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$\rho-$Statistical Convergence of Order α in Partial Metric Spaces

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Abstract

In the present study, we introduce and study the notions of $\rho-$statistically convergence and $\rho-$statistically boundedness of order α in a partial metric space (X, p) and a relation between these two concepts is established, where $\rho = (\rho_n)$ is a non decreasing sequence of positive real numbers. Apart from this, we explored the inclusion relations between $\rho-$statistical convergence and $\rho-$statistical boundedness with the variation of the sequence $\rho = (\rho_n)$.

Keywords: Statistical convergence; $\rho-$statistical convergence; partial metric space; statistical boundedness.

2020 Mathematics Subject Classification: 46A45, 40A05, 40A35.

1. Introduction

Fast [11] in 1951, introduced the notion of statistical convergence of real sequences which is in fact a generalization of usual convergence of real sequences. Later on this concept was studied as "convergence in density" by Buck [7] in 1953. It is also a part of monograph by Zygmund [35] and referred as almost convergence. Steinhaus [32] and Schoenberg [29] introduced and studied this concept independently in connection with summability of sequences. Later on this concept of statistical convergence and its various extensions have been explored by many more mathematicians. In the present work, we mainly concerns with the concepts of statistical convergence, ρ-statistical convergence and partial metric space. Most of the work in the field of sequence space is dominated by the sequence of scalars (real or complex). Through this study, we contribute to sequence space by adding sequences from an arbitrary non-empty set X, via partial metric. To go through, let us first recall some basic tools. The natural density is the main pillar of the concept of statistical convergence and is defined as:

Definition 1.1. *The natural density of a subset K of $\mathbb{N}$ is denoted by $\delta(K)$ and defined as*

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n}\text{card}(\{k \in K : k \leq n\}),$$

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provided the limit exists. It is easily verified that $\delta(K) = 0$ *for finite subset* K *of* $\mathbb{N}$ *and* $\delta(K) + \delta(\mathbb{N} - K) = 1$ *for every* $K \subseteq \mathbb{N}$ *. For a detailed account of natural density, one may peep into Niven and Zuckerman [24].*

Definition 1.2 ([27])**.** *A real valued sequence* (z_m) *is statistically convergent to* $L \in \mathbb{R}$ *if for each* $\varepsilon > 0$, $\delta(\{|z_m - L| \geq \varepsilon\}) = 0$*, i.e.,* $\lim_{n\to\infty} \frac{1}{n}\text{card}(\{m \leq n : |z_m - L| > \varepsilon\}) = 0$*. Here L is referred as statistical limit of* (z_m)*. We write* $z_m \to L(S)$ *and by* $S(c)$ *we denote the set of all statistically convergent real sequences.*

With the passage of time, various generalization of this notion, that is, $\lambda-$statistical convergence, $A-$statistical convergence, $I-$convergence, rough statistical convergence, $\mu-$statistical convergence, lacunary statistical convergence, deferred statistical convergence, convergence of order α etc. have been appeared and also these generalizations have been studied for vector valued sequences by many more mathematicians. One may refer to [4,6,8,9,13–15,19,21,22,34].

Definition 1.3 ([1])**.** *A real valued sequence* (z_m) *is said to be ρ-statistical convergent to L or we can say* $z_m \to L(S_\rho)$ *if for every* $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{\rho_n}\text{card}(\{m \leq n : |z_m - L| \geq \varepsilon\}) = 0.$$

Here and in what follows, $\rho = (\rho_n)$ *is a non decreasing sequence of positive real numbers tending to infinity, satisfying the following conditions:*

$$\limsup_n \frac{\rho_n}{n} < \infty,\ \Delta\rho_n = \rho_{n+1} - \rho_n = O \tag{1}$$

By $S_\rho(c)$ *we notate the class of all ρ-statistical convergent sequences of reals, i.e.,*

$$S_\rho(c) = \{(z_m) : z_m \to L(S_\rho)\ for\ some\ L\}.$$

As statistical convergence has its deep root in the notion of natural density, similar is the relation of ρ-statistical convergence with $\rho-$*density).*

