International Journal of
Mathematics And its Applications
ISSN: 2347-1557

Int. J. Math. And Appl., **14(2)**(2026), 267–272
Available Online: http://ijmaa.in

Convolution Properties of a Subclass of Univalent Harmonic Mappings

Deepali Khurana[1,*], Shalini Tomar[2]

[1]*Department of Mathematics, Hans Raj Mahila Maha Vidyalaya, Jalandhar, Punjab, India*

[2]*Department of Mathematics, Savitri Bai Phule Government Girls College, Kharkhoda, Sonipat, Haryana, India*

Abstract

Sudharshan et al. in [8] studied a subclass $TS_H^*(m,n,\lambda,\gamma)$ of univalent harmonic mappings of the form $f_m = h + \overline{g_m}$, where

$$h(z) = z - \sum_{k=2}^{\infty} |a_k| z^k, \quad g_m(z) = (-1)^{m-1} \sum_{k=1}^{\infty} |b_k| \overline{z}^k$$

using Salagean modified differential operator. They proved that for $0 \leq \gamma_1 \leq \gamma_2 < 1$, if $f_m \in TS_H^*(m,n,\lambda,\gamma_1)$ and $F_m \in TS_H^*(m,n,\lambda,\gamma_2)$, then $f_m * F_m \in TS_H^*(m,n,\lambda,\gamma_1)$. In the present article, we improved the above stated result.

Keywords: Univalent harmonic mappings; Harmonic functions; harmonic univalent; harmonic convex; harmonic starlike; convolution of harmonic functions.
2020 Mathematics Subject Classification: 30C45.

1. Introduction

A continuous function $f = u + \imath v$ is a complex valued harmonic function in $\mathbb{C}$ if both u and v are real valued harmonic in $\mathbb{C}$. In any simply connected domain $\mathbb{D}$, a complex valued harmonic function $f = u + \imath v$ can be written as $f = h + \overline{g}$, where h and g are analytic in $\mathbb{D}$. We call h the analytic part and g the co-analytic part of f. We denote by $\mathcal{H}$ the class of complex valued harmonic mappings $f = h + \overline{g}$ defined in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by $h(0) = g(0) = h'(0) - 1 = 0$. Such mappings have the following power series representation

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}. \tag{1}$$

A necessary and sufficient condition for $f \in \mathcal{H}$ to be locally univalent and sense-preserving in $\mathbb{D}$ is that $|h'(z)| > |g'(z)|$ for all $z \in \mathbb{D}$ (see[5]). Let S_H denote the subclass of $\mathcal{H}$ consisting of all

*Corresponding author (deepali.02.08.88@gmail.com)

sense-preserving univalent harmonic mappings f and S_H^0 is the subclass of S_H whose members satisfy additional condition $f_{\bar{z}}(0) = 0$, i.e. $g'(0) = 0$. Denote by K_H^0, S_H^{*0} and C_H^0 the subclasses of S_H^0 whose functions map $\mathbb{D}$ onto convex, starlike and close-to-convex domains, respectively.

In 1984 Clunie and Sheil-Small in their landmark paper [1], investigated some geometric properties of the class S_H and its subclasses. Since then researchers defined many subclasses of the class S_H and studied their various properties e.g. see ([2, 3, 4, 6 and 7]) and references therein. In particular, Sudharshan et al. [8] studied a class $S_H^*(m,n,\lambda,\gamma)$ of harmonic functions. For $0 \leq \gamma < 1$ and $0 \leq \lambda \leq 1$, α real, $m \in \mathbb{N}$, $n \in \mathbb{N}_0$, $m > n$ and $z \in \mathbb{D}$, let $f \in S_H^*(m,n,\lambda,\gamma)$ consists the family of harmonic functions of the form (1) such that

$$Re\left(\frac{(1+e^{i\alpha})D^m f(z)}{\lambda D^m f(z) + (1-\lambda)D^n f(z)} - e^{i\alpha}\right) \geq \gamma. \tag{2}$$

where $D^m f(z)$ is modified Salagean differential operator defined as

$$D^m f(z) = D^m h(z) + (-1)^m \overline{D^m g(z)} \tag{3}$$

here

$$D^m h(z) = z + \sum_{k=2}^{\infty} k^m a_k z^k \ \text{ and } \ D^m g(z) = \sum_{k=1}^{\infty} k^m b_k z^k.$$

