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Generalized Polygonal Sum Labeling of Some Families of Graphs

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Abstract

A graph G with "p" vertices and "q" edges is called a Polygonal sum graph(PSG) if it admits a labeling known as Polygonal sum labeling (PSL). PSL is an injective function $h : V(G) \rightarrow N$, where N represents the set of all non-negative integers that induces a bijection $h^+ : E(G) \rightarrow \{P_K(1), P_K(2), \ldots, P_K(q)\}$ of the edges of G defined by $h^+(uv) = h(u) + h(v)$ for every $e = uv \in E(G)$, where $P_K(1), P_K(2), \ldots, P_K(q)$ are the first "q" polygonal numbers. In this paper we prove that Olive tress, Caterpillars $S(n_1, n_2, \ldots, n_m)$ Shrub $St(n_1, n_2, \ldots, n_m)$, Banana tree $Bt(n_1, n_2, \ldots, n_m)$ and H-graphs are $PSG's$.

Keywords: Polygonal numbers; Polygonal sum labeling; Polygonal sum graph.
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1. Introduction

In this paper, we consider non - trivial finite, simple, undirected graphs. The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$ respectively. We adopt the graph theoretic notation and terminology from Bonday and Murty [2] and the number theory concepts from Burton [1]. Numerous types of labeling's have been introduced and explored by various researchers, and an excellent review of graph labeling can be found in [3]. In 2018 R. Sureshkumar [6] introduced a labeling called polygonal sum labeling (PSL), a graph which admits PSL is called a polygonal sum graph (PSG). In this paper we prove some families of graphs which admits PSL.

2. Main Results

Definition 2.1. *Number of the form* $P_K(n) = \frac{n}{2}\left[(K-2)n - (K-4)\right]$ *for all* $n \geq 1$ *are called Polygonal numbers, where* $K \geq 3$ *is the number of sides of the polygon. For* $K = 3$ *it gives triangular numbers, for* $K = 4$ *it gives square numbers and so on.*

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Definition 2.2. *PSL is an injective function $h : V(G) \to N$, where N represents the set of all non- negative integers that induces a bijection $h^+ : E(G) \to \{P_K(1), P_K(2), \ldots, P_K(q)\}$ of the edges of G defined by $h^+(uv) = h(u) + h(v)$ for every $e = uv \in E(G)$, where $P_K(1), P_K(2), \ldots, P_K(q)$ are the first q polygonal numbers. For $K = 3$ the above labeling gives Triangular sum labeling, for $K = 4$ it gives square sum labeling and so on.*

Definition 2.3. *A graph G is called a PSG if it admits PSL.*

Theorem 2.4. *Olive trees are $PSG's$.*

Proof. Let u_0 be the root vertex. Let $u_{11}, u_{12}, \ldots, u_{1n}$ be the vertices in the first level. Let $u_{22}, u_{23}, \ldots, u_{2n}$ be the $n-1$ vertices in the second level and so on. Let u_{nn} be the unique vertex in the n^{th} level. Then $u_0u_{1r}, 1 \le r \le n$; $u_{1r}u_{2r}, 2 \le r \le n$; $u_{2r}u_{3r}, 3 \le r \le n; \ldots u_{n-1n}u_{nn}$ are the edges in the corresponding levels. Then G has $\frac{n(n+1)}{2}$ edges and $\frac{n(n+1)}{2} + 1$ vertices.
Define $h : V(G) \to N$ by

