

Strong Hub Number of a Fuzzy Graph

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Abstract

In the study of fuzzy graphs, which generalize classical graphs by allowing varying degrees of membership for nodes and arcs, the concept of centrality becomes crucial for understanding the structural importance of nodes. One such centrality measure is the hub number, the minimum number of nodes (hubs) required. Every other node in the fuzzy graph is strongly adjacent to at least one hub with a degree of membership exceeding a given threshold. This paper introduces the formal definition of the hub number in the context of fuzzy graphs, explores its properties, and provides methods for its computation. Applications of the hub number include network design, information dissemination, and fuzzy social network analysis. The study offers theoretical insights and practical algorithms, contributing to the broader understanding of centrality in uncertain and imprecise network structures.

Keywords: Fuzzy graph theory; Hub set; Fuzzy connectivity; Fuzzy path; Membership threshold; Fuzzy network optimization; NP-completeness.

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1. Introduction

Fuzzy graph theory provides a robust framework for modeling systems characterized by uncertainty, partial connectivity, or imprecise relationships. Unlike classical graphs, where edges either exist or do not, fuzzy graphs incorporate edge membership functions that quantify the strength or reliability of connections. This makes them particularly suitable for applications such as transportation planning, communication networks, and decision-making under uncertainty. In classical graph theory, the concept of a hub set ensures that all communication between non-hub vertices can occur via a small, designated subset of nodes called hubs. This concept is relevant in scenarios such as the design of rapid-transit systems, where the goal is to place stations (hubs) in a minimal number of locations so that travel between any two non-station nodes is still feasible through these hubs. This paper extends the concept of hub set to fuzzy graphs.

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2. Preliminaries

We summarize briefly some basic definitions in fuzzy graphs which are presented in [1,2,4,11,13,16,19, 21].

A **fuzzy graph** is denoted by $G : (V, \sigma, \mu)$ where V is a node set, σ is a fuzzy subset of V and μ is a fuzzy relation on σ . i.e., $\mu(x, y) \leq \sigma(x) \wedge \sigma(y)$ for all $x, y \in V$. We call σ the fuzzy node set of G and μ the fuzzy arc set of G , respectively. We consider fuzzy graph G with no loops and assume that V is finite and nonempty, μ is reflexive (i.e., $\mu(x, x) = \sigma(x)$, for all x) and symmetric (i.e., $\mu(x, y) = \mu(y, x)$, for all (x, y)). In all the examples σ is chosen suitably. Also, we denote the underlying crisp graph by $G^* : (\sigma^*, \mu^*)$ where $\sigma^* = \{u \in V : \sigma(u) > 0\}$ and $\mu^* = \{(u, v) \in V \times V : \mu(u, v) > 0\}$. Throughout we assume that $\sigma^* = V$. The fuzzy graph $H : (\tau, \nu)$ is said to be a **partial fuzzy subgraph** of $G : (\sigma, \mu)$ if $\nu \subseteq \mu$ and $\tau \subseteq \sigma$. In particular we call $H : (\tau, \nu)$, a fuzzy subgraph of $G : (\sigma, \mu)$ if $\tau(u) = \sigma(u)$ for all $u \in \tau^*$ and $\nu(u, v) = \mu(u, v)$ for all $(u, v) \in \nu^*$. A fuzzy subgraph $H : (\tau, \nu)$ spans the fuzzy graph $G : (V, \sigma, \mu)$ if $\tau = \sigma$. The fuzzy graph $H : (P, \tau, \nu)$ is called an induced fuzzy subgraph of $G : (V, \sigma, \mu)$ induced by P if $P \subseteq V$ and $\tau(u) = \sigma(u)$ for all $u \in P$ and $\nu(u, v) = \mu(u, v)$ for all $u, v \in P$. We shall use the notation $\langle P \rangle$ to denote the fuzzy subgraph induced by P . $G : (V, \sigma, \mu)$ is called trivial if $|\sigma^*| = 1$.

