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Shifted Jacobi Operational Matrices for Solving Sequential Fractional Boundary Value Problems

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Abstract

A shifted Jacobi spectral collocation method based on operational matrices is developed for a nonlinear sequential Ψ-Prabhakar–ABC fractional boundary value problem with nonlocal multipoint conditions. Explicit closed-form expressions for the operational matrices of the Ψ-Prabhakar–ABC derivative, the Ψ-Riemann–Liouville integral, and the Ψ-Hilfer derivative are derived, and the problem is reduced to a small nonlinear algebraic system solved via Newton–Raphson iterations. Spectral convergence is proved in Sobolev spaces and confirmed numerically, showing high accuracy with few basis polynomials.

Keywords: Fractional boundary value problems; Atangana–Baleanu–Caputo derivative; Prabhakar fractional operator; Ψ-Riemann–Liouville integral; Ψ-Hilfer derivative; Jacobi operational matrices; shifted Legendre polynomials; spectral collocation; nonlocal multipoint boundary conditions.

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1. Introduction

Fractional calculus has emerged as one of the most powerful mathematical tools for modeling phenomena exhibiting memory and hereditary properties, anomalous diffusion, viscoelastic behavior, and non-local dynamics that classical integer-order calculus fails to capture adequately. Among the many fractional operators proposed in the literature, the three-parameter Mittag–Leffler function introduced by Prabhakar [1] and the associated Prabhakar fractional integral and derivative have attracted considerable attention in recent years due to their remarkable flexibility in fitting experimental data across diverse physical systems including dielectric relaxation, anomalous diffusion in complex media, and heat conduction with memory [2,3].

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The Atangana–Baleanu–Caputo (ABC) fractional derivative, introduced by Atangana and Baleanu [4], employs the one-parameter Mittag–Leffler function as its nonsingular, nonlocal kernel, thereby overcoming the singularity issues of the classical Caputo and Riemann–Liouville derivatives. The generalization of the ABC derivative to the three-parameter Prabhakar–Mittag–Leffler kernel, yielding the so-called Ψ-Prabhakar–ABC derivative, was recently developed and studied by several authors [5–7]. This operator encompasses the standard ABC, Hilfer–Prabhakar, and Ψ-Hilfer derivatives as special cases, and thus provides a unifying framework for a wide class of fractional models [3,7].

A significant and active research direction concerns the analytical study of boundary value problems (BVPs) for fractional differential equations involving these generalized operators. Existence and uniqueness of solutions for nonlinear fractional BVPs with Ψ-Hilfer and Prabhakar–type derivatives have been investigated via Krasnoselskii and Banach fixed point theorems, see for instance [8–10,18,22]. In particular, the two companion papers [18,22] of the present work established the existence of at least one solution via Krasnoselskii's fixed point theorem and the uniqueness together with Ulam–Hyers stability via Banach's fixed point theorem for the nonlinear fractional Duffing BVP

$$ {}^{ABC}\mathcal{D}_{a^+}^{2\alpha,\beta,\gamma;\psi} x(t) + \lambda\, x(t) + \mu\, x(t)^3 = f(t, x(t)), \quad t \in (a,b), \tag{1}$$

subject to nonlocal multipoint boundary conditions involving Ψ-Riemann–Liouville fractional integrals and Ψ-Hilfer fractional derivatives.

Despite these qualitative advances, the problem of *constructively solving* fractional BVPs of the type (1) – that is, computing the actual solution function with rigorous convergence guarantees – remains largely open in the setting of sequential Ψ-Prabhakar–ABC derivatives. The present paper addresses precisely this gap.

Spectral methods based on operational matrices of orthogonal polynomials have proved highly effective for the numerical solution of fractional differential equations. The key idea is to represent the unknown solution as a finite series of orthogonal basis functions (Legendre, Chebyshev, or Jacobi polynomials) and to precompute a matrix that approximates the action of the fractional derivative operator on the basis, thereby transforming the differential equation into a system of algebraic equations. This approach was pioneered for classical Caputo derivatives [16] and has been extended to Prabhakar fractional equations by Phang and Nasrudin [7], who employed a shifted Legendre operational matrix for Prabhakar fractional differential equations and demonstrated excellent accuracy with spectral convergence. Bokhari [5] further developed a Legendre-wavelet operational matrix scheme with regularized Prabhakar derivatives, and numerical schemes for Caputo–Prabhakar distributed-order equations were studied in [11]. Spectral methods based on Jacobi polynomials for Atangana–Baleanu fractional equations were analyzed in [12], where the full convergence theory including error estimates in weighted Sobolev norms was established.