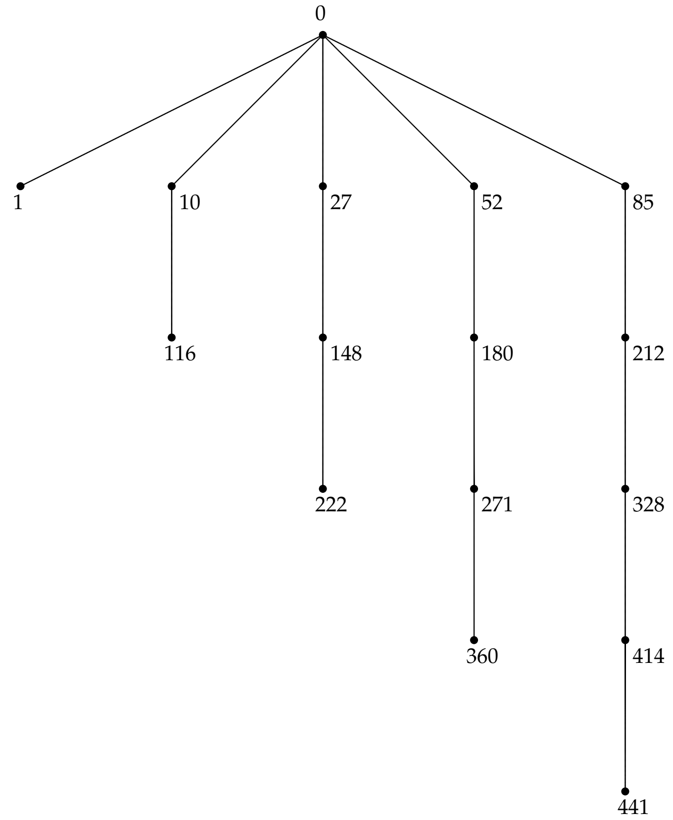
$$\begin{aligned} h(u_0) &= 0 \\ h(u_{mr}) &= P_K\left[(m-1)n - \frac{m(m-1)}{2} + 1\right] - h(u_{m-1r}) \end{aligned}$$

where, m (represent each level) $= 1, 2, 3, \ldots, n; r = m, m+1, \ldots, n$. This induces the edge labels as follows:

$$\begin{aligned} h^+(u_0u_{1r}) &= h(u_0) + h(u_{1r}) \\ &= 0 + P_K(r) \\ &= P_K(r), \qquad 1 \le r \le n \\ h^+(u_{1r}u_{2r}) &= h(u_{1r}) + h(u_{2r}) \\ &= h(u_{1r}) + P_K[n-1+r] - h(u_{1r}) \\ &= P_K[n-1+r], \qquad 1 \le r \le n-1 \\ h^+(u_{2r}u_{3r}) &= h(u_{2r}) + h(u_{3r}) \\ &= h(u_{2r}) + P_K[2n-3+r] - h(u_{2r}) \\ &= P_K[2n-3+r], \qquad 1 \le r \le n-2 \\ &\vdots \\ h^+(u_{n-1n}u_{nn}) &= h(u_{n-1n}) + h(u_{nn}) \\ &= h(u_{n-1n}) + P_K\left[(n-1)n - \frac{(n-1)n}{2} + n\right] - h(u_{n-1n}) \\ &= P_K\left[\frac{n(n+1)}{2}\right] \end{aligned}$$

Hence the injective function h induces the first $P_K\left[\frac{n(n+1)}{2}\right]$ edge labels. Therefore Olive trees are $PSG's$. □

Example 2.5. *The decagonal sum labeling of Olive tree with 16 vertices is shown in the following figure.*



Theorem 2.6. *Shrub graph $St(n_1, n_2, \ldots, n_m)$ is a PSG.*

Proof. Let $V(St(n_1, n_2, \ldots, n_m)) = \{v, v_r, v_{rs} : 1 \leq r \leq m, 1 \leq s \leq n_r\}$ and $E(St(n_1, n_2, \ldots, n_m)) = \{vv_r, v_rv_{rs} : 1 \leq r \leq m, 1 \leq s \leq n_r\}$. There fore $St(n_1, n_2, \ldots, n_m)$ has $m + n_1 + n_2 + \cdots + n_m + 1$ vertices and $m + n_1 + n_2 + \cdots + n_m$ edges.
Define $h : V(St(n_1, n_2, \ldots, n_m)) \to N$ by