In a fuzzy graph $G : (V, \sigma, \mu)$, a **path** P of length n is a sequence of distinct nodes u_0, u_1, \dots, u_n such that $\mu(u_{i-1}, u_i) > 0, i = 1, 2, \dots, n$ and the degree of membership of a weakest arc is defined as its strength. If $u_0 = u_n$ and $n \geq 3$ then P is called a cycle and P is called a fuzzy cycle, if it contains more than one weakest arc. The strength of a cycle is the strength of the weakest arc in it. The strength of connectedness between two nodes x and y is defined as the maximum of the strengths of all paths between x and y and is denoted by $CONN_G(x, y)$. A fuzzy graph $G : (\sigma, \mu)$ is **connected** if for every x, y in σ^* , $CONN_G(x, y) > 0$. An arc (u, v) of a fuzzy graph is called an **effective arc** (M-strong arc) if $\mu(u, v) = \sigma(u) \wedge \sigma(v)$. Then u and v are called effective neighbors. The set of all effective neighbors of u is called effective neighborhood of u and is denoted by $EN(u)$. A fuzzy graph G is said to be **complete** if $\mu(u, v) = \sigma(u) \wedge \sigma(v)$, for all $u, v \in \sigma^*$ and is denoted by K_σ . The **order** p and **size** q of a fuzzy graph $G : (\sigma, \mu)$ are defined to be $p = \sum_{x \in V} \sigma(x)$ and $q = \sum_{(x,y) \in V \times V} \mu(x, y)$. Let $G : (V, \sigma, \mu)$ be a fuzzy graph and $S \subseteq V$. Then the **scalar cardinality** of S is defined to be $\sum_{v \in S} \sigma(v)$ and it is denoted by $|S|$. Let p denotes the scalar cardinality of V , also called the order of G .

An arc of a fuzzy graph is called **strong** if its weight is at least as great as the strength of connectedness of its end nodes when it is deleted. Depending on $CONN_G(x, y)$ of an arc (x, y) in a fuzzy graph G , Sunil Mathew and Sunitha [21] defined three different types of arcs. Note that $CONN_{G-(x,y)}(x, y)$ is the the strength of connectedness between x and y in the fuzzy graph obtained from G by deleting the arc (x, y) . An arc (x, y) in G is α - **strong** (Fuzzy Bridge) if $\mu(x, y) > CONN_{G-(x,y)}(x, y)$. An arc (x, y) in G is β - **strong** if $\mu(x, y) = CONN_{G-(x,y)}(x, y)$. An arc (x, y) in G is δ - **arc** if $\mu(x, y) < CONN_{G-(x,y)}(x, y)$. Thus an arc (x, y) is a strong arc if it is either α - strong or β - strong. A path P is called strong path if P contains only strong arcs. A fuzzy graph G is said to be **bipartite** [19] if the node set V can be

partitioned into two non empty sets V_1 and V_2 such that $\mu(v_1, v_2) = 0$ if $v_1, v_2 \in V_1$ or $v_1, v_2 \in V_2$. Further if $\mu(u, v) = \sigma(u) \wedge \sigma(v)$ for all $u \in V_1$ and $v \in V_2$ then G is called a **complete bipartite graph** and is denoted by K_{σ_1, σ_2} , where σ_1 and σ_2 are respectively the restrictions of σ to V_1 and V_2 . A connected fuzzy graph $G = (V, \sigma, \mu)$ is called a **fuzzy tree** if it has a fuzzy spanning subgraph $F : (\sigma, \nu)$, which is a tree [spanning tree], where for all arcs (x, y) not in F there exists a path from x to y in F whose strength is more than $\mu(x, y)$ [16]. Note that here F is a tree which contains all nodes of G and hence is a spanning tree of G . A **maximum spanning tree** of a connected fuzzy graph $G : (V, \sigma, \mu)$ is a fuzzy spanning subgraph $T : (\sigma, \nu)$, such that T is a tree, and for which $\sum_{u \neq v} \nu(u, v)$ is maximum. A node which is not an endnode of T is called an internal node of T [11]. A node u is said to be **isolated** if $\mu(u, v) = 0$ for all $v \neq u$.

3. Hub Sets in Fuzzy Graphs

Fuzzy graph theory provides a robust framework for modeling systems characterized by uncertainty, partial connectivity, or imprecise relationships. Unlike classical graphs, where edges either exist or do not, fuzzy graphs incorporate edge membership functions that quantify the strength or reliability of connections. This makes them particularly suitable for applications such as transportation planning, communication networks, and decision-making under uncertainty.