Definition 1.4. *For a given sequence* $\rho = (\rho_n)$*, the* $\rho-$*density) of* $K \subseteq \mathbb{N}$ *is defined as*

$$\delta^\rho(K) = \lim_{n\to\infty} \frac{1}{\rho_n} card(\{m \leq n : m \in K\})$$

Definition 1.5. *For a given sequence* $\rho = (\rho_n)$*, if the set of indicies m 's for which* (z_m) *does not satisfy property P has zero ρ-density, then we say* (z_m) *satisfies P for "almost all m with respect to ρ" abbreviated as "a.a. m w.r.t. ρ"*

ρ-statistical convergence, now may be redefined as:

Definition 1.6. *A sequence* (z_m) *of reals is ρ-statistical convergent to* $L \in \mathbb{R}$ *if for* $\varepsilon > 0$,

$$\delta^\rho(\{m \in \mathbb{N} : |z_m - L| \geq \varepsilon\}) = 0, \ \ i.e.,\ |z_m - L| < \varepsilon \quad a.a.\ m\ w.r.t\ \rho.$$

The credit of introducing the idea of partial metric space goes to Matthews [20] in 1994. Initially the concept of partial metric space was used in the field of computer science, but now it has been extensively used in biological sciences, information science and fixed point theory etc.

Definition 1.7 ([20]). *Let $X \neq \emptyset$. A function $p : X \times X \to \mathbb{R}$ satisfying the following*

(p_1) $0 \leq p(x,x) \leq p(x,y)$

(p_2) $p(x,x) = p(x,y) = p(y,y) \Longleftrightarrow x = y$

(p_3) $p(x,y) = p(y,x)$

(p_4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$ *for all* $x, y, z \in X$,

is said to be a partial metric on X and (X, p) is called a partial metric space.

It is clear from axiom (p_1) that $|p(x_k, x) - p(x,x)|$ and $p(x_k, x) - p(x,x)$ are the same thing, for any sequence $\langle x_k \rangle$ in X and $x \in X$. In comparison to a metric on X, we can say a partial metric p is precisely metric $p : X \times X \to \mathbb{R}$ such that $\forall\ x \in X,\ p(x,x) = 0$. That is, in the definition of partial metric space, only one side axiom of metric is preserved, i.e., $\forall\ x, y \in X,\ p(x,y) = 0 \Rightarrow x = y$ and other half that is, $x = y \Rightarrow p(x,y) = 0$ need not hold good. For a detailed description of partial metric space, one may refer [20,23,28]. Nuray [25], Bayram et al. [5] and Kumar et al. [18] stepped into partial metric space via statistical convergence and introduced notion of statistical convergence in partial metric space. We call this notion as statistical p-convergence.

Definition 1.8. *A sequence (z_m) in partial metric (X, p) is said to be statistically p-convergent to some $a \in X$ if for given $\varepsilon > 0$, $\delta(\{m \in \mathbb{N} : p(z_m, a) \geq p(a,a) + \varepsilon\}) = 0$ and we write it as $z_m \xrightarrow{p} a(S)$. By $S(c^p)$, we notate the class of all statistically convergent sequence from (X, p).*

Definition 1.9. *Let (z_m) be sequence in (X, p) and $a \in X$. If for given $\varepsilon > 0$, $\exists$ a positive integer m_0 such that $p(z_m, a) \leq p(a,a) + \varepsilon$ for all $m \geq m_0$, then we say (z_m) is convergent to a. We write $z_m \xrightarrow{p} a$ and c^p for the class of all convergent sequences.*

Definition 1.10. *A sequence (z_m) in partial metric space (X, p) is said to be p-bounded if $\exists$ some $a \in X$ and $M > 0$ such that $p(z_m, a) < p(a,a) + M\ \forall\ m \geq 1$. We write b^p as the class of all bounded sequences.*

In the present paper, we are going to explore ρ-statistical convergence of order α in partial metric space (X, p) by introducing the notion of ρ-statistical convergence and strongly ρ-Cesaro summability of order α. We recall [4,10,17], a scalar sequence space E is

(i) Solid (normal) if $(\eta_m) \in E$ whenever $|\eta_m| \leq |\xi_m|$, $m \geq 1$, for $(\xi_m) \in E$.