$$\begin{aligned}
h(v) &= 0 \\
h(v_r) &= \frac{1}{2}[(K-2)r^2 - (K-4)r], \qquad 1 \leq r \leq m. \\
h(v_{1s}) &= \frac{1}{2}[(k-2)(m+s)^2 - (K-2) - (k-4)(m+s-1)], \quad 1 \leq s \leq n_1. \\
h(v_{2s}) &= \frac{1}{2}[(k-2)(m+n_1+s)^2 - (K-2) - 4(k-4)(m+n_1+s-2)], \qquad 1 \leq s \leq n_2. \\
&\vdots \\
h(v_{rs}) &= \frac{1}{2}[(k-2)(m+n_1+n_2+\cdots+n_{r-1}+s)^2 - (K-2)r^2 - \\
&\quad (K-4)(m+n_1+n_2+\cdots+n_{r-1}+s-r)], \qquad 1 \leq r \leq m, 1 \leq s \leq n_r
\end{aligned}$$

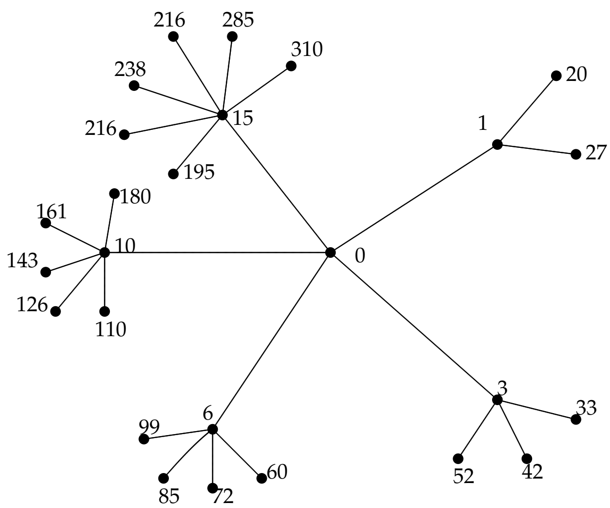
This induces the edge labels

$$\begin{aligned}
h^+(vv_r) &= h(v) + h(v_r) \\
&= 0 + \frac{1}{2}[(K-2)r^2 - (K-4)r] = P_K(r), \quad 1 \leq r \leq m
\end{aligned}$$

$$\begin{aligned}
h^{+}(v_1v_{1s}) &= h(v_1)+h(v_{1s}) \\
&= \frac{1}{2}[(K-2)-(K-4)]+\frac{1}{2}[(K-2)(m+s)^2-(K-2)-(K-4)(m+s-1)] \\
&= P_K(m+s), \quad 1\le s\le n_1 \\
h^{+}(v_2v_{2s}) &= h(v_2)+h(v_{2s}) \\
&= \frac{1}{2}[(K-2)4-(K-4)2]+\frac{1}{2}[(K-2)(m+n_1+s)^2-4(K-2)-(K-4)(m+n_1+s-2)] \\
&= P_K(m+n_1+s), \qquad 1\le s\le n_2 \\
&\vdots \\
h^{+}(v_mv_{ms}) &= \frac{1}{2}[(K-2)m^2-(K-4)m]+\frac{1}{2}[(K-2)(m+n_1+n_2+\cdots+n_{m-1}+s)^2-(K-2)m^2- \\
&\quad (K-4)(n_1+n_2+\cdots+n_{m-1}+s)] \\
&= P_K(m+n_1+n_2+\cdots+n_{m-1}+s), \qquad 1\le s\le n_r
\end{aligned}$$

Clearly the edges labels are the first $m+n_1+n_2+\cdots+n_m$ polygonal numbers. Hence Shrub graph $St(n_1,n_2,\ldots,n_m)$ is a PSG. □

Example 2.7. *The triangular sum labeling of $St(2,3,4,5,6)$ as shown in the following figure.*



Theorem 2.8. *Banana tree $Bt(n_1,n_2,\ldots,n_m)$ is a PSG.*

Proof. Let $V(Bt(n_1,n_2,\ldots,n_m))=\{v,v_r,w_r,w_{rs}:1\le r\le m,1\le s\le n_r-1\}$, $E(Bt(n_1,n_2,\ldots,n_m))=\{vv_r,v_rw_r,w_rw_{rs}:1\le r\le m,1\le s\le n_r-1\}$, $Bt(n_1,n_2,\ldots,n_m)$ has $2m+n_1+n_2+\cdots+n_m+1$ vertices and $2m+n_1+n_2+\cdots+n_m$ edges.

