International Journal of
Mathematics And its Applications
ISSN: 2347-1557

Int. J. Math. And Appl., **14(2)**(2026), 331–341
Available Online: http://ijmaa.in

On Fourth Leap Zagreb Index of Some Graph Operations

Hasan J H Abuowda[1,*], Veena Mathad[1]

[1] *Department of Studies in Mathematics, University of Mysore, Mysuru, Karnataka, India*

Abstract

In this research work, we use the new version of the Fourth leap Zagreb index of graph depending on the vertices for computing the values of this index of some graph operations containing the Cartesian product, Composition, Corona product, Disjunction.

Keywords: Second degree of vertex; Leap Zagreb indices; fourth leap Zagreb index; graph operations.

2020 Mathematics Subject Classification: 05C07, 05C12, 05C76.

1. Introduction

In this paper, we consider with simple graph, let G be connected, which has set $V(G)$ for its vertices and set $E(G)$ for its edges. The complement $\overline{\mathrm{G}}$ of G is a graph with vertex set $V(G)$ and two vertices of $\overline{\mathrm{G}}$ are adjacent if and only if they are not adjacent in G. The degree of a vertex v in G, denoted by $d(v)$, is the total number of edges incident with v [9]. According to [1], topological indices play a key role in QSPR and QSAR modeling in chemical graph theory. If two vertices u and v of G are adjacent, the edge connecting them will be indicated as uv. If $u,v \in V(G)$, the distance $d_G(u,v)$ between u and v is defined as the length of the shortest $u-v$ path in G, and a shortest u - v path is often called a geodesic. The diameter of a connected graph G is the length of any longest geodesic, represented by $diam(G)$. In a graph G, we can define the first and second degrees of a vertex v, respectively using the distance as follows: $d_1(v/G) = |\{u \in V(G) : d(u,v) = 1\}|$, $d_2(v/G) = |\{u \in V(G) : d(u,v) = 2\}|$. The maximum and minimum degrees of G are denoted by $\Delta = \Delta(G)$ an $\delta = \delta(G)$, respectively. The Zagreb indices $M_1(G)$ and $M_2(G)$ are the most widely investigated topological indices and they are related to the most common molecular descriptors. These have been defined more than forty years ago [7,8], which are defined as follows:

$$M_1(G) = \sum_{v\in V(G)} d_1^2(v/G)$$

---

[*] Corresponding author (hasan1992jamal@gmail.com)

$$M_2(G) = \sum_{uv \in E(G)} d_1(u/G) d_1(v/G).$$

For more details on the Zagreb indices, we refer to [3–5,10,14,15]. The Forgotten index [7] is given by $F(G) = \sum_{v \in V(G)} d_1^3(v/G) = \sum_{uv \in E(G)} [d_1^2(u/G) + d_1^2(v/G)]$. For more information on Zagreb and beyond topological indices, readers can refer to the survey [6]. In 2017, Naji et al. [13] have introduced new distance-degree-based topological indices depending on the second degrees of vertices, called leap Zagreb indices of G and are defined as:

$$LM_1(G) = \sum_{v \in V(G)} d_2^2(v/G),$$
$$LM_2(G) = \sum_{uv \in E(G)} d_2(u/G) d_2(v/G),$$
$$LM_3(G) = \sum_{v \in V(G)} d_1(v/G) d_2(v/G).$$

In 2022, Alsinai et al. [2] have introduced a new version of leap Zagreb index called fourth leap Zagreb index of G defined by:

$$LM_4(G) = \sum_{uv \in E(G)} \Big[d_1(u/G) d_2(u/G) + d_1(v/G) d_2(v/G) \Big].$$

Also, Alsinai et al. [2] proved that for a graph G with diameter at most two,

$$LM_4(G) = \sum_{v \in V(G)} d_1^2(v/G) d_2(v/G).$$

The precise formulas for the fourth leap Zagreb index of few graph operations, including cartesian product, composition, disjunction, and corona product of graphs, are presented in this study by using the new version fourth leap Zagreb index depending on the vertices.

Lemma 1.1 ([13])**.** *Let G be a connected graph with n vertices and m edges. Then for every vertex $v \in V(G)$,*

$$d_2(v/G) \leq \Big(\sum_{u \in N(v)} d_1(u/G) \Big) - d_1(v/G).$$

Equality holds if and only if G is a $CF-$graph, where $CF-$graph means $\{C_3, C_4\}-$ free graph.

Lemma 1.2 ([13])**.** *Let G be a connected graph with n vertices. Then for every vertex $v \in V(G)$,*

$$d_2(v/G) \leq d_1(v/\overline{G}) = n - 1 - d_1(v/G).$$

Equality holds if and only if G has diameter at most two.