In classical graph theory, the concept of a hub set ensures that all communication between non-hub vertices can occur via a small, designated subset of nodes, called hubs. This concept is relevant in scenarios such as the design of rapid-transit systems, where the goal is to place stations (hubs) in a minimal number of locations so that travel between any two non-station nodes is still feasible through these hubs. This paper extends the concept of hub set to fuzzy graphs.

Definition 3.1. Let $G : (V, \sigma, \mu)$ be a fuzzy graph. Given a threshold $\theta \in (0, 1]$, a path $P = (v_0, v_1, \dots, v_k)$ is said to be a fuzzy S -path if all intermediate nodes $v_1, v_2, \dots, v_{k-1} \in S$ and all arcs (v_i, v_{i+1}) satisfy $\mu(v_i, v_{i+1}) \geq \theta$. A subset $S \subseteq V$ is called a fuzzy hub set with respect to θ if, for every pair $x, y \in V - S$, there exists a fuzzy S -path connecting x and y . The fuzzy hub number, denoted by $h_f(G, \theta)$, is defined as the minimum cardinality of such a fuzzy hub set.

This parameter generalizes the classical hub number and provides a means to evaluate the minimal infrastructure required to maintain threshold-based connectivity in uncertain or imprecise networks. The objective of this work is to formally introduce the fuzzy hub set concept, analyze its structural properties, and explore computational methods for determining $h_f(G, \theta)$ under various fuzzy graph configurations. This contributes to the broader understanding of optimization in fuzzy environments and opens new avenues for research in fuzzy network design.

Example 3.2. Consider the fuzzy graph given in Fig. 1.

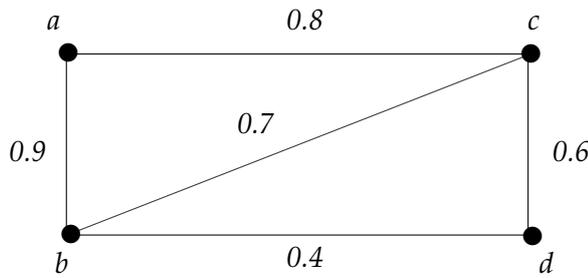


Fig. 1 Illustration of Hub set in a fuzzy graph

In this fuzzy graph, assume a threshold $\theta = 0.6$ for paths to be considered valid (i.e., only arcs with membership value ≥ 0.6 can be used in paths). We only keep arcs with $\mu \geq 0.6$. Such arcs are $(a, b) = 0.9$, $(a, c) = 0.8$, $(b, c) = 0.7$, $(c, d) = 0.6$. Arc $(b, d) = 0.4$ is removed. Now we find a fuzzy hub set $S \subseteq V$ such that for every pair $x, y \in V - S$, there is an S -path between them (i.e., a path whose intermediate nodes are in S and arcs are above threshold θ).

Candidate Hub Set: $S = \{c\}$.

Let's test whether this works:

Check all pairs in $V - S = \{a, b, d\}$

$a - b$: direct arc (a, b) with $0.9 \geq 0.6 \rightarrow$ valid \Rightarrow trivial S -path

$a - d$: path $a \rightarrow c \rightarrow d$

$(a, c) = 0.8, (c, d) = 0.6 \rightarrow$ both ≥ 0.6

Intermediate node = $c \in S \Rightarrow$ valid

$b - d$: path $b \rightarrow c \rightarrow d$

$(b, c) = 0.7, (c, d) = 0.6 \rightarrow$ both ≥ 0.6

Intermediate node = $c \in S \Rightarrow$ valid. All pairs in $V - S$ are connected via fuzzy S -paths $\Rightarrow S = \{c\}$ is a fuzzy hub set. Therefore fuzzy hub number $h_f(G, 0.6) = 1$.

Theorem 3.3. For any connected fuzzy graph $G : (V, \sigma, \mu)$ and threshold $\theta \in (0, 1]$, a fuzzy hub set exists.

Proof. Since $G : (V, \sigma, \mu)$ is connected, for every pair $x, y \in V$, there exists a path with positive membership values. Let $S = V$. Then, trivially, for all $x, y \in V - S = \phi$, the hub condition is satisfied. Hence, $S = V$ is always a fuzzy hub set. Therefore, at least one fuzzy hub set exists. □

Theorem 3.4. For any connected fuzzy graph $G : (V, \sigma, \mu)$ with n nodes, $h_f(G, \theta) \leq n - 2$.