Define $h:V(Bt(n_1,n_2,\ldots,n_m))\to N$ by

$$h(v) = 0,$$

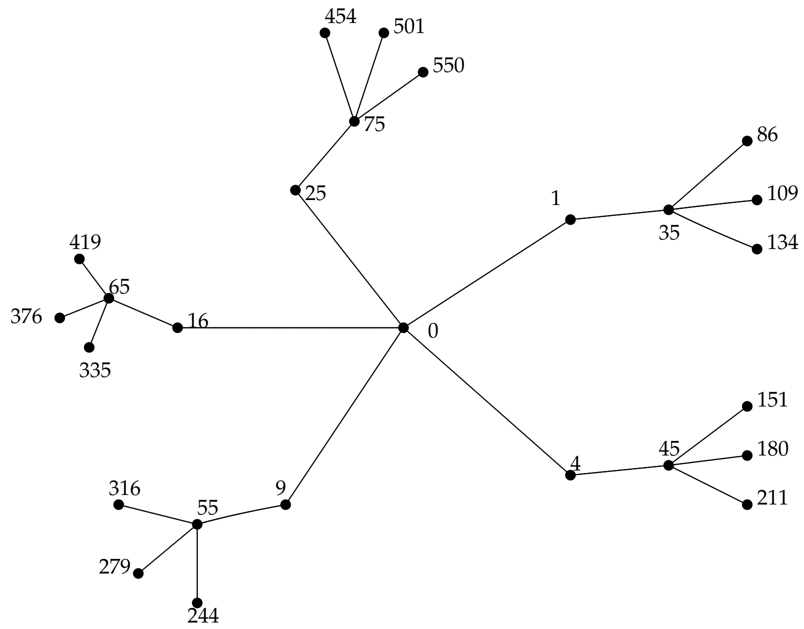
$$\begin{aligned}
h(v_r) &= \frac{1}{2}[(K-2)r^2-(k-4)r], \quad 1 \le r \le m \\
h(w_r) &= \frac{1}{2}[(K-2)(m^2+2mr)-(K-4)m], \qquad 1 \le r \le m \\
h(w_{1s}) &= P_K(2m+s)-h(w_1), \qquad 1 \le s \le n_1-1 \\
h(w_{2s}) &= P_K(2m+n_1-1+s)-h(w_2), \qquad 1 \le s \le n_2-1 \\
h(w_{3s}) &= P_K(2m+n_1+n_2-2+s)-h(w_3), \qquad 1 \le s \le n_3-1 \\
&\vdots \\
h(w_{rs}) &= P_K(2m+s+n_1+n_2+\cdots+n_{r-1}-(r-1))-h(w_r), \quad 1 \le r \le m \quad ; 1 \le s \le n_r-1
\end{aligned}$$

The induced edge labels are

$$\begin{aligned}
h^+(vv_r) &= h(v)+h(v_r) \\
&= 0+\frac{1}{2}[(K-2)r^2-(K-4)r] = P_K(r), \qquad 1 \le r \le m \\
h^+(v_rw_r) &= h(v_r)+h(w_r) \\
&= \frac{1}{2}[(K-2)r^2-(k-4)r]+\frac{1}{2}[(K-2)(m^2+2mr)-(K-4)m] \\
&= P_K(2r), \qquad 1 \le r \le m \\
h^+(w_1w_{1s}) &= h(w_1)+h(w_{1s}) \\
&= h(w_1)+P_K(2m+s)-h(w_1) \\
&= P_K(2m+s), \qquad 1 \le s \le n_1-1 \\
h^+(w_2w_{2s}) &= h(w_2)+h(w_{2s}) \\
&= h(w_2)+P_K(2m+n_1-1+s)-h(w_2) \\
&= P_K(2m+n_1-1+s), \qquad 1 \le s \le n_2-1 \\
&\vdots \\
h^+(w_rw_{rs}) &= h(w_r)+h(w_{rs}) \\
&= h(w_r)+P_K(2m+s+n_1+n_2+\cdots+n_{r-1}-(r-1))-h(w_r) \\
&= P_K(2m+s+n_1+n_2+\cdots+n_{r-1}-(r-1)), \qquad 1 \le r \le m,\ 1 \le s \le n_r-1
\end{aligned}$$