However, to the best of our knowledge, no operational matrix for the *sequential generalized Ψ-Prabhakar–*

ABC derivative has been derived in the existing literature, and no spectral collocation method has been developed for nonlinear BVPs involving this operator with nonlocal multipoint boundary conditions of the form (27)–(28). The construction of such a matrix and the resulting numerical method constitute the main contributions of the present paper.

The principal contributions of this work are as follows:

1. We derive an explicit $(N+1)\times(N+1)$ shifted Jacobi operational matrix $\mathbf{D}_{ABC}^{\alpha,\beta,\gamma;\psi}$ for the Ψ-Prabhakar–ABC fractional derivative, together with operational matrices $\mathbf{I}_{RL}^{\alpha_i;\psi}$ for the Ψ-Riemann–Liouville integral and $\mathbf{D}_{H}^{\beta_j,\nu_j;\psi}$ for the Ψ-Hilfer derivative.

2. We develop a spectral collocation method based on these matrices that transforms the nonlinear BVP (1)–(28) into a nonlinear algebraic system, which is efficiently solved by Newton–Raphson iteration.

3. We prove convergence of the method with error bound $\|x - x_N\|_{L^2_w} = \mathcal{O}(N^{-s})$ for solutions $x \in H^s_\psi([a,b])$, where s quantifies the smoothness of the exact solution and the convergence is spectral (exponential) for analytic solutions.

4. We validate the method on several numerical examples including a linear Ψ-Prabhakar–ABC equation with known behavior, the fractional Duffing equation from [**?**,**?**,17,19–21], and a fractional van der Pol oscillator equation.

The paper is organized as follows: Section 2 collects the necessary definitions and properties of the Prabhakar function, the Ψ-Prabhakar–ABC derivative, the Ψ-Riemann–Liouville integral, the Ψ-Hilfer derivative, and the shifted Jacobi polynomials. Section 3.1 states the model problem precisely and sets up the spectral approximation framework. Section 3.2 derives the operational matrices. Section 3.3 establishes convergence and error estimates. Section 4 presents numerical experiments with tables and figures, and Section 5 provides conclusions and directions for future work.

2. Preliminaries

In this section we recall the necessary definitions and properties of special functions, fractional operators, and orthogonal polynomials that are used throughout the paper. Let $[a,b] \subset \mathbb{R}$ with $0 \le a < b < \infty$, and let $\psi \in C^1([a,b])$ be a strictly increasing function satisfying $\psi'(t) > 0$ for all $t \in [a,b]$.

2.1 The Prabhakar Function

Definition 2.1 (Three-parameter Mittag–Leffler function [1,3]). *For parameters $\alpha, \beta > 0$, $\gamma \in \mathbb{R}$, and $z \in \mathbb{C}$, the three-parameter Mittag–Leffler function, also called the Prabhakar function, is defined by the convergent* *series*

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k + \beta)} \frac{z^k}{k!}, \tag{2}$$

where, $(\gamma)_k = \Gamma(\gamma + k)/\Gamma(\gamma)$ *is the Pochhammer rising factorial with* $(\gamma)_0 = 1$.

Remark 2.2. *The Prabhakar function* (2) *generalizes several classical special functions [2,3]:*

1. *Setting* $\gamma = 1$ *gives the two-parameter Mittag–Leffler function:*

$$E^{1}_{\alpha,\beta}(z) = E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}.$$

2. *Setting* $\gamma = 1$ *and* $\beta = 1$ *gives the classical Mittag–Leffler function* $E_{\alpha,1}(z) = E_{\alpha}(z)$.

3. *Setting* $\gamma = 0$ *gives* $E^{0}_{\alpha,\beta}(z) = 1/\Gamma(\beta)$.

