partial \omega}{\partial y} + 4(y)^{-4}(y')^2(y'')^3 y'''\frac{\partial^2 \omega}{\partial x^2}\\
&+ 8(y')^3(y)^{-4}(y'')^3 y'''\frac{\partial^2 \omega}{\partial x \partial y} + 4(y')^4(y)^{-4}(y'')^3 y'''\frac{\partial^2 \omega}{\partial y^2} + 4(y'')^4(y)^{-4}(y')^2 y'''\frac{\partial \omega}{\partial y} - y'\frac{\partial^3 \omega}{\partial x^3}\\
&- 3(y')^2\frac{\partial^3 \omega}{\partial x^2 \partial y} - 3y'y''\frac{\partial^2 \omega}{\partial x \partial y} - 3(y')^3\frac{\partial^3 \omega}{\partial x \partial y^2} - 3y''(y')^2\frac{\partial^2 \omega}{\partial y^2} - (y')^4\frac{\partial^3 \omega}{\partial y^3} - y'y''\frac{\partial \omega}{\partial y} = 0
\end{aligned} \tag{56}$$

where (56) forms an identity in x, y, y', y'', y'''. Given that ω and ϕ are functions in x and y alone, by equating the combinations of coefficients of the powers of y', y'', y''' to zero and integrating yields:

$$\omega = A_1 y + A_2 \tag{57}$$

where A_1 and A_2 are arbitrary functions of x.

$$\phi = A_1' y^2 + A_3 y + A_4 \tag{58}$$

where A_3 and A_4 are arbitrary functions of x.

$$3A_1'' y + 2A_3' - A_2'' = 0 \tag{59}$$

By equating the coefficients of powers of y^0 and y^1 to zero in (59) and substituting yields:

$$A_1''' y^{-2} + A_3''(y)^{-3} + A_4''(y)^{-4} = 0 \tag{60}$$

By equating the coefficients of powers of y^{-4}, y^{-3} and y^{-2} to zero and solving yields:

$$A_1 = B_1 x + B_2, A_3 = B_3 x + B_4, A_2 = B_3 x^2 + B_5 x + B_6 \;\; and \;\; A_4 = B_7 x + B_8$$

where $B_1, B_2, B_3, B_4, B_5, B_6, B_7, B_8$ are arbitrary constants. From $\omega = A_1 y + A_2$ and substituting A_1 and A_2 gives:

$$\omega = B_1 xy + B_2 y + B_3 x^2 + B_5 x + B_6 \tag{61}$$

From $\phi = A_1' y^2 + A_3 y + A_4$ then by substituting A_1, A_3 and A_4:

$$\phi = (B_1 x + B_2)' \, y^2 + (B_3 x + B_4)\, y + B_7 x + B_8 \tag{62}$$

Now the infinitesimal generator G is of the form: $G = \omega \frac{\partial}{\partial x} + \phi \frac{\partial}{\partial y}$ By substituting ω and ϕ, this form is then given as:

$$G = \left(B_1 xy + B_2 y + B_3 x^2 + B_5 x + B_6\right) \frac{\partial}{\partial x} + \left(B_1 y^2 + B_3 xy + B_4 y + B_7 x + B_8\right) \frac{\partial}{\partial y}$$

Therefore

$$\begin{aligned} G = & B_1 \left(xy \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y}\right) + B_2 \left(y \frac{\partial}{\partial x}\right) + B_3 \left(x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}\right) + B_4 \left(y \frac{\partial}{\partial y}\right) \\ & + B_5 \left(x \frac{\partial}{\partial x}\right) + B_6 \left(\frac{\partial}{\partial x}\right) + B_7 \left(x \frac{\partial}{\partial y}\right) + B_8 \left(\frac{\partial}{\partial y}\right) \end{aligned} \tag{63}$$

which is eight parameter symmetry. We can generate an $eight - one$ parameter symmetry given by:

$$\begin{aligned} & G_1 = \frac{\partial}{\partial x}, G_2 = \frac{\partial}{\partial y}, G_3 = x \frac{\partial}{\partial x}, G_4 = y \frac{\partial}{\partial x}, G_5 = y \frac{\partial}{\partial y}, \\ & G_6 = x \frac{\partial}{\partial y}, G_7 = xy \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y}, G_8 = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y} \end{aligned} \tag{64}$$

Now, the non-zero Lie brackets are given as follows:

$$[G_1, G_3] = G_1, [G_1, G_6] = G_2, [G_2, G_4] = G_1, [G_2, G_5] = G_2, [G_2, G_8] = G_6,$$

$$[G_3, G_4] = -G_4, [G_3, G_6] = G_6, [G_4, G_5] = -G_4 \, [G_4, G_8] = G_7, [G_5, G_7] = G_7$$

The process of finding the symmetries of ordinary differential equations is highly systematic. Thus let

$$S_1 = \frac{\partial}{\partial x}, S_3 = x \frac{\partial}{\partial x} \tag{65}$$

which are the Lie solvable algebra of the admitted eight-one parameter symmetry (64). By solving using prolongation:

$$G^{[3]} = G^{[2]} + \left(\phi''' - 3y'''\omega' - 3y''\omega'' - y'\omega'''\right) \frac{\partial}{\partial y'''} \tag{66}$$

Consider the third order prolongation for the differential invariant operator: $G = S = \omega \frac{\partial}{\partial x} + \phi \frac{\partial}{\partial y}$. It then follows that: $S_1^{[0]} = \frac{\partial}{\partial x}$, $S_1^{[1]} = \frac{\partial}{\partial x}$, $S_1^{[2]} = \frac{\partial}{\partial x}$,

$$S_1^{[3]} = 1 \bullet \frac{\partial}{\partial x} + 0 \bullet \frac{\partial}{\partial y} \tag{67}$$