Clearly the edge labels are the first $2m+n_1+n_2+\cdots+n_m$Polygonal numbers. Hence $Bt(n_1,n_2,\ldots,n_m)$ is a PSG □

Example 2.9. *The square sum labeling of $Bt(4,4,4,4,4)$ as shown in the following figure.*



Theorem 2.10. *Caterpillars $S(n_1, n_2, \ldots, n_m)$ is a PSG's for all $n_r \geq 1$.*

Proof. Let $P_m : u_1u_2u_3 \ldots u_m$ be a path with m vertices. From each vertex $u_s, s = 1, 2, 3, \ldots, m$ there are $n_r, r = 1, 2, 3 \ldots, m$ pendent vertices say $u_{s1}, u_{s2}, u_{s3}, \ldots, u_{sn_r}$. The resulting graph is known as caterpillar and is represented as $S(n_1, n_2, \ldots, n_m)$.

The caterpillar graph $S(n_1, n_2, \ldots, n_m) = G$ has $m + n_1 + n_2 + n_3 + \cdots + n_m$ vertices and $q = m - 1 + n_1 + n_2 + n_3 + \cdots + n_m$ edges.

Define $h : V(G) \to N$ by

$$\begin{aligned}
h(u_1) &= 0 \\
h(u_s) &= \frac{1}{2}[(K-2)(s-1)^2 - (k-4)s] - h(u_{s-1}), \qquad 2 \leq s \leq m \\
h(u_{1r}) &= \frac{1}{2}[(K-2)(m-1+r)^2 - (K-4)(m-1+r))], \qquad 1 \leq r \leq n_1 \\
h(u_{sr}) &= \frac{1}{2}[(K-2)(m-1+n_1+n_2+n_3+\cdots+n_{s-1}+r)^2 - \\
&\quad (K-4)(m-1+n_1+n_2+n_3+\cdots+n_{s-1}+r)] - h(u_s), \;\; r = 2,3,4,\ldots,n_s\,; s = 2,3,\ldots,m.
\end{aligned}$$

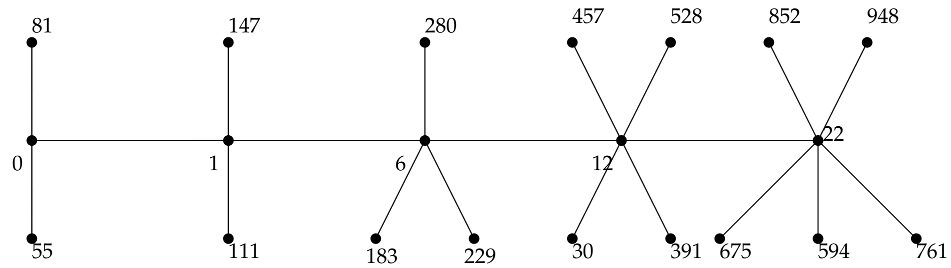
and the induced edge labels are

$$\begin{aligned}
h^+(u_1u_2) &= P_K(1) \\
h^+(u_su_{s+1}) &= h(u_s) + h(u_{s+1}) \\
&= P_K(s), \qquad s = 2,3,4,\ldots,m-1 \\
h^+(u_1u_{1r}) &= h(u_1) + h(u_{1r})
\end{aligned}$$

$$
\begin{aligned}
&= P_K(m-1+r), \qquad 1 \le r \le n_1 \\
h^+(u_s u_{sr}) &= h(u_s) + h(u_{sr}) = P_K(m-1+n_1+n_2+n_3+\cdots+n_{s-1}+r)
\end{aligned}
$$

where, $r = 1,2,3,\ldots,n_s$ $s = 2,3,4,\ldots,m$. As a result the induced edge labels represent the first q polygonal numbers. Hence Caterpillars $S(n_1,n_2,\ldots,n_m)$ are PSG for all $n_r \geq 1$ □