(ii) Symmetric if $(\eta_m) \in E$ implies $(\eta_{\sigma_m}) \in E$, where (σ_m) is permutation on m.

Motivating from above definition, for an arbitrary sequence space E in *p.m.s.* (X,p) we say E is solid (normal) if $(\eta_m) \in E$ whenever $p(\eta_m, a) \leq p(\xi_m, a)$, $m \geq 1$ for $(\xi_m) \in E$ for all $a \in X$.
In this paper, we study the sequence space by using sequences from an arbitrary non-empty set X, equipped with a partial metric. Throughout the paper, (X,p) will denote the partial metric space abbreviated as *p.m.s.*

2. Main Results

In this section we study the ρ-statistical convergence of order α for sequences from an arbitrary partial metric space (X,p) and its relation with ρ-strongly Cesàro summability of order α.

Definition 2.1. *A sequence (z_m) in p.m.s. (X,p) is said to be statistically convergent of order α $(0 < \alpha \leq 1)$ to $z_0 \in X$ if for $\varepsilon > 0$,*

$$\lim_{n\to\infty} \frac{1}{n^\alpha} \text{card}(\{m \leq n : p(z_m, z_0) \geq p(z_0, z_0) + \varepsilon\}) = 0$$

and we write $z_m \to z_0(S^\alpha)$. We shall denote the set of all statistically convergent sequences of order α by $S^\alpha(c^p)$.

For $\alpha = 1$, we call a statistically convergent sequence of order α simply as a statistically convergent sequence and corresponding space is denoted by $S(c^p)$.

Definition 2.2. *Let $\rho = (\rho_n)$ be the sequence defined above and $0 < \alpha \leq 1$. A sequence (z_m) in p.m.s. (X,p) is said to be ρ-statistically convergent of order α to $z_0 \in X$ if for $\varepsilon > 0$,*

$$\lim_{n\to\infty} \frac{1}{\rho_n^\alpha} \text{card}(\{m \leq n : p(z_m, z_0) \geq p(z_0, z_0) + \varepsilon\}) = 0$$

and we write $z_m \xrightarrow{p} z_0(S_\rho^\alpha)$. By $S_\rho^\alpha(c^p)$ we shall denote the class of all ρ-statistically convergent sequences of order α.

Definition 2.3. *For a given sequence $\rho = (\rho_n)$, the (α,ρ)-density (or $\rho_\alpha-$density) of $K \subseteq \mathbb{N}$ is defined as $\delta_\rho^\alpha(K) = \lim_{n\to\infty} \frac{1}{\rho_n^\alpha} \text{card}(\{m \in \mathbb{N} : m \in K\})$.*

Definition 2.4. *For a given sequence $\rho = (\rho_n)$, if the set of indicies $m^{'}s$ for which (z_m) does not satisfy property P has zero (α,ρ)-density, then we say (z_m) satisfies P for "almost all m with respect to ρ_α" abbreviated as "a.a. m w.r.t. ρ_α."*

Now we may also redefined ρ-statistical convergence of order α in *p.m.s.* (X,p), as

Definition 2.5. *Let $\rho = (\rho_n)$ be the sequence defined above and $0 < \alpha \leq 1$. A sequence (z_m) in p.m.s. (X,p) is said to be ρ-statistically convergent of order α to $z_0 \in X$ if for $\varepsilon > 0$,*

$$\delta_\rho^\alpha(\{m \leq n : |p(z_m, z_0) - p(z_0, z_0)| \geq \varepsilon\}) = 0, \ \textit{i.e.,} \ |p(z_m, z_0) - p(z_0, z_0)| < \varepsilon \quad \textit{a.a. } m \textit{ w.r.t.} \quad \rho_\alpha.$$