2. Cartesian Product

Definition 2.1 ([11]). *For given disjoint graphs G_1 and G_2 their caetesion product, denoted by $G_1 \times G_2$ is defined as the graph on the vertex set $V(G_1) \times V(G_2)$,and vertices u=(u_1,u_2) and v=(v_1,v_2) of $G_1 \times G_2$ are connected by an edge if and only if either $u_1 = v_1$ and $u_2v_2 \in E(G_2)$ or u_2= v_2 and $u_1v_1 \in E(G_1)$. The distance between any two vertices u=(u_1,u_2) and v=$(v_1,v_2) \in G_1 \times G_2$ is given by $d_{G_1\times G_2}(u,v)$ = $d_G(u_1,v_1)$ + $d_{G_2}(u_2,v_2)$ and $|V(G_1 \times G_2)|$ =$|V(G_1)||V(G_2)|$.*

Lemma 2.2 ([12]). *Let G_1 and G_2 be two nontrivial disjoint connected graphs with orders n_1,n_2,respectively. Then for any vertex $(u,v) \in V(G_1 \times G_2)$,*

(1) $d_1((u,v)/G_1 \times G_2) = d_1(u/G_1) + d_1(v/G_2)$;

(2) $d_2((u,v)/G_1 \times G_2) = d_2(u/G_1) + d_1(u/G_1)\, d_1(v/G_2) + d_2(v/G_2)$.

Theorem 2.3. *Let G_1 and G_2 be two nontrivial disjoint connected graphs with n_1, n_2 vertices and m_1, m_2 edges, respectively. Then*

$$\begin{aligned} LM_4(G_1 \times G_2) = {} & n_2LM_4(G_1) + 2m_2F(G_1) + M_1(G_1) \sum_{v\in V(G_2)} d_2(v/G_2) \\ & + 4m_2LM_3(G_1) + 2M_1(G_1)M_1(G_2) + 4m_1LM_3(G_2) \\ & + M_1(G_2) \sum_{u\in V(G_1)} d_2(u/G_1) + 2m_1F(G_2) + n_1LM_4(G_2) \end{aligned}$$

Proof. Let G_1 and G_2 be two nontrivial connected graphs with n_1, n_2 vertices and m_1, m_2 edges. respectively Then by definition of fourth leap Zagrab index and Lemma 2.2, we have

$$\begin{aligned} LM_4(G_1 \times G_2) &= \sum_{(u,v)\in V(G_1\times G_2)} (d_1((u,v)/G_1 \times G_2))^2 d_2((u,v)/G_1 \times G_2) \\ &= \sum_{u\in V(G_1)} \sum_{v\in V(G_2)} \Bigg[\Big(d_1(u/G_1) + d_1(v/G_2)\Big)^2 \Big(d_2(u/G_1) \\ & + d_1(u/G_1)d_1(v/G_2) + d_2(v/G_2)\Big)\Bigg] \\ &= \sum_{u\in V(G_1)} \sum_{v\in V(G_2)} \Bigg[\Big(d_1^2(u/G_1) + 2d_1(u/G_1)d_1(v/G_2) + d_1^2(v/G_2)\Big) \\ & \Big(d_2(u/G_1) + d_1(u/G_1)d_1(v/G_2) + d_2(v/G_2)\Big)\Bigg] \\ &= \sum_{u\in V(G_1)} \sum_{v\in V(G_2)} \Big[d_1^2(u/G_1)d_2(u/G_1) + d_1^2(u/G_1)d_1(u/G_1)d_1(v/G_2) + d_1^2(u/G_1)d_2(v/G_2)\Big] \\ & + \sum_{u\in V(G_1)} \sum_{v\in V(G_2)} \Big[2d_1(u/G_1)d_1(v/G_2)d_2(u/G_1) + 2d_1(u/G_1)d_1(v/G_2) \\ & d_1(u/G_1)d_1(v/G_2) + 2d_1(u/G_1)d_1(v/G_2)d_2(v/G_2)\Big] \end{aligned}$$

$$
\begin{aligned}
&+\sum_{u\in V(G_1)}\sum_{v\in V(G_2)}\Big[d_2(u/G_1)d_1^2(v/G_2)+d_1(u/G_1)d_1(v/G_2)d_1^2(v/G_2)+d_1^2(v/G_2)d_2(v/G_2)\Big]\\
&=\sum_{u\in V(G_1)}\Big[n_2d_1^2(u/G_1)d_2(u/G_1)+d_1^3(u/G_1)(2m_2)+d_1^2(u/G_1)\sum_{v\in V(G_2)}d_2(v/G_2)\Big]\\
&+\sum_{u\in V(G_1)}\Big[4m_2d_1(u/G_1)d_2(u/G_1)+2d_1^2(u/G_1)M_1(G_2)+2d_1(u/G_1)LM_3(G_2)\\
&+\sum_{u\in V(G_1)}\Big[d_2(u/G_1)M_1(G_2)+d_1(u/G_1)F(G_2)+LM_4(G_2)\Big]\\
&=n_2LM_4(G_1)+2m_2F(G_1)+M_1(G_1)\sum_{v\in V(G_2)}d_2(v/G_2)\\
&+4m_2LM_3(G_1)+2M_1(G_1)M_1(G_2)+4m_1LM_3(G_2)\\
&+M_1(G_2)\sum_{u\in V(G_1)}d_2(u/G_1)+2m_1F(G_2)+n_1LM_4(G_2).
\end{aligned}
$$