Further, $TS_H^*(m,n,\lambda,\gamma)$ denotes the subclass of $S_H^*(m,n,\lambda,\gamma)$ consisting of harmonic functions of the form $f_m = h + \overline{g_m}$ such that h and g_m are given by

$$h(z) = z - \sum_{k=2}^{\infty} |a_k| z^k, \quad g_m(z) = (-1)^{m-1} \sum_{k=1}^{\infty} |b_k| \bar{z}^k. \tag{4}$$

In [8], authors studied many basic properties of classes $S_H^*(m,n,\lambda,\gamma)$ and $TS_H^*(m,n,\lambda,\gamma)$ including convolution properties. For harmonic functions

$$f_m(z) = z - \sum_{k=2}^{\infty} |a_k| z^k + (-1)^{m-1} \sum_{k=1}^{\infty} |b_k| \bar{z}^k$$

and

$$F_m(z) = z - \sum_{k=2}^{\infty} |A_k| z^k + (-1)^{m-1} \sum_{k=1}^{\infty} |B_k| \bar{z}^k.$$

The convolution of f_m and F_m is given by

$$(f_m * F_m)(z) = f_m(z) * F_m(z) = z - \sum_{k=2}^{\infty} |a_k A_k| z^k + (-1)^{m-1} \sum_{k=1}^{\infty} |b_k B_k| \bar{z}^k.$$

Sudharshan et al. in [8] established the following result of convolution

Theorem 1.1. *For $0 \leq \beta \leq \gamma < 1$, let $f_m \in TS_H^*(m,n,\lambda,\gamma)$ and $F_m \in TS_H^*(m,n,\lambda,\beta)$. Then $f_m * F_m \in TS_H^*(m,n,\lambda,\gamma) \subset TS_H^*(m,n,\lambda,\beta)$.*

In the present work we studied the convolution of harmonic functions from the class $TS_H^*(m,n,\lambda,\gamma)$ and also our result improved the Theorem 1.1.

2. Main Results

In order to prove our main theorem we need the following lemmas. Lemma 2.1 and 2.2 are due to Sudharshan et al. [8].

Lemma 2.1. *Let $f_m = h + \overline{g_m}$ be given by (4). Then $f_m \in TS_H^*(m,n,\lambda,\gamma)$ if and only if*

$$\sum_{k=1}^{\infty} \Big(\left((2-\lambda(1+\gamma))\, k^m - (1-\lambda)(1+\gamma)k^n \right) |a_k| + \left((2-\lambda(1+\gamma))\, k^m - (-1)^{(m-n)}(1-\lambda)(1+\gamma)k^n \right) |b_k| \Big) \leq 2(1-\gamma).$$

Lemma 2.2. *For $0 \leq \beta \leq \gamma < 1$, let $f_m \in TS_H^*(m,n,\lambda,\gamma)$ and $F_m \in TS_H^*(m,n,\lambda,\beta)$. Then $f_m * F_m \in TS_H^*(m,n,\lambda,\gamma) \subset TS_H^*(m,n,\lambda,\beta)$.*

Lemma 2.3. *If $m_1, m_2 \in \mathbb{N}$, then*
(i) $TS_H^*(m_1+m_2, m_2+n, \lambda, \gamma) \subseteq TS_H^*(m_1, n, \lambda, \gamma)$,(if m_1+m_2 and m_1 both are even or odd). (ii) $TS_H^*(m_1+m_2-1, m_2+n-1, \lambda, \gamma) \subseteq TS_H^*(m_1, n, \lambda, \gamma)$,(if m_1+m_2 is even but m_1 is odd or visa versa).