Theorem 2.6. *Let $\alpha, \beta \in (0,1]$ such that $\alpha \leq \beta$. Then $S_\rho^\alpha(c^p) \subset S_\rho^\beta(c^p)$, converse may not be true in general.*

Proof. For given $\varepsilon > 0$, we have

$$0 \leq \frac{1}{\rho_n^{\beta}} \text{card}\left(\{m \leq n : |p(z_m, z_0) - p(z_0, z_0)| \geq \varepsilon\}\right)$$
$$\leq \frac{1}{\rho_n^{\alpha}} \text{card}\left(\{m \leq n : |p(z_m, z_0) - p(z_0, z_0)| \geq \varepsilon\}\right).$$

Taking limit $n \to \infty$, we get required result.

For reverse inclusion, consider the following example:

Let $X = \mathbb{R}$ with partial metric p defined as $p(\xi, \eta) = |\xi - \eta| \;\; ; \;\; \xi, \eta \in \mathbb{R}$ and $\rho_n = n$. Construct a sequence (z_m) such that

$$z_m = \begin{cases} \sqrt{n} & \text{for } m = n^2 \\ 0 & \text{otherwise,} \end{cases}$$

This implies, $\text{card}\left(\{m \leq n : |p(z_m, z_0) - p(z_0, z_0)| \geq \varepsilon\}\right) \leq \sqrt{n}$. Thus if we consider $\frac{1}{2} < \beta \leq 1$, then we have

$$\lim_{n\to\infty} \frac{1}{\rho_n^{\beta}} \text{card}\left(\{m \leq n : |p(z_m, z_0) - p(z_0, z_0)| \geq \varepsilon\}\right) \leq \lim_{n\to\infty} \frac{\sqrt{n}}{\rho_n^{\beta}} \longrightarrow 0.$$

On the other hand for $0 < \alpha < \frac{1}{2}$, we have

$$\lim_{n\to\infty} \frac{1}{\rho_n^{\alpha}} \text{card}\left(\{m \leq n : |p(z_m, 0) - p(0, 0)| \geq \varepsilon\}\right) \leq \lim_{n\to\infty} \frac{\sqrt{n}}{l_r^{\alpha}} \nrightarrow 0.$$

Hence inclusion is strict for $\alpha < \beta$ with $0 < \alpha < \frac{1}{2}$ and $\frac{1}{2} < \beta \leq 1$. □

Corollary 2.7. *Let $\rho = (\rho_n)$ be the sequence defined above and $0 < \alpha \leq \beta \leq 1$. Then we have following*

(i) $S_\rho^\alpha(c^p) = S_\rho^\beta(c^p)$ *iff* $\alpha = \beta$.

(ii) $S_\rho^\alpha(c^p) = S_\rho(c^p)$ *iff* $\alpha = 1$.

Definition 2.8. *Let $\rho = (\rho_n)$ be the sequence defined above and $0 < \alpha \leq 1$. A sequence (z_m) in p.m.s. (X, p) is said to be ρ- statistically bounded of order α if there exist some $z_0 \in X$ and $M > 0$ such that*

$$\lim_{n\to\infty} \frac{1}{\rho_n^{\alpha}} \text{card}(\{m \leq n : |p(z_m, z_0) - p(z_0, z_0)| \geq M\}) = 0,$$

$$\textit{i.e.,} \quad |p(z_m, z_0) - p(z_0, z_0)| < M \quad \textit{a.a. } m \textit{ w.r.t. } \rho_\alpha$$

and we write $S_\rho^\alpha(b^p)$ for the class of all ρ-statistically bounded sequences of order α.

Theorem 2.9.

(i) $S_\rho^\alpha(b^p)$ *is not symmetric.*

(ii) $S_\rho^\alpha(b^p)$ *is normal.*

Proof. (i) By taking $\rho = (\rho_n) = n$, $X = \mathbb{R}$, $p(\xi, \eta) = |\xi - \eta|$ and for $\alpha = 1$.