□

3. Composition

Definition 3.1 ([11])**.** *The composition $G_1[G_2]$ of graphs G_1 and G_2 with disjoint vertex sets and edge sets is a graph on vertex set $V(G_1)\times V(G_2)$ in which (u_1,v_1) is adjacent with (u_2,v_2) whenever $[u_1$ is adjacent with $u_2]$ or $[u_1=u_2$ and v_1 is adjacent with $v_2]$.*

Lemma 3.2 ([12])**.** *Let G_1 and G_2 be two disjoint graphs with n_1 and n_2 vertices and with m_1 and m_2 edges, respectively. Then*

(1) $d_1((u,v)/G_1[G_2]) = n_2d_1(u/G_1) + d_1(v/G_2)$;

(2) $d_2((u,v)/G_1[G_2])) = n_2d_2(u/G_1) + d_1(v/\overline{G_2})$.

Theorem 3.3. *Let G_1 and G_2 be disjoint graphs with n_1 and n_2 vertices and edges m_1 and m_2 edges, respectively. Then*

$$
\begin{aligned}
LM_4(G_1[G_2]) &= n_2^4LM_4(G_1)+M_1(G_1)[n_2^4-n_2^3-2n_2^2m_2]+4n_2^2m_2LM_3(G_1)\\
&\quad+8n_2m_2m_1[n_2-1]+M_1(G_2)[-4n_2m_1+n_2\sum_{u\in V(G_1)}d_2(u/G_1)+n_2n_1-n_1F(G_2)].
\end{aligned}
$$

Proof. Let G_1 and G_2 be disjoint graphs with n_1 and n_2 vertices and m_1 and m_2 edges, respectively. Then by Lemma 3.2 we have

$$
\begin{aligned}
LM_4(G_1[G_2]) &= \sum_{(u,v)\in V(G_1\times G_2)} d_1^2((u,v)/G_1[G_2])\ d_2((u,v)/G_1[G_2]))\\
&=\sum_{u\in V(G_1)}\sum_{v\in V(G_2)}\Big[\Big(n_2d_1(u/G_1)+d_1(v/G_2)\Big)^2\Big(n_2d_2(u/G_1)+d_1(v/\overline{G_2})\Big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{u\in V(G_1)} \sum_{v\in V(G_2)} \Bigg[\Bigg(n_2^2 d_1^2(u/G_1) + 2n_2 d_1(u/G_1) d_1(v/G_2) + d_1^2(v/G_2)\Bigg) \\
&\Bigg(n_2 d_2(u/G_1) + d_1(v/\overline{G_2})\Bigg)\Bigg] \\
&= \sum_{u\in V(G_1)} \sum_{v\in V(G_2)} \Bigg[n_2^2 d_1^2(u/G_1)\Bigg(n_2 d_2(u/G_1) + d_1(v/\overline{G_2})\Bigg) \\
&+ 2n_2 d_1(u/G_1) d_1(v/G_2)\Bigg(n_2 d_2(u/G_1) + d_1(v/\overline{G_2})\Bigg) \\
&+ d_1^2(v/G_2)\Bigg(n_2 d_2(u/G_1) + d_1(v/\overline{G_2})\Bigg)\Bigg] \\
&= \sum_{u\in V(G_1)} \sum_{v\in V(G_2)} \Bigg[n_2^3 d_1^2(u/G_1) d_2(u/G_1) + n_2^2 d_1^2(u/G_1) d_1(v/\overline{G_2}) \\
&+ 2n_2^2 d_1(u/G_1) d_1(v/G_2) d_2(u/G_1) + 2n_2 d_1(u/G_1) d_1(v/G_2) d_1(v/\overline{G_2}) \\
&+ n_2 d_1^2(v/G_2) d_2(u/G_1) + d_1^2(v/G_2) d_1(v/\overline{G_2})\Bigg] \\
&= \sum_{u\in V(G_1)} \sum_{v\in V(G_2)} \Bigg[n_2^3 d_1^2(u/G_1) d_2(u/G_1) + n_2^2 d_1^2(u/G_1)\Bigg(n_2 - 1 - d_1(v/G_2)\Bigg) \\
&+ 2n_2^2 d_1(u/G_1) d_1(v/G_2) d_2(u/G_1) + 2n_2 d_1(u/G_1) d_1(v/G_2)\Bigg(n_2 - 1 - d_1(v/G_2)\Bigg) \\
&+ n_2 d_1^2(v/G_2) d_2(u/G_1) + d_1^2(v/G_2)\Bigg(n_2 - 1 - d_1(v/G_2)\Bigg)\Bigg] \\
&= \sum_{u\in V(G_1)} \sum_{v\in V(G_2)} \Bigg[n_2^3 d_1^2(u/G_1) d_2(u/G_1) + n_2^3 d_1^2(u/G_1) - n_2^2 d_1^2(u/G_1) \\
&- n_2^2 d_1^2(u/G_1) d_1(v/G_2) + 2n_2^2 d_1(u/G_1) d_1(v/G_2) d_2(u/G_1) \\
&+ 2n_2^2 d_1(u/G_1) d_1(v/G_2) - 2n_2 d_1(u/G_1) d_1(v/G_2) \\
&- 2n_2 d_1(u/G_1) d_1^2(v/G_2) + n_2 d_1^2(v/G_2) d_2(u/G_1) + n_2 d_1^2(v/G_2) - d_1^2(v/G_2) - d_1^3(v/G_2)\Bigg] \\
&= \sum_{u\in V(G_1)} \Bigg[n_2^4 d_1^2(u/G_1) d_2(u/G_1) + n_2^4 d_1^2(u/G_1) - n_2^3 d_1^2(u/G_1) \\
&- 2n_2^2 m_2 d_1^2(u/G_1) + 4n_2^2 m_2 d_1(u/G_1) d_2(u/G_1) + 4n_2^2 m_2 d_1(u/G_1) \\
&- 4n_2 m_2 d_1(u/G_1) - 2n_2 d_1(u/G_1) M_1(G_2) + n_2 M_1(G_2) d_2(u/G_1) \\
&+ n_2 M_1(G_2) - M_1(G_2) F(G_2)\Bigg] \\
&= n_2^4 LM_4(G_1) + n_2^4 M_1(G_1) - n_2^3 M_1(G_1) - 2n_2^2 m_2 M_1(G_1) \\
&+ 4n_2^2 m_2 LM_3(G_1) + 8n_2^2 m_2 m_1 - 8n_2 m_2 m_1 - 4n_2 m_1 M_1(G_2) \\
&+ n_2 M_1(G_2) \sum_{u\in V(G_1)} d_2(u/G_1) + n_2 n_1 M_1(G_2) - n_1 M_1(G_2) F(G_2). \\
&= n_2^4 LM_4(G_1) + M_1(G_1)[n_2^4 - n_2^3 - 2n_2^2 m_2] + 4n_2^2 m_2 LM_3(G_1) \\
&+ 8n_2 m_2 m_1 [n_2 - 1] + M_1(G_2)[-4n_2 m_1 + n_2 \sum_{u\in V(G_1)} d_2(u/G_1) + n_2 n_1 - n_1 F(G_2)].
\end{aligned}
$$