Proof. Take any two nodes $x, y \in V$. To ensure they can communicate via hubs, we may choose $S = V - \{x, y\}$, which has size $n - 2$. This guarantees the condition of an S -path between all $x, y \in V - S$ (i.e., just one pair). So $h_f(G, \theta) \leq n - 2$. □

Theorem 3.5. Let $\theta_1 < \theta_2$. Then $h_f(G, \theta_1) \leq h_f(G, \theta_2)$.

Proof. Any arc $e \in E$ that satisfies $\mu(e) \geq \theta_2$ also satisfies $\mu(e) \geq \theta_1$ (since $\theta_1 < \theta_2$). Therefore, any fuzzy path valid under θ_2 is also valid under θ_1 . Let S_2 be a minimum fuzzy hub set under threshold θ_2 . Then S_2 also ensures the fuzzy hub condition under θ_1 though not necessarily optimally. Thus $h_f(G, \theta_1) \leq h_f(G, \theta_2)$. \square

Theorem 3.6. *Let $e = (u, v)$ be a fuzzy bridge in the threshold fuzzy graph G_f^θ (that is, the fuzzy graph containing only arcs with $\mu(u, v) \geq \theta$). Then any fuzzy hub set S must include either u or v .*

Proof. In G_f^θ , removing e disconnects or reduces the strength of connectedness between u and v of the graph. So any path between a node x in the component containing u , and a node y in the component containing v , must pass through u or v . If both $u, v \notin S$, then no fuzzy S -path exists between $x, y \in V - S$, violating the fuzzy hub set condition. Hence, at least one of u or v must be in S . \square

Theorem 3.7. *The Fuzzy Hub Set Problem is NP-complete. Specifically, given a fuzzy graph $G_f = (V, \mu_E)$, a threshold $\theta \in (0, 1]$, and an integer k , determining whether there exists a fuzzy hub set $S \subseteq V$ such that $|S| \leq k$ is NP-complete.*

Proof. We prove this in two parts.

(1) The problem is in NP: Given a candidate hub set $S \subseteq V$ with $|S| \leq k$, we can verify whether it satisfies the fuzzy hub set property in polynomial time. For each pair $x, y \in V \setminus S$, we can check for the existence of an S -path between them. An S -path requires that all intermediate vertices are in S and all edges used have $\mu_E \geq \theta$. This can be checked using a modified BFS or DFS algorithm restricted to paths that satisfy these two conditions, which runs in polynomial time. Hence, the problem is in NP.

(2) The problem is NP-hard: We reduce from the classical *Minimum Vertex Separator Problem*, which is known to be NP-complete. Given a graph $G = (V, E)$ and integer k , the vertex separator problem asks whether there exists a set $S \subseteq V$, $|S| \leq k$, such that all pairs of vertices in $V \setminus S$ are disconnected in G . We transform this into an instance of the fuzzy hub set problem as follows:

- Construct a fuzzy graph $G_f = (V, \mu_E)$ where $\mu_E(u, v) = 1$ if $(u, v) \in E$, and $\mu_E(u, v) = 0$ otherwise.
- Set the threshold $\theta = 1$.
- Ask whether there exists a fuzzy hub set S with $|S| \leq k$ in G_f .

In this construction, any edge with $\mu_E < \theta$ is effectively removed. Thus, the fuzzy hub set problem becomes: is there a set S of size at most k such that every pair of vertices in $V \setminus S$ can be connected by a path through vertices in S using only edges from the original graph G ? This is essentially the complement of the separator problem. Therefore, solving the fuzzy hub set problem under these conditions would also solve the minimum vertex separator problem, implying NP-hardness.

Conclusion: Since the fuzzy hub set problem is both in NP and NP-hard, it is NP-complete. \square

4. Fuzzy Hub Sets in Classes of Fuzzy Graphs

The notion of a fuzzy hub set naturally extends to various classes of fuzzy graphs. For a given threshold $\theta \in (0, 1]$, the fuzzy hub number $h_f(G_f, \theta)$ represents the minimum number of vertices required such that every pair of non-hub vertices is connected via an S -path (i.e., a path whose internal vertices belong to S and all edges satisfy $\mu_E \geq \theta$).