Let $(z_m) = (1, 0, 0, 2, 0, 0, 0, 0, 3, 0, 0, 0, 0, 0, 0, 4, \ldots.)$. Then $(z_m) \in S_\rho^\alpha(b^p)$. Consider (z'_m) be a sequence which is obtained by rearrangement of (z_m) as follows

$$(z'_m) = (z_1, z_2, z_4, z_3, z_9, z_5, z_{16}, z_6, z_{25}, z_7, z_{36}, z_8, z_{49}, z_{10}, \ldots)$$
$$= (1, 0, 2, 0, 3, 0, 4, 0, 5, 0, 6, 0, 7, 0, 8, 0, \ldots).$$

Then for any $M > 0$, we have

$$\lim_{n\to\infty} \frac{1}{\rho_n^\alpha} \text{card}(\{m \le n : |p(z'_m, z_0) - p(z_0, z_0)| \ge M\}) \ne 0$$

so $(z'_m) \notin S_\rho^\alpha(b^p)$ and hence $S_\rho^\alpha(b^p)$ is not symmetric.

(ii) Let $(z_m) \in S_\rho^\alpha(b^p)$ and (z'_m) be a sequence such that $p(z'_m, a) \le p(z_m, a)$ for all $m \in \mathbb{N}$ and for all $a \in X$. As $(z_m) \in S_\rho^\alpha(b^p)$ so there exists some $z_0 \in X$ such that

$$\lim_{n\to\infty} \frac{1}{\rho_n^\alpha} \text{card}(\{m \le n : |p(z_m, z_0) - p(z_0, z_0)| \ge M\}) = 0.$$

Clearly

$$\text{card}(\{m : |p(z'_m, z_0) - p(z_0, z_0)| \ge M\}) \le \text{card}(\{m \ : |p(z_m, z_0) - p(z_0, z_0)| \ge M\})$$

so $(z'_m) \in S_\rho^\alpha(b^p)$. Hence $S_\rho^\alpha(b^p)$ is normal.

□

Theorem 2.10. *In p.m.s.* (X, p), *ρ-statistically convergence of order α implies ρ-statistically boundedness of order α. Converse may not be true in general.*

Proof. Let $\rho = (\rho_n)$ be the sequence defined above and let (z_m) be a ρ-statistically convergent of order α to some $z_0 \in X$. Then for given $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{\rho_n^\alpha} \text{card}(\{m \le n : p(z_m, z_0) > p(z_0, z_0) + \varepsilon\}) = 0.$$

Now for sufficiently large M, we may assert that,

$$\text{card}(\{m \le : p(z_m, z_0) > p(z_0, z_0) + M\}) \le \text{card}(\{m \le n : p(z_m, z_0) > p(z_0, z_0) + \varepsilon\})$$

and hence the result follows.

For converse part, let $X = \mathbb{R}$ and p be the partial metric defined by $p(\xi, \eta) = |\xi - \eta|$; $\xi, \eta \in \mathbb{R}$ and

$\alpha = 1$. Let $\rho = (\rho_n) = n$. Consider a sequence (z_m) in X as

$$z_m = \begin{cases} -1 & \text{for } m = 2n \\ 1 & \text{for } m = 2n+1 \end{cases}$$

Now $\frac{1}{\rho_n}\text{card}(\{m \leq n : p(z_m, 1) > p(1,1) + \varepsilon\}) \neq 0$ and hence (z_m) is not ρ-statistically convergent to 1. Similar can be proved for -1. As every bounded sequence is ρ-statistically bounded of order α, hence the result follows. $\square$

For the characterizing the ρ-statistically φ-convergent of order α we have the following definition.

Definition 2.11. *A sequence* (z_{m_n}), $n \in \mathbb{N}$ *is said to be* ρ_α*-statistical dense if* $\delta_\rho^\alpha(B) = 1$, *where* $B = \{m_1 < m_2 < m_3 < \ldots\}$, *i.e.,* $\lim_{n\to\infty} \frac{1}{\rho_n^\alpha}\text{card}(\{m \leq n : m \in B\} = 1$.