□

4. Corona Product

Definition 4.1 ([11]). *Let G_1 and G_2 be two disjoint graphs on disjoint vertex sets with n_1 and n_2 vertices, respectively. The corona $G_1 \circ G_2$ of G_1 and G_2 is defined as the graph obtained by taking one copy of G_1 and n_1 copies of G_2, and then joining the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2, We denote i^{th} copy of G_2 as G_2^j.*

Lemma 4.2 ([12]). *Let G_1 and G_2 be two disjoint graphs with n_1 and n_2 vertices and m_1 and m_2 edges, respectively . Assume that $1 \leq j \leq n$ then,*

$$d_1(v/(G_1 \circ G_2)) = \begin{cases} d_1(v/G_1) + n_2, & \text{if } v \in V(G_1)\,; \\ d_1(v/G_2) + 1, & \text{if } v \in V(G_1)\,. \end{cases}$$

$$d_2(v/(G_1 \circ G_2)) = \begin{cases} d_2(v/G_1) + n_2 d_1(v/G_1), & \text{if } v \in V(G_1)\,; \\ d_1(v_j/G_1) + n_2 - 1 + d_1(v/G_2^j), & \text{if } v \in V(G_2^j)\,. \end{cases}$$

Theorem 4.3. *Let G_1 and G_2 be two disjoint graphs with n_1 and n_2 vertices and m_1 and m_2 edges, respectively. Then*

$$\begin{aligned} LM_4(G_1 \circ G_2) = {} & LM_4(G_1) + n_2 F(G_1) + n_1 F(G_2) + 2n_2 LM_3(G_1) + 2n_2^2 M_1(G_1) \\ & + n_2 \sum_{v \in V(G_1)} d_2(v/G_1) + M_1(G_2)[2m_1 + n_1 n_2 - n_1 + 2n_1] \\ & + n_1 n_2 [n_2 - 1] + 2n_2 m_1 [1 + n^2] + 4m_2 n_1 [n_2 - 1] + 2m_2 [4m_1 + n_1]. \end{aligned}$$

Proof. Let G_1 and G_2 be two disjoint graphs with n_1 and n_2 vertices and m_1 and m_2 edges, respectively. Assuming that $1 \leq j \leq n_1$, by Lemma 4.2, we have,