Proof. **Proof of Lemma 2.3 (i)**:
Let $f_{m_1+m_2} \in TS_H^*(m_1+m_2, m_1+n, \lambda, \gamma)$ then by using Lemma 2.1, we have

$$\sum_{k=2}^{\infty} \frac{k^{m_1+m_2}(2-\lambda(1+\gamma)) - k^{m_1+n}(1-\lambda)(1+\gamma)}{(1-\gamma)} |a_k| + \sum_{k=1}^{\infty} \frac{k^{m_1+m_2}(2-\lambda(1+\gamma)) - (-1)^{m_2-n}k^{m_1+n}(1-\lambda)(1+\gamma)}{(1-\gamma)} |b_k| \leq 1. \tag{5}$$

Now

$$\begin{aligned} &\sum_{k=2}^{\infty} \frac{k^{m_2}(2-\lambda(1+\gamma)) - k^{n}(1-\lambda)(1+\gamma)}{(1-\gamma)} |a_k| + \sum_{k=1}^{\infty} \frac{k^{m_2}(2-\lambda(1+\gamma)) - (-1)^{m_2-n}k^{n}(1-\lambda)(1+\gamma)}{(1-\gamma)} |b_k| \\ \leq\ & \sum_{k=2}^{\infty} \frac{k^{m_1}\left(k^{m_2}(2-\lambda(1+\gamma)) - k^{n}(1-\lambda)(1+\gamma)\right)}{(1-\gamma)} |a_k| + \sum_{k=1}^{\infty} \frac{k^{m_1}\left(k^{m_2}(2-\lambda(1+\gamma)) - (-1)^{m_2-n}k^{n}(1-\lambda)(1+\gamma)\right)}{(1-\gamma)} |b_k| \\ \leq\ & 1. \quad \textit{using } (5) \end{aligned}$$

So, $f_m \in TS_H^*(m_2,n,\lambda,\gamma)$. Thus, the proof of (i) is established. The proof of (ii) of Lemma 2.3 is similar to (i), hence it will be omitted. □

Theorem 2.4. *For* $0 \leq \gamma_2 \leq \gamma_1 < 1$*, let the functions*

$$f_{m_1}(z) = z - \sum_{k=2}^{\infty} a_k z^k + (-1)^{m_1-1} \sum_{k=1}^{\infty} b_k \overline{z}^k \quad (a_k,\ b_k \geq 0)$$

and

$$F_{m_2}(z) = z - \sum_{k=2}^{\infty} A_k z^k + (-1)^{m_2-1} \sum_{k=1}^{\infty} B_k \overline{z}^k \quad (A_k,\ B_k \geq 0)$$

belongs to $TS_H^*(m_1, n, \lambda, \gamma_1)$ *and* $TS_H^*(m_2, n, \lambda, \gamma_2)$*, respectively. Then* $f_{m_1} * F_{m_2} \in TS_H^*(m_1 + m_2, m_2 + n, \lambda, \gamma_1)$ *(if both* $m_1 + m_2$ *and* m_1 *are even or odd) and* $f_{m_1} * F_{m_2} \in TS_H^*(m_1 + m_2 - 1, m_2 + n - 1, \lambda, \gamma)$*,(if* $m_1 + m_2$ *is even but* m_1 *is odd or visa versa).*

Proof. Here, we only prove the Theorem for the case when both $m_1 + m_2$ and m_1 are even. For the case when $m_1 + m_2$ is even but m_1 is odd or visa versa one can prove in similar way. Therefore it is omitted. Since $f_{m_1} \in TS_H^*(m_1, n, \lambda, \gamma_1)$, then by using Lemma 2.2 we have

$$\sum_{k=2}^{\infty} \frac{k^{m_1}(2 - \lambda(1+\gamma_1)) - k^n(1-\lambda)(1+\gamma_1)}{(1-\gamma_1)} |a_k| + \sum_{k=1}^{\infty} \frac{k^{m_1}(2 - \lambda(1+\gamma_1)) - (-1)^{m_1-n} k^n(1-\lambda)(1+\gamma_1)}{(1-\gamma_1)} |b_k| \leq 1. \tag{6}$$