The next theorem characterizes the ρ-statistically convergent sequence of order α in term of $\rho_\alpha$statistically dense convergent subsequences.

Theorem 2.12. *A sequence* (z_m) *is* ρ*-statistically convergent of order* α *iff every* ρ_α*-statistically dense subsequence of* (z_m) *is* ρ*-statistically convergent of order* α.

Proof. Let (z_m) is ρ-statistically convergent to z_0 of order α. So for given $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \frac{1}{\rho_n^\alpha}\text{card}(\{m \leq n : p(z_m, z_0) > p(z_0, z_0) + \varepsilon\}) = 0, \ \ i.e., \ \ \delta_\rho^\alpha(A) = 0$$

where $A = \{m \in \mathbb{N} : p(z_m, z_0) > p(z_0, z_0) + \varepsilon\}$ and $(z_{m_n})_{n\in\mathbb{N}}$ is ρ-statistically dense subsequence of (z_m) which is not ρ-statistically convergent of order α, i.e.,

$$\liminf_{n\to\infty} \frac{1}{\rho_n^\alpha}\text{card}(\{m_n \leq : p(z_{m_n}, z_0) > p(z_0, z_0) + \varepsilon\}) = d, \ where \ d \in (0,1).$$

Now $\text{card}(\{m \leq : p(z_m, z_0) > p(z_0, z_0) + \varepsilon\}) \geq \text{card}(\{m_n \leq : p(z_{m_n}, z_0) > p(z_0, z_0) + \varepsilon\})$. So

$$\liminf_{n\to\infty} \frac{1}{\rho_n^\alpha}\text{card}(\{m \leq n : p(z_m, z_0) > p(z_0, z_0) + \varepsilon\}) \geq d \neq 0$$

a contradiction to given.
Converse part follows from the fact that every sequence is a α-lacunary statistical dense subsequence of itself. $\square$

Theorem 2.13. *A sequence* (z_m) *in* (X, p) *is* ρ*-statistically convergent of order* α *to some* $z_0 \in X$ *iff there exists a sequence* (y_m), *convergent to* z_0 *such that* $y_m = z_m$ *a.a.* m *w.r.t.* ρ_α.

Proof. Let (z_m) be a ρ-statistically convergent to some $z_0 \in X$. So for each $\varepsilon > 0$, we get $\rho_\theta^\alpha(K) = 0$

where $K = \{m \in \mathbb{N} : p(z_m, z_0) > p(z_0, z_0) + \varepsilon\}$. Let

$$y_m = \begin{cases} z_m & \text{for } m \in \mathbb{N} - K \\ z_0 & \text{for } m \in K. \end{cases}$$

Now $\{m \in \mathbb{N} : y_m \neq z_m\} \subseteq K$ and so $y_m = z_m$ *a.a. m w.r.t.* ρ_α. Also

$$p(y_m, z_0) = \begin{cases} p(z_m, z_0) & \text{for } m \in \mathbb{N} - K \\ p(z_0, z_0) & \text{for } m \in K \end{cases}$$

implies $p(y_m, z_0) < p(z_0, z_0) + \varepsilon$ for all $m \geq 1$. Thus (y_m) is convergent sequence and $y_m = z_m$ *a.a. m w.r.t.* ρ_α.

Conversely, let there exists $m_0 \in \mathbb{N}$ such that $p(y_m, z_0) < p(z_0, z_0) + \varepsilon$ for all $m \geq m_0$. The result follows from the inclusion relation $\{m : p(z_m, z_0) \geq p(z_0, z_0) + \varepsilon\} \subseteq K \cup \{1, 2, 3, \ldots, m_0 - 1\}$. □

Following on the similar lines, we have

Theorem 2.14. *A sequence* (z_m) *is ρ-statistically bounded of order α iff there exists a bounded sequence* (y_m) *such that* $y_m = z_m$ *a.a. m w.r.t* ρ_α.