$$\begin{aligned} LM_4(G_1 \circ G_2) &= \sum_{v \in V(G_1 \circ G_2)} d_1^2(v/G_1 \circ G_2) d_2(v/G_1 \circ G_2) \\ &= \sum_{v \in V(G_1)} d_1^2(v/G_1 \circ G_2) d_2(v/G_1 \circ G_2) \\ &+ \sum_{j=1}^{n_1} \sum_{v \in V(G_2^j)} d_1^2(v/G_1 \circ G_2) d_2(v/G_1 \circ G_2) \\ &= \sum_{v \in V(G_1)} \left[\Big(d_1(v/G_1) + n_2 \Big)^2 \Big(d_2(v/G_1) + n_2 (d_1(v/G_1) \Big) \right] \\ &+ \sum_{j=1}^{n_1} \sum_{v \in V(G_2^j)} \left[\Big(d_1(v/G_2) + 1 \Big)^2 \Big(d_1(v_j/G_1) + n_2 - 1 + d_1(v/G_2^j) \Big) \right] \\ &= \sum_{v \in V(G_1)} \Big(d_1^2(v/G_1) + 2n_2 d_1(v/G_1) + n_2^2 \Big) \Big(d_2(v/G_1) + n_2 d_1(v/G_1) \Big) \\ &+ \sum_{j=1}^{n_1} \sum_{v \in V(G_2^j)} \Big(d_1^2(v/G_2) + 2d_1(v/G_2) + 1 \Big) \Big(d_1(v_j/G_1) + n_2 - 1 + d_1(v/G_2^j) \Big) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{v\in V(G_1)} \Bigg[\Bigg(d_1^2(v/G_1)d_2(v/G_1) + n_2 d_1^3(v/G_1) \\
&+ 2n_2 d_1(v/G_1)d_2(v/G_1) + 2n_2^2 d_1^2(v/G_1) + n_2^2 d_2(v/G_1) + n_2^3 d_1(v/G_1)\Bigg) \\
&+ \sum_{j=1}^{n_1} \sum_{v\in V(G_2^j)} \Bigg[d_1^2(v/G_2)d_1(v_j/G_1) + n_2 d_1^2(v/G_2) - d_1^2(v/G_2) \\
&+ d_1^2(v/G_2)d_1(v/G_2^j) + 2d_1(v/G_2)d_1(v_j/G_1) + 2n_2 d_1(v/G_2) \\
&- 2d_1(v/G_2) + 2d_1(v/G_2)d_1(v/G_2^j) + d_1(v_j/G_1) + n_2 - 1 + d_1(v/G_2^j)\Bigg] \\
&= LM_4(G_1) + n_2 F(G_1) + 2n_2 LM_3(G_1) + 2n_2^2 M_1(G_1) \\
&+ n_2^2 \sum_{v\in V(G_1)} d_2(v/G_1) + 2n_2^3 m_1 \\
&+ \sum_{j=1}^{n_1} \Bigg[d_1(v_j/G_1)M_1(G_2) + n_2 M_1(G_2) - M_1(G_2) + F(G_2) + 4d_1(v_j/G_1)m_2 \\
&+ 4n_2 m_2 - 4m_2 + 2M_1(G_2) + n_2 d_1(v_j/G_1) + n_2^2 - n_2 + 2m_2 \Bigg] \\
&= \Bigg[LM_4(G_1) + n_2 F(G_1) + 2n_2 LM_3(G_1) + 2n_2^2 M_1(G_1) \\
&+ n_2^2 \sum_{v\in V(G_1)} d_2(v/G_1) + 2n_2^3 m_1 + 2m_1 M_1(G_2) + n_2 n_1 M_1(G_2) \\
&- n_1 M_1(G_2) + n_1 F(G_2) + 8m_2 m_1 + 4n_2 n_1 m_2 - 4m_2 n_1 \\
&+ 2n_1 M_1(G_2) + 2n_2 m_1 + n_2^2 n_1 - n_1 n_2 + 2n_1 m_2 \Bigg] \\
&= LM_4(G_1) + n_2 F(G_1) + 2n_2 LM_3(G_1) + 2n_2^2 M_1(G_1) + n_2^2 \sum_{v\in V(G_1)} d_2(v/G_1) \\
&+ 2n_2^3 m_1 + 2m_1 M_1(G_2) + n_1 n_2 M_1(G_2) - n_1 M_1(G_2) + n_1 F(G_2) \\
&+ 8m_1 m_2 + 4n_2 n_1 m_2 - 4m_2 n_1 + 2n_1 M_1(G_2) \\
&+ 2n_2 m_1 + n_2^2 n_1 - n_1 n_2 + 2n_1 m_2 \\
&= LM_4(G_1) + n_2 F(G_1) + n_1 F(G_2) + 2n_2 LM_3(G_1) + 2n_2^2 M_1(G_1) \\
&+ n_2^2 \sum_{v\in V(G_1)} d_2(v/G_1) + M_1(G_2)[2m_1 + n_1 n_2 - n_1 + 2n_1] \\
&+ n_1 n_2 [n_2 - 1] + 2n_2 m_1 [1 + n_2^2] + 4m_2 n_1 [n_2 - 1] + 2m_2 [4m_1 + n_1].
\end{aligned}
$$

□

5. Disjunction

Definition 5.1 ([11]). *The disjunction $G_1 \vee G_2$ of two graphs G_1 and G_2 with disjoint vertex sets and edge sets is the graph with vertex set $V(G_1) \times V(G_2)$ in which (u_1, v_1) is adjacent with (u_2, v_2) whenever u_1 is adjacent with u_2 in G_2.*

Lemma 5.2 ([12]). *Let G_1 and G_2 be two disjoint graphs with n_1 and n_2 vertices and m_1 and m_2 edges, respectively. Then*

(1) $d_1((u,v)/G_1 \vee G_2) = n_2 d_1(u/G_1) + n_1 d_1(v/G_2) - d_1(u/G_1)d_1(v/G_2)$.