Similarly, $F_{m_2}(z) \in TS_H^*(m_2, n, \lambda, \gamma_2)$, we have

$$\sum_{k=2}^{\infty} \frac{k^{m_2}(2 - \lambda(1+\gamma_2)) - k^n(1-\lambda)(1+\gamma_2)}{(1-\gamma_2)} |A_k| + \sum_{k=1}^{\infty} \frac{k^{m_2}(2 - \lambda(1+\gamma_2)) - (-1)^{m_2-n} k^n(1-\lambda)(1+\gamma_2)}{(1-\gamma_2)} |B_k| \leq 1. \tag{7}$$

Therefore,

$$\frac{k^{m_2}(2 - \lambda(1+\gamma_2)) - k^n(1-\lambda)(1+\gamma_2)}{(1-\gamma_2)} |A_k| \leq 1, \quad \forall k = 1, 2.... \tag{8}$$

and

$$\frac{k^{m_2}(2 - \lambda(1+\gamma_2)) - (-1)^{m_2-n} k^n(1-\lambda)(1+\gamma_2)}{(1-\gamma_2)} |B_k| \leq 1 \;\; \forall k = 1, 2, \cdots. \tag{9}$$

Now for the convolution function $f_m * F_m$, we have:

$$\sum_{k=2}^{\infty} \frac{k^{m_1+m_2}(2 - \lambda(1+\gamma_1)) - k^n(1-\lambda)(1+\gamma_1)}{(1-\gamma_1)} |a_k A_k| + \sum_{k=1}^{\infty} \frac{k^{m_1+m_2}(2 - \lambda(1+\gamma_1)) - (-1)^{m_1-n}(1-\lambda)(1+\gamma_1)}{(1-\gamma_1)} |b_k B_k|$$
$$= \sum_{k=2}^{\infty} \frac{k^{m_2}\,(k^{m_1}(2 - \lambda(1+\gamma_1)) - k^n(1-\lambda)(1+\gamma_1))}{(1-\gamma_1)} |a_k A_k|$$

$$+\sum_{k=1}^{\infty}\frac{k^{m_2}\left(k^{m_1}(2-\lambda(1+\gamma_1))-(-1)^{m_1-n}k^{n}(1-\lambda)(1+\gamma_1)\right)}{(1-\gamma_1)}|b_kB_k|$$
$$\leq\sum_{k=2}^{\infty}\frac{k^{m_2}(2-\lambda(1+\gamma_2))-k^{n}(1-\lambda)(1+\gamma_2)}{(1-\gamma_2)}|A_k|\frac{k^{m_1}(2-\lambda(1+\gamma_1))-k^{n}(1-\lambda)(1+\gamma_1)}{(1-\gamma_1)}|a_k|$$
$$+\sum_{k=1}^{\infty}\frac{k^{m_2}(2-\lambda(1+\gamma_2))-(-1)^{m_2-n}k^{n}(1-\lambda)(1+\gamma_2)}{(1-\gamma_2)}|B_k|$$
$$\frac{k^{m_1}(2-\lambda(1+\gamma_1))-(-1)^{m_1-n}k^{n}(1-\lambda)(1+\gamma_1)}{(1-\gamma_1)}|b_k|$$
$$\leq\sum_{k=2}^{\infty}\frac{k^{m_1}(2-\lambda(1+\gamma_1))-k^{n}(1-\lambda)(1+\gamma_1)}{(1-\gamma_1)}|a_k|$$
$$\sum_{k=1}^{\infty}\frac{k^{m_1}(2-\lambda(1+\gamma_1))-(-1)^{m_1-n}k^{n}(1-\lambda)(1+\gamma_1)}{(1-\gamma_1)}|b_k|$$
$$\leq 1.(\text{using}6)$$

□

By setting $m_1 = m_2$ in Theorem 2.4, we obtained the next result.