Using the same technique as in Theorem 2.12, we give a characterization of the lacunary statistically φ-boundedness of order α in terms of the α-lacunary statistically dense φ-bounded subsequences of it, in terms of following.

Theorem 2.15. *A sequence* (z_m) *is lacunary statistically φ-bounded of order α iff every α-lacunary statistically dense subsequence of* (z_m) *is lacunary statistically φ-bounded of order α.*

Definition 2.16. *Let* (z_m) *be a sequence in p.m.s.* (X, p) *and* $0 < \alpha \leq 1$. *The sequence* (z_m) *is strongly ρ-Cesàro summable of order α to* $z_0 \in X$ *if*

$$\lim_{n\to\infty} \frac{1}{\rho_n^\alpha} \sum_{m=1}^{n} |p(z_m, z_0) - p(z_0, z_0)| = 0.$$

We notate $|\sigma_1|_\rho^\alpha(c^p)$ *for the set of all strongly ρ-Cesàro summable sequences of order α. We write* $|\sigma_1|_\rho^\alpha(c^\varphi)$ *for* $\alpha = 1$ *as* $|\sigma_1|_\rho(c^\varphi)$.

Theorem 2.17. *For* $\alpha \in (0, 1]$, $|\sigma_1|_\rho^\alpha(c^p) \subset S_\rho^\alpha(c^p)$, *i.e., every strongly ρ-Cesàro summable sequence of order α is ρ-statistically convergent of order α to same limit.*

Proof. For $\varepsilon > 0$ and $0 < \alpha \leq 1$, we have

$$\begin{aligned} \frac{1}{\rho_n^\alpha} \sum_{m=1}^{n} |p(z_m, z_0) - p(z_0, z_0)| &\geq \frac{1}{\rho_n^\alpha} \sum_{\substack{m \leq n \\ |p(z_m, z_0) - p(z_0, z_0)| > \varepsilon}} |p(z_m, z_0) - p(z_0, z_0)| \\ &\geq \varepsilon . \frac{1}{\rho_n^\alpha} \text{card}(\{mleqn : |p(z_m, z_0) - p(z_0, z_0)| \geq \varepsilon\} \end{aligned}$$

and hence the result follows by applying the limit $n \to \infty$ in the above inequality. □

Theorem 2.18. *Let $\rho = (\rho_n)$ and $\tau = (\tau_n)$ be two sequence such that $\rho_n \leq \tau_n$, $0 < \alpha \leq \beta \leq 1$ and $\lim_{n\to\infty} \inf \frac{\rho_n}{\tau_n} > 0$. Then $S^{\alpha}_{\tau}(c^p) \subseteq S^{\beta}_{\rho}(c^p)$.*

Proof. As $\rho \leq \tau_n$ for all $n \in \mathbb{N}$, so for $\varepsilon > 0$, we have

$$\begin{aligned}
\frac{1}{{\tau_n}^{\alpha}}\text{card}\left(\{m \leq n : |p(z_m, z_0) - p(z_0, z_0)| \geq \varepsilon\}\right) &\geq \frac{1}{{tau_n}^{\alpha}}\text{card}\left(\{m \leq n : |p(z_m, z_0) - p(z_0, z_0)| \geq \varepsilon\}\right)\\
&\geq \frac{1}{{tau_n}^{\beta}}\text{card}\left(\{m \leq n : |p(z_m, z_0) - p(z_0, z_0)| \geq \varepsilon\}\right)\\
&= \left(\frac{\rho_n}{tau_n}\right)^{\beta}\frac{1}{\rho_n^{\beta}}\text{card}\left(\{m \leq n : |p(z_m, z_0) - p(z_0, z_0)| \geq \varepsilon\}\right).
\end{aligned}$$

Hence the proof. □

Corollary 2.19. *If $0 < \alpha \leq 1$ and $\lim\inf \frac{\rho_n}{\tau_n} > 0$, then*

(i) $S^{\alpha}_{\tau_n}(c^p) \subseteq S_{\rho}(c^p)$.

(ii) $S_{\tau_n}(c^p) \subseteq S_{\rho}(c^p)$.

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