(2) $d_2((u,v)/G_1 \vee G_2) = (n_1n_2 - 1) - n_2 d_1(u/G_1) - n_1 d_1(v/G_2) + d_1(u/G_1)d_1(v/G_2)$.

Theorem 5.3. *Let G_1 and G_2 be two disjoint graphs with n_1 and n_2 vertices and m_1 and m_2 edges, respectively. Then*

$$\begin{aligned}
LM_4(G_1 \vee G_2) = {} & M_1(G_1)\Big[n_1n_2^4 - n_2^3 - 10n_1n_2^2m_2 + 7n_1n_2M_1(G_2) - 3n_1F(G_2) - M_1(G_2) + 4n_2m_2\Big] \\
& + F(G_1)\Big[- n_2^4 + 6n_2^2m_2 - 3n_2M_1(G_2) + F(G_2)\Big] \\
& + M_1(G_2)\Big[- 10n_2n_1^2m_1 + n_1^4n_2 - n_1^3 + 4n_1m_1\Big] \\
& + n_1m_1\Big[8n_1n_2^2m_2 - 8n_2m_2 + 6n_1F(G_2)\Big] - n_1^4F(G_2).
\end{aligned}$$

Proof. Let G_1 and G_2 be two disjoint graphs with n_1 and n_2 vertices and m_1 and m_2 edges, Then by Lemma 5.2, we obtain

$$\begin{aligned}
LM_4(G_1 \vee G_2) = {} & \sum_{(u,v)\in V(G_1\vee G_2)} \Big(d_1((u,v)/G_1 \vee G_2)\Big)^2 \Big(d_2((u,v)/G_1 \vee G_2)\Big) \\
= {} & \sum_{v\in V(G_2)} \sum_{u\in V(G_1)} \bigg[\Big(n_2d_1(u/G_1) + n_1d_1(v/G_2) - d_1(u/G_1)d_1(v/G_2)\Big)^2 \\
& \Big((n_1n_2 - 1) - n_2d_1(u/G_1) - n_1d_1(v/G_2) + d_1(u/G_1)d_1(v/G_2)\Big)\bigg] \\
= {} & \sum_{v\in V(G_2)} \sum_{u\in V(G_1)} \bigg[\Big((n_2d_1(u/G_1) + n_1d_1(v/G_2)\Big)^2 \\
& - 2\Big(n_2d_1(u/G_1) + n_1d_1(v/G_2)\Big)\Big(d_1(u/G_1)d_1(v/G_2)\Big) + d_1^2(u/G_1)d_1^2(v/G_2)\bigg] \\
& \Big((n_1n_2 - 1) - n_2d_1(u/G_1) - n_1d_1(v/G_2) + d_1(u/G_1)d_1(v/G_2)\Big) \\
= {} & \sum_{v\in V(G_2)} \sum_{u\in V(G_1)} \bigg[\Big(n_2^2d_1^2(u/G_1) + 2n_2n_1d_1(u/G_1)d_1(v/G_2) \\
& + n_1^2d_1^2(v/G_2) - 2n_2d_1(u/G_1)d_1(u/G_1)d_1(v/G_2) \\
& - 2n_1d_1(v/G_2)d_1(u/G_1)d_1(v/G_2) + d_1^2(u/G_1)d_1^2(v/G_2)\Big)\bigg] \\
& \Big((n_1n_2 - 1) - n_2d_1(u/G_1) - n_1d_1(v/G_2) + d_1(u/G_1)d_1(v/G_2)\Big) \\
= {} & \sum_{v\in V(G_2)} \sum_{u\in V(G_1)} \bigg[n_1n_2^3d_1^2(u/G_1) - n_2^2d_1^2(u/G_1) - n_2^3d_1^2(u/G_1) \\
& d_1(u/G_1) - n_1n_2^2d_1^2(u/G_1)d_1(v/G_2) + n_2^2d_1^2(u/G_1)d_1(u/G_1)d_1(v/G_2) \\
& + 2n_1^2n_2^2d_1(u/G_1)d_1(v/G_2) - 2n_2n_1d_1(u/G_1)d_1(v/G_2)
\end{aligned}$$