Example 2.11. *The Heptagonal sum labeling of $S(2,2,3,4,5)$ as shown in the following figure*



Theorem 2.12. *$H-$ graphs are $PSG's$.*

Proof. Let $u_1,u_2,u_3,\ldots,u_n$ be the n vertices of the first copy of P_n and $v_1,v_2,v_3,\ldots,v_n$ be the vertices in the second copy of P_n. Let $u_r v_s \in \{1,2,3,\ldots n\}$ be the connecting edge. This graph has $2n-1$ edges. Define $h : V(H) \to N$ by

$$
\begin{aligned}
h(u_r) &= 0 \\
h(u_{r-i}) &= P_K(i) - h(u_{r-i+1}), \qquad i = 1,2,3,\ldots r-1 \\
h(u_{r+1}) &= P_K(r+i-1) - h(u_{r+i-1}), \qquad i = 1,2,3,\ldots n-r \\
h(v_s) &= P_K(n) \\
h(v_{s-i}) &= P_K(n+i) - h(v_{s-i+1}), \qquad i = 1,2,3,\ldots,s-1 \\
h(v_{s+i}) &= P_K(n+s-1+i) - h(v_{s+i-1}), \qquad i = 1,2,3,\ldots,n-s
\end{aligned}
$$

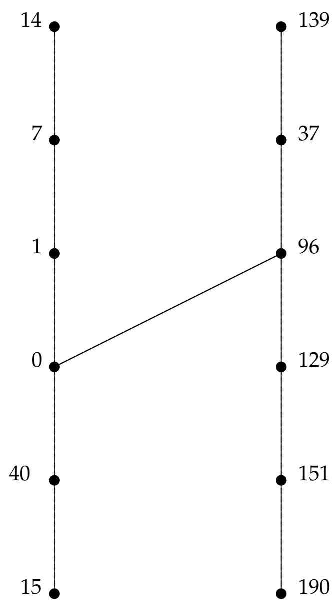
and the induced edge labels are

$$
\begin{aligned}
h^+(u_r u_{r-1}) &= h(u_r) + h(u_{r-1}) = P_K(1) \\
h^+(u_{r-i} u_{r-(i+1)}) &= h(u_{r-i}) + h(u_{r-(i+1)}) \\
&= P_K(i+1) \qquad ;\ i = 1,2,3,\ldots r-2 \\
h^+(u_r u_{r+1}) &= h(u_r) + h(u_{r+1}) \\
&= P_K(r) \\
h^+(u_{r+i} u_{r+(i+1)}) &= h(u_{r+i}) + h(u_{r+(i+1)}) \\
&= P_K(n) \\
h^+(v_s v_{s-1}) &= h(v_s) + h(v_{s-1})
\end{aligned}
$$

$$
\begin{aligned}
&= P_K(n+1)\\
h^+(v_{s-i}v_{s-(i+1)}) &= h(v_{s-i}) + h(v_{s-(i+1)})\\
&= P_K(n+(i+1)) \qquad ;\ i = 1,2,3,\ldots s-2\\
h^+(v_s v_{s+1}) &= h(v_s) + h(v_{s+1})\\
&= P_K(n+s)\\
h^+(v_{s+i}v_{s+(i+1)}) &= h(v_{s+i}) + h(v_{s+(i+1)})\\
&= P_K(n+s+i) \qquad ;\ i = 1,2,3,\ldots n-(s+1)
\end{aligned}
$$

Clearly h is an injective function that establishes $h^+(E(G)) = P_K(1), P_K(2), P_K(3), \ldots P_K(2n-1)$. □

Example 2.13. *The Octagonal sum labeling of $H-$ graph obtained from two isomorphic copies of P_6 as shown in the following figure*



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