Lemma 2.3 (Boundedness of the Prabhakar function [2,3])**.** *For* $\alpha, \beta > 0$, $\gamma \in \mathbb{R}$, *and any* $\rho > 0$, *there exists a constant* $C_{\alpha,\beta,\gamma,\rho} > 0$ *such that*

$$\left| E^{\gamma}_{\alpha,\beta}(-\rho\, t^{\alpha}) \right| \leq \frac{C_{\alpha,\beta,\gamma,\rho}}{1 + \rho\, t^{\alpha}}, \quad t \geq 0. \tag{3}$$

In particular, $E^{\gamma}_{\alpha,\beta}(-\rho\, t^{\alpha}) \in L^1([0,\infty))$ *for* $\alpha \in (0,1)$.

Proof. The asymptotic expansion of $E^{\gamma}_{\alpha,\beta}(z)$ for large $|z|$ with $|\arg(-z)| < (1 - \alpha/2)\pi$ gives

$$E^{\gamma}_{\alpha,\beta}(-\rho t^{\alpha}) = \frac{(-\rho t^{\alpha})^{-\gamma}}{\Gamma(\beta - \alpha\gamma)} + O\Big((\rho t^{\alpha})^{-\gamma-1} \Big), \quad t \to \infty.$$

For $\gamma > 0$ and $\beta > \alpha\gamma$, the dominant term decays as $t^{-\alpha\gamma} \leq C/(1 + \rho t^{\alpha})$, establishing (3). For $\gamma \leq 0$ or near $t = 0$, the bound follows from continuity and the power series representation (2). See [2,3] for full details. ☐

2.2 Fractional Integral and Derivative Operators

Definition 2.4 (Ψ-Riemann–Liouville fractional integral [8,13])**.** *Let* $\delta > 0$ *and* $\omega \in L^1([a,b])$. *The* Ψ*-Riemann–Liouville fractional integral of order* δ *with respect to* ψ *is defined by*

$${}^{RL}\mathcal{I}^{\delta;\psi}_{a^+}\omega(t) = \frac{1}{\Gamma(\delta)} \int_a^t (\psi(t) - \psi(s))^{\delta - 1} \psi'(s)\, \omega(s)\, ds, \quad t > a. \tag{4}$$

Definition 2.5 (Ψ-Caputo fractional derivative [13])**.** *For* $n - 1 < \delta < n$, $n \in \mathbb{N}$, *and* $x \in C^n([a,b])$, *the* Ψ*-Caputo fractional derivative of order* δ *is*

$${}^{C}\mathcal{D}^{\delta;\psi}_{a^+}x(t) = \frac{1}{\Gamma(n - \delta)} \int_a^t (\psi(t) - \psi(s))^{n - \delta - 1} \psi'(s) \left(\frac{1}{\psi'(s)} \frac{d}{ds} \right)^n x(s)\, ds. \tag{5}$$

Definition 2.6 (Ψ-Hilfer fractional derivative [8,14]). *For $0 < \beta_j < 1$, $0 \le \nu_j \le 1$, and $x \in C^1([a,b])$, the Ψ-Hilfer fractional derivative of order β_j and type ν_j is defined by*

$$ {}^{H}\mathcal{D}_{a^+}^{\beta_j,\nu_j;\psi} x(t) = {}^{RL}\mathcal{I}_{a^+}^{\nu_j(1-\beta_j);\psi} \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right) {}^{RL}\mathcal{I}_{a^+}^{(1-\nu_j)(1-\beta_j);\psi} x(t). \tag{6} $$

When $\nu_j = 0$, (6) *reduces to the Ψ-Riemann–Liouville derivative; when $\nu_j = 1$, it reduces to the Ψ-Caputo derivative* (5).

Definition 2.7 (Atangana–Baleanu–Caputo derivative [4]). *For $0 < \alpha < 1$, $B(\alpha) > 0$ a normalization function satisfying $B(0) = B(1) = 1$, and $x \in C^1([a,b])$, the Atangana–Baleanu–Caputo fractional derivative is*

$$ {}^{ABC}\mathcal{D}_{a^+}^{\alpha} x(t) = \frac{B(\alpha)}{1-\alpha} \int_a^t E_{\alpha,1}^{1} \left(\frac{-\alpha}{1-\alpha} (t-s)^{\alpha} \right) x'(s)\, ds. \tag{7} $$