$$
\begin{aligned}
&- 2n_2^2n_1d_1(u/G_1)d_1(v/G_2)d_1(u/G_1) - 2n_2n_1^2d_1(u/G_1)d_1(v/G_2)d_1(v/G_2)\\
&+ 2n_1n_2d_1(u/G_1)d_1(v/G_2)d_1(u/G_1)d_1(v/G_2)\\
&+ n_1^3n_2d_1^2(v/G_2) - n_1^2d_1^2(v/G_2) - n_1^2n_2d_1^2(v/G_2)d_1(u/G_1)\\
&- n_1^2n_1d_1^2(v/G_2)d_1(v/G_2) + n_1^2d_1^2(v/G_2)d_1(u/G_1)d_1(v/G_2)\\
&- 2n_1n_2^2d_1(u/G_1)d_1(u/G_1)d_1(v/G_2) + 2n_2d_1(u/G_1)d_1(u/G_1)d_1(v/G_2)\\
&+ 2n_2^2d_1(u/G_1)d_1(u/G_1)d_1(v/G_2)d_1(u/G_1)\\
&+ 2n_1n_2d_1(u/G_1)d_1(u/G_1)d_1(v/G_2)d_1(v/G_2)\\
&- 2n_2d_1(u/G_1)d_1(u/G_1)d_1(v/G_2)d_1(u/G_1)d_1(v/G_2)\\
&- 2n_1^2n_2d_1(v/G_2)d_1(u/G_1)d_1(v/G_2) + 2n_1d_1(v/G_2)d_1(u/G_1)d_1(v/G_2)\\
&+ 2n_1n_2d_1(v/G_2)d_1(u/G_1)d_1(v/G_2)d_1(u/G_1)\\
&+ 2n_1^2d_1(v/G_2)d_1(u/G_1)d_1(v/G_2)d_1(v/G_2)\\
&- 2n_1d_1(v/G_2)d_1(u/G_1)d_1(v/G_2)d_1(u/G_1)d_1(v/G_2)\\
&+ n_1n_2d_1^2(u/G_1)d_1^2(v/G_2) - d_1^2(u/G_1)d_1^2(v/G_2)\\
&- n_2d_1^2(u/G_1)d_1^2(v/G_2)d_1(u/G_1) - n_1d_1^2(u/G_1)d_1^2(v/G_2)d_1(v/G_2)\\
&+ d_1^2(u/G_1)d_1^2(v/G_2)d_1(u/G_1)d_1(v/G_2)\bigg]\\
&= \sum_{v\in V(G_2)}\bigg[n_1n_2^3M_1(G_1) - n_2^2M_1(G_1) - n_2^3F(G_1) - n_1n_2^2d_1(v/G_2)M_1(G_1)\\
&+ n_2^2d_1(v/G_2)F(G_1)\\
&+ 4n_1^2n_2^2m_1d_1(v/G_2) - 4n_1n_2m_1d_1(v/G_2) - 2n_2^2n_1M_1(G_1)d_1(v/G_2)\\
&- 4n_2n_1^2m_1d_1(v/G_2)d_1(v/G_2) + 2n_1n_2M_1(G_1)d_1(v/G_2)d_1(v/G_2)\\
&+ n_1^4n_2d_1^2(v/G_2) - n_1^3d_1^2(v/G_2) - 2n_1^2n_2m_1d_1^2(v/G_2)\\
&- n_1^4d_1^2(v/G_2)d_1(v/G_2) + 2n_1^2m_1d_1^2(v/G_2)d_1(v/G_2)\\
&- 2n_1n_2^2M_1(G_1)d_1(v/G_2) + 2n_2M_1(G_1)d_1(v/G_2) + 2n_2^2F(G_1)d_1(v/G_2)\\
&+ 2n_1n_2M_1(G_1)d_1(v/G_2)d_1(v/G_2) - 2n_2F(G_1)d_1(v/G_2)d_1(v/G_2)\\
&- 4n_1^2n_2m_1d_1(v/G_2)d_1(v/G_2) + 4n_1m_1d_1(v/G_2)d_1(v/G_2)\\
&+ 2n_1n_2M_1(G_1)d_1(v/G_2)d_1(v/G_2) + 4n_1^2m_1d_1(v/G_2)d_1(v/G_2)d_1(v/G_2)\\
&- 2n_1M_1(G_1)d_1(v/G_2)d_1(v/G_2)d_1(v/G_2)\\
&+ n_1n_2M_1(G_1)d_1^2(v/G_2) - M_1(G_1)d_1^2(v/G_2)\\
&- n_2F(G_1)d_1^2(v/G_2) - n_1M_1(G_1)d_1^2(v/G_2))d_1(v/G_2) + F(G_1)d_1^2(v/G_2))d_1(v/G_2)\bigg]\\
&= \bigg[n_1n_2^4M_1(G_1) - n_2^3M_1(G_1) - n_2^4F(G_1) - 2n_1n_2^2m_2M_1(G_1)\\
&+ 2n_2^2m_2F(G_1) + 8n_1^2n_2^2m_1m_2 - 8n_1n_2m_1m_2 - 4n_2^2n_1M_1(G_1)m_2 - 4n_2n_1^2m_1M_1(G_2)
\end{aligned}
$$