Complete Fuzzy Graphs: In a complete fuzzy graph, each pair of distinct vertices is connected with edge membership $\mu_E(u, v) > 0$. If $\mu_E(u, v) \geq \theta$ for all $u, v \in V$, then:

$$h_f(G_f, \theta) = 0.$$

No hubs are needed as all vertices are directly reachable.

Path Fuzzy Graphs P_n^f : In fuzzy path graphs, if some edges have $\mu_E < \theta$, connectivity may be broken. Therefore, hub vertices must bridge segments. In general, $h_f(P_n^f, \theta)$ increases with the number of such weak edges.

Cycle Fuzzy Graphs C_n^f : If all edge weights are at least θ , then every vertex pair remains connected:

$$h_f(C_n^f, \theta) = 0.$$

If one or more edges fall below threshold, at most 1 or 2 hub vertices may restore connectivity.

Star Fuzzy Graphs S_n^f : Let v_0 be the center vertex with peripheral vertices v_1, v_2, \dots, v_n . If $\mu_E(v_0, v_i) \geq \theta$ for all i , then:

$$h_f(S_n^f, \theta) = 1,$$

with v_0 as the unique hub vertex.

Complete Bipartite Fuzzy Graphs $K_{m,n}^f$: In bipartite fuzzy graphs, to ensure connectivity among vertices within the same partition, at least one hub is required from each part:

$$h_f(K_{m,n}^f, \theta) \geq 2.$$

Tree Fuzzy Graphs: In trees, connectivity is fragile to edge drops below threshold. Hence, articulation points (cut-vertices) whose incident edges satisfy $\mu_E \geq \theta$ become necessary:

$$h_f(T_f, \theta) = \text{number of such articulation points.}$$

Grid and Mesh Fuzzy Graphs: Due to their redundant path structures, grid fuzzy graphs are resilient to occasional weak edges. Typically:

$$h_f(G_{\text{grid}}^f, \theta) \ll |V|.$$

Summary Table

Fuzzy Graph Class	Typical $h_f(G_f, \theta)$	Conditions
Complete Fuzzy Graph	0	All edge memberships $\geq \theta$
Path Fuzzy Graph	0 to $n-2$	Based on number of weak links
Cycle Fuzzy Graph	0-2	Depending on $\min \mu_E$
Star Fuzzy Graph	1	Center must satisfy threshold
Bipartite Fuzzy Graph	≥ 2	One hub from each partition
Tree Fuzzy Graph	≥ 1	Depends on articulation points
Grid/Mesh Fuzzy Graph	Low	High redundancy maintains connectivity

Table 1: Fuzzy hub number across fuzzy graph classes under threshold θ

5. Conclusion

In this paper, it is introduced the concept of fuzzy hub sets in fuzzy graphs as a generalization of the classical hub set problem. By incorporating a membership threshold parameter $\theta \in (0, 1]$, it is defined the fuzzy hub number $h_f(G_f, \theta)$ to represent the smallest number of hub nodes required to maintain connectivity between all pairs of non-hub nodes via paths composed of sufficiently strong fuzzy edges. It is examined the behavior of fuzzy hub sets across several important classes of fuzzy graphs, including complete, path, cycle, star, bipartite, tree, and grid fuzzy graphs. The analysis demonstrated that the fuzzy hub number is closely tied to both the topological structure of the graph and the distribution of arc membership values. For instance, complete fuzzy graphs require no hubs when all arcs meet the threshold, while tree and bipartite fuzzy graphs often necessitate specific hub placements due to their hierarchical or partitioned nature. Furthermore, it is established that the NP-completeness of the fuzzy hub set problem highlights its computational intractability in the general case. This finding underscores the practical importance of developing efficient heuristics, approximation algorithms, or class-restricted solutions for real-world applications involving uncertain or imprecise data. Future research may focus on algorithmic strategies for fuzzy hub identification, the extension of this framework to dynamic and time-varying fuzzy graphs, and the incorporation of higher-order fuzzy systems such as type-2 fuzzy sets. These directions are expected to enhance the theoretical depth and practical utility of fuzzy hub sets in fields such as communication networks, transportation systems, and decision support under uncertainty.

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