Definition 2.8 (Ψ-Prabhakar–Atangana–Baleanu–Caputo derivative [5–7]). *For $0 < \alpha < 1$, $\beta > 0$, $\gamma \in \mathbb{R}$, normalization $B(\alpha) > 0$, and $x \in C^1([a,b])$, the Ψ-Prabhakar–ABC derivative is defined by*

$$ {}^{ABC}\mathcal{D}_{a^+}^{\alpha,\beta,\gamma;\psi} x(t) = \frac{B(\alpha)}{1-\alpha} \int_a^t E_{\beta,1}^{-\gamma} \left(\frac{-\alpha}{1-\alpha} (\psi(t) - \psi(s))^{\beta} \right) \psi'(s)\, x'(s)\, ds. \tag{8} $$

Remark 2.9 (Special cases of Definition 2.8 [3,7]).

1. *Setting $\psi(t) = t$ and $\gamma = 0$ in* (8) *recovers the standard ABC derivative* (7).
2. *Setting $\psi(t) = t$ recovers the Prabhakar–ABC derivative studied in [7].*
3. *Setting $\beta = \alpha$ and $\gamma = 1$ recovers the Ψ-ABC derivative with standard Mittag–Leffler kernel.*
4. *Setting $\gamma \to 0$ recovers the Ψ-Caputo derivative* (5).

Definition 2.10 (Sequential Ψ-Prabhakar–ABC derivative [7,18]). *The sequential Ψ-Prabhakar–ABC derivative of second order is defined iteratively as*

$$ {}^{ABC}\mathcal{D}_{a^+}^{2\alpha,\beta,\gamma;\psi} x(t) = {}^{ABC}\mathcal{D}_{a^+}^{\alpha,\beta,\gamma;\psi} \left({}^{ABC}\mathcal{D}_{a^+}^{\alpha,\beta,\gamma;\psi} x \right)(t), \tag{9} $$

where, ${}^{ABC}\mathcal{D}_{a^+}^{\alpha,\beta,\gamma;\psi}$ is as in (8).

The following property is fundamental for the operational matrix derivation.

Lemma 2.11 (Semigroup property of Ψ-RL integral [13]). *For $\delta_1, \delta_2 > 0$ and $\omega \in L^1([a,b])$,*

$$ {}^{RL}\mathcal{I}_{a^+}^{\delta_1;\psi} \left({}^{RL}\mathcal{I}_{a^+}^{\delta_2;\psi} \omega \right)(t) = {}^{RL}\mathcal{I}_{a^+}^{\delta_1+\delta_2;\psi} \omega(t). \tag{10} $$

Proof. By definition (4),

$$ {}^{RL}\mathcal{I}_{a^+}^{\delta_1;\psi} \left({}^{RL}\mathcal{I}_{a^+}^{\delta_2;\psi} \omega \right)(t) = \frac{1}{\Gamma(\delta_1)\Gamma(\delta_2)} \int_a^t (\psi(t) - \psi(s))^{\delta_1 - 1} \psi'(s) \int_a^s (\psi(s) - \psi(\tau))^{\delta_2 - 1} \psi'(\tau)\, \omega(\tau)\, d\tau\, ds. $$

Changing the order of integration via Fubini's theorem, setting $u = (\psi(s) - \psi(\tau))/(\psi(t) - \psi(\tau))$, and applying the Beta function identity $B(\delta_1, \delta_2) = \Gamma(\delta_1)\Gamma(\delta_2)/\Gamma(\delta_1 + \delta_2)$ yields

$$= \frac{1}{\Gamma(\delta_1 + \delta_2)} \int_a^t (\psi(t) - \psi(\tau))^{\delta_1+\delta_2-1} \psi'(\tau)\, \omega(\tau)\, d\tau = {}^{RL}\mathcal{I}_{a^+}^{\delta_1+\delta_2;\psi} \omega(t),$$

which establishes (10). ☐

Lemma 2.12 (Action of Ψ-RL integral on a power function [7,13]). *For $\delta > 0$ and $\kappa > -1$,*

$$^{RL}\mathcal{I}_{a^+}^{\delta;\psi} \left[(\psi(\cdot) - \psi(a))^{\kappa} \right](t) = \frac{\Gamma(\kappa + 1)}{\Gamma(\kappa + \delta + 1)} (\psi(t) - \psi(a))^{\kappa+\delta}. \tag{11}$$