$$\begin{aligned}
&+2n_1n_2M_1(G_1)M_1(G_2)+n_1^4n_2M_1(G_2)-n_1^3M_1(G_2)-2n_1^2n_2m_1M_1(G_2)-n_1^4F(G_2)\\
&+2n_1^2m_1F(G_2)-4n_1n_2^2m_2M_1(G_1)+4n_2m_2M_1(G_1)+4n_2^2m_2F(G_1)\\
&+2n_1n_2M_1(G_1)M_1(G_2)-2n_2F(G_1)M_1(G_2)-4n_1^2n_2m_1M_1(G_2)+4n_1m_1M_1(G_2)\\
&+2n_1n_2M_1(G_1)M_1(G_2)+4n_1^2m_1F(G_2)-2n_1M_1(G_1)F(G_2)+n_1n_2M_1(G_1)M_1(G_2)\\
&-M_1(G_1)M_1(G_2)-n_2F(G_1)M_1(G_2)-n_1M_1(G_1)F(G_2)+F(G_1)F(G_2)\Big]\\
&=M_1(G_1)\Big[n_1n_2^4-n_2^3-2n_1n_2^2m_2-4n_1n_2^2m_2+2n_1n_2M_1(G_2)\\
&-4n_1n_2^2m_2+4n_2m_2+2n_1n_2M_1(G_2)+2n_1n_2M_1(G_2)\\
&-2n_1F(G_2)+n_1n_2M_1(G_2)-M_1(G_2)-n_1F(G_2)\Big]\\
&+F(G_1)\Big[-n_2^4+2n_2^2m_2+4n_2^2m_2-2n_2M_1(G_2)-n_2M_1(G_2)+F(G_2)\Big]\\
&+M_1(G_2)\Big[-4n_2n_1^2m_1+n_1^4n_2-n_1^3-2n_1^2n_2m_1-4n_1^2n_2m_1+4n_1m_1\Big]\\
&+n_1m_1\Big[8n_1n_2^2m_2-8n_2m_2+2n_1F(G_2)+4n_1F(G_2)\Big]-n_1^4F(G_2)\\
&=M_1(G_1)\Big[n_1n_2^4-n_2^3-10n_1n_2^2m_2+7n_1n_2M_1(G_2)\\
&-3n_1F(G_2)-M_1(G_2)+4n_2m_2\Big]\\
&+F(G_1)\Big[-n_2^4+6n_2^2m_2-3n_2M_1(G_2)+F(G_2)\Big]\\
&+M_1(G_2)\Big[-10n_2n_1^2m_1+n_1^4n_2-n_1^3+4n_1m_1\Big]\\
&+n_1m_1\Big[8n_1n_2^2m_2-8n_2m_2+6n_1F(G_2)\Big]-n_1^4F(G_2)
\end{aligned}$$

□

References

[1] A. R. Ashrafi, M. Saheli and M. Ghorbani, *The eccentric connectivity index of nanotubes and nanotori*, Journal of Computational and Applied Mathematics, 235(2011), 4561-4566.

[2] A. Alsinai, A. Saleh, H. Anmed, L. N. Mishra and N. D. Soner, *On fourth leap Zagreb index of graphs*, Discrete Mathematics, Algorithms and Applications, 15(2)(2022), 19 pages.

[3] B. Borovičanin, K. C. Das, B. Furtula and I. Gutman, *Bounds for Zagreb indices,* MATCH Communications in Mathematical and in Computer Chemistry, 78(2017), 17-100.

[4] K. C. Das and I. Gutman, *Some properties of the second Zagreb index*, MATCH Communications in Mathematical and in Computer Chemistry, 52(2004), 103-112.

[5] I. Gutman and K. C. Das, *The first Zagreb index 30 years after*, MATCH Communications in Mathematical and in Computer Chemistry, 50(2004), 83-92.

[6] I. Gutman, E. Milovanovic and I. Milovanovic, *Beyond the Zagreb indices*, AKCE International Journal of Graphs and Combinatorics, 17(1)(2020), 74-85.

[7] I. Gutman, B. Rušcic, N. Trinajstic and C. F. Wilcox, *Graph theory and molecular orbitals. XII. Acyclic polyenes*, The Journal of Chemical Physics, 62(1975), 3399-3405.

[8] I. Gutman and N. Trinajstic, *Graph theory and molecular orbitals. Total f-electron energy of alternant hydrocarbons*, Chemical Physics Letters, 17(4)(1972), 535-538.

[9] F. Harary, *Graph Theory*, Narosa Publishing House, New Delhi, (2001).

[10] A. Ilić and D. Stevanović, *On comparing Zagreb indices*, MATCH Communications in Mathematical and in Computer Chemistry, 62(2009), 681-687.

[11] W. Imrich and S. Klavzar, *Product Graphs: Structure and Recognition*, John Wiley and Sons, New York, USA, (2000).

[12] A. M. Naji and N. D. Soner, *The First Leap Zagreb Index of Some Graph Operations*, International journal of Applied Graph Theory, 2(1)(2018), 7-18.

[13] A. M. Naji, N. D. Soner and Ivan Gutman, *On leap Zagreb indices of graphs*, Communications in Combinatorics and Optimization, 2(2)(2017), 99-117.

[14] S. Nikolić, G. Kovaćević, A. Milićević and N. Trinajstić, *The Zagreb indices 30 years after*, Croatica Chemica Acta, 76(2)(2003), 113-124.

[15] M. Veylaki, M. J. Nikmehr and H. A. Tavallaee, *The third and hyper-Zagreb coindices of some graph operations*, Journal Applied Mathematics Computational, 50(2016), 315-325.