Corollary 2.5. *For $0 \leq \gamma_2 \leq \gamma_1 < 1$, let the functions*

$$f_m(z) = z - \sum_{k=2}^{\infty} a_k z^k + (-1)^{m-1}\sum_{k=1}^{\infty} b_k \bar{z}^k \quad (a_k,\ b_k \geq 0)$$

and

$$F_m(z) = z - \sum_{k=2}^{\infty} A_k z^k + (-1)^{m-1}\sum_{k=1}^{\infty} B_k \bar{z}^k \quad (A_k,\ B_k \geq 0)$$

belongs to $TS_H^*(m,n,\lambda,\gamma_1)$ and $TS_H^*(m,n,\lambda,\gamma_2)$, respectively. Then $f_m * F_m \in TS_H^*(2m, m+n, \lambda, \gamma_1)$ (if m is an even integer) and $f_m * F_m \in TS_H^*(2m-1, m+n-1, \lambda, \gamma_1)$ (if m is and odd integer).

Proof. The proof of Corollary is easily followed by the proof of Theorem 2.4, by setting $m_1 = m_2$ in Lemma 2.3. □

Remark 2.6. *In this remark, we consider the following two cases and in each case, we observe that Corollary 2.5 improve the Theorem 1.1.*

Case (i) *When m is an even integer: For m an even integer our Corollary states that $f_m * F_m \in TS_H^*(2m, m+n, \lambda, \gamma_1)$, whereas Theorem 1.1 gives $f_m * F_m \in TS_H^*(m,n,\lambda,\gamma_1)$. But by Lemma 2.2 and (i) of Lemma 2.3 (by taking $m_1 = m_2 = m$, m an even integer) we have $TS_H^*(2m, m+n, \lambda, \gamma_1) \subseteq TS_H^*(m,n,\lambda,\gamma_1) \subseteq TS_H^*(m,n,\lambda,\gamma_2)$. Therefore, our result provides smaller class in comparison to the class given by Sudharshan et al. to which $f_m * F_m(z)$ belongs.*

Case (ii) *When m is an odd integer: For m an odd integer our Corollary states that $f_m * F_m \in TS_H^*(2m-1, m+n-1, \lambda, \gamma_1)$, whereas Theorem 1.1 gives $f_m * F_m \in TS_H^*(m,n,\lambda,\gamma_1)$. But by Lemma 2.2 and (iI) of Lemma 2.3 (by taking $m_1 = m_2 = m$, m an odd integer) we have $TS_H^*(2m-1, m+n-1, \lambda, \gamma_1) \subseteq TS_H^*(m,n,\lambda,\gamma_1) \subseteq TS_H^*(m,n,\lambda,\gamma_2)$. Our class provides better estimates in this case too.*

Hence we conclude that for all $m \in \mathbb{N} = 1, 2, 3....$, our result improves Theorem 1.1.

References

[1] J. Clunie and T. Sheil-Small, *Harmonic univalent functions*, Ann. Acad. Sci. Fenn. Ser. A I Math., 9(1984), 3-25.

[2] J. M. Jahangiri, G. Murugusundaramoorthy and K. Vijaya, *Salagean type harmonic functions*, South J. Pure Appl. Math., 2(2002), 77-82.

[3] R. Kumar, S. Gupta and S. Singh, *A class of univalent harmonic functions defined by multiplier transformation*, Rev. Roumaine Math. Pures Appl., 57(1)(2012), 371-382.

[4] T. Rosy, B. A. Stephen and K. G. Subramanian, *Goodman-Ronning type harmonic univalent functions*, Kyungpook Math. J., 41(2001), 45-54.

[5] H. Lewy, *On the non-vanishing of the Jacobian in certain one-to-one mappings*, Bull. Amer. Math. Soc., 42(10)(1936), 689-692.

[6] G. S. Salagean, *Subclasses of univalent functions*, Lecture Notes in Math., Springer - Verlag, Heidlberg, 1013(1983), 362-372.

[7] E. Yasar and S. Yalcin, *Harmonic univalent functions starlike or convex of complex order*, Tamsui Oxford Journal of Information and Mathematical Sciences, 27(3)(2011), 269-277.

[8] T. V. Sudharshan, R. Ezhilarasi, A. Stephen and K. G. Subarmnian, *A new subclass of Salagean-type harmonic univalent functions*, Chamchuri J. Math., 2(1)(2010), 81-92.