By solving for the characteristic: $\frac{dx}{1} = \frac{dy}{0}$, then $dy = 0$ and integrating yields the differential invariant:

$$y = U \tag{68}$$

where U is a constant, a function of x. Again, consider the third order prolongation of the differential invariant operator: $G = S = \omega \frac{\partial}{\partial x} + \phi \frac{\partial}{\partial y}$ and when $S_3 = x \frac{\partial}{\partial x}$ it follows that:

$$\begin{aligned} & S_3^{[0]} = x \frac{\partial}{\partial x}, \quad S_3^{[1]} = x \frac{\partial}{\partial x} - y' \frac{\partial}{\partial y'}, \quad S_3^{[2]} = x \frac{\partial}{\partial x} - y' \frac{\partial}{\partial y'} - 2y'' \frac{\partial}{\partial y''}, \\ & S_3^{[3]} = x \frac{\partial}{\partial x} - y' \frac{\partial}{\partial y'} - 2y'' \frac{\partial}{\partial y''} - 3y''' \frac{\partial}{\partial y'''} \end{aligned} \tag{69}$$

By solving for the characteristics:

$$\frac{dy}{1} = \frac{dy'}{-y'} = \frac{dy''}{-2y''} = \frac{dy'''}{-3y'''} \tag{70}$$

Then by integrating (70) the differential invariants are as follows:

(i). $\frac{dy}{1} = \frac{dy'}{-y'}$. Therefore

$$y = \ln\left|\frac{C_1}{y'}\right| \tag{71}$$

where C_1 is a constant.

(ii). $\frac{dy'}{-y'} = \frac{dy''}{-2y''}$, $t_1 = \frac{1}{C_2}$. Therefore

$$t_1 = \frac{y''}{(y')^2} \tag{72}$$

where $C_2$and t_1 are constants.

(iii). $\frac{dy'}{-y'} = \frac{dy'''}{-3y'''}$, $t_2 = \frac{1}{C_3}$. Therefore

$$t_2 = \frac{y'''}{(y')^3} \tag{73}$$

where $C_3$and $t_2$are constants.

(iv). $\frac{dy''}{-2y''} = \frac{dy'''}{-3y'''}$, $t_3 = \frac{1}{C_4}$. Therefore

$$t_3 = \frac{(y''')^2}{(y'')^3} \tag{74}$$

where C_4 and t_3 are constants. By taking (68) and (72) such that:

$y = U$, $t_1 = \frac{y''}{(y')^2}$ and if we let $t_1 = V$ then t_1 becomes:

$$V = \frac{y''}{(y')^2} \tag{75}$$

Now reducing (1) to first order ODE yields:

$$\begin{aligned}\frac{dV}{dy} &= \frac{D_x(V)}{D_x(y)} = \frac{D_x\left(\frac{y''}{y'^2}\right)}{D_x(y)} = \frac{y'''(y')^2}{(y')^5} - \frac{2y'y''y''}{(y')^5} = \frac{y'''}{(y')^3} - \frac{2y''y''}{(y')^4} = \frac{y'(y'')^4(y)^{-4}}{(y')^3} - \frac{2y''y''}{(y')^2(y')^2}\\ &= \frac{y'(y'')^4(y)^{-4}}{(y')^3} - \frac{2(y'')(y'')}{(y')^2(y')^2} = (y'')^4(y')^{-2}(y)^{-4} - \frac{2(y'')(y'')}{(y')^2(y')^2}\end{aligned} \tag{76}$$

From (68) and (75) through substitution leads to: $\frac{dV}{dy} = (y'')^4(y')^{-2}(y)^{-4} - 2V^2$. Therefore

$$\frac{dV}{dy} + 2V^2 = (y'')^4(y')^{-2}(y)^{-4} \tag{77}$$

Then (77) is of the form:

$$\frac{dV}{dy} + P(y)V = Q(y) \tag{78}$$

implying that we have managed to reduce a third order equation (1) to a simple first order linear equation (77) that is easily solvable by other known simpler methods. If $P(y) = 2V$ and $Q(y) = (y'')^4(y')^{-2}(y)^{-4}$, then (1) reduces to (78) which can be easily integrated using integrating factors given by: $I(y)$. Thus

$$I(y) = e^{\int P(y)dy} \tag{79}$$

$$\begin{aligned}I(y) &= e^{\int 2V dy} = e^{2\int V dy} = e^{2\int\left(\frac{y''}{(y')^2}\right)dy} = e^{2\ln|y'|^2 + C} = e^{\ln|y'|^4} \bullet e^C\\ &= Me^{\ln|y'|^4}\end{aligned}$$

(If $e^C = M) = e^{\ln|y'|^4}$ (Since $C = 0$, $M = 1$) then therefore $I(y) = e^{\ln|y'|^4} = (y')^4$ where C and M are constants. From the form:

$$V = \frac{1}{I(y)}\int (y')^4 Q(y)\,dy \tag{80}$$

Then it follows that: $V = \frac{1}{(y')^4} \int (y')^4 \left[(y'')^4\ (y')^{-2}\ (y)^{-4}\right] dy$ whose simplification leads to

$$V = \frac{1}{(y')^4} \int (y'')^4\, (y')^2\, (y)^{-4}\, dy \tag{81}$$

which now completes the process of integration hence (81) is a simple first order form of the required mathematical solution of the special type wave equation (1).

5. Conclusion

Our interest was to get a mathematical solution whose form can be used by other mathematicians, engineers and researchers in science to solve specific wave equations which forms the basis of future predictions aiming at saving human loss of life and properties. For example, once the range of amplitude or velocity of a particular wave across the ocean has been calculated, a decision can be made to clear the vicinity including human evacuation in order to minimize damages.

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