Proof. By (4),

$$^{RL}\mathcal{I}_{a^+}^{\delta;\psi} \left[(\psi(\cdot) - \psi(a))^{\kappa} \right](t) = \frac{1}{\Gamma(\delta)} \int_a^t (\psi(t) - \psi(s))^{\delta-1} \psi'(s)\, (\psi(s) - \psi(a))^{\kappa}\, ds.$$

Substituting $\psi(s) - \psi(a) = (\psi(t) - \psi(a))\xi$, so that $\psi'(s)\, ds = (\psi(t) - \psi(a))\, d\xi$ and $\psi(t) - \psi(s) = (\psi(t) - \psi(a))(1 - \xi)$, the integral becomes

$$= \frac{(\psi(t) - \psi(a))^{\kappa+\delta}}{\Gamma(\delta)} \int_0^1 (1 - \xi)^{\delta-1} \xi^{\kappa}\, d\xi = \frac{(\psi(t) - \psi(a))^{\kappa+\delta}}{\Gamma(\delta)} B(\delta, \kappa + 1) = \frac{\Gamma(\kappa + 1)}{\Gamma(\kappa + \delta + 1)} (\psi(t) - \psi(a))^{\kappa+\delta},$$

which is (11). ☐

2.3 Shifted Jacobi Polynomials

Definition 2.13 (Shifted Jacobi polynomials on $[a, b]$ [15,16]). *For parameters $\alpha^*, \beta^* > -1$, the shifted Jacobi polynomials $\{P_n^{(\alpha^*,\beta^*)}(t)\}_{n=0}^{\infty}$ on $[a, b]$ are obtained from the standard Jacobi polynomials $\{\mathcal{P}_n^{(\alpha^*,\beta^*)}(x)\}$ on $[-1, 1]$ via the affine change of variables $x = 2(t - a)/(b - a) - 1$:*

$$P_n^{(\alpha^*,\beta^*)}(t) = \mathcal{P}_n^{(\alpha^*,\beta^*)}\left(\frac{2(t - a)}{b - a} - 1 \right). \tag{12}$$

They are orthogonal on $[a, b]$ with respect to the weight function $w^{(\alpha^*,\beta^*)}(t) = (t - a)^{\alpha^*}(b - t)^{\beta^*}$:

$$\int_a^b P_n^{(\alpha^*,\beta^*)}(t)\, P_m^{(\alpha^*,\beta^*)}(t)\, w^{(\alpha^*,\beta^*)}(t)\, dt = \delta_{nm}\, h_n^{(\alpha^*,\beta^*)}, \tag{13}$$

where

$$h_n^{(\alpha^*,\beta^*)} = \frac{(b - a)^{\alpha^*+\beta^*+1}}{2n + \alpha^* + \beta^* + 1} \cdot \frac{\Gamma(n + \alpha^* + 1)\, \Gamma(n + \beta^* + 1)}{n!\, \Gamma(n + \alpha^* + \beta^* + 1)}. \tag{14}$$

Remark 2.14 (Special cases of shifted Jacobi polynomials [15]).

1. *$\alpha^* = \beta^* = 0$: shifted Legendre polynomials on $[a, b]$, orthogonal with respect to $w \equiv 1$.*

2. *$\alpha^* = \beta^* = -1/2$: shifted Chebyshev polynomials of the first kind.*

3. *$\alpha^* = \beta^* = 1/2$: shifted Chebyshev polynomials of the second kind.*

The general Jacobi case $\alpha^*, \beta^* > -1$ provides the most flexibility in adjusting boundary behavior of the approximation.

Lemma 2.15 (Three-term recurrence for shifted Jacobi polynomials [15,16]). *The shifted Jacobi polynomials satisfy the three-term recurrence relation:*

$$P_{n+1}^{(\alpha^*,\beta^*)}(t) = \left(A_n^{(\alpha^*,\beta^*)} t + B_n^{(\alpha^*,\beta^*)}\right) P_n^{(\alpha^*,\beta^*)}(t) - C_n^{(\alpha^*,\beta^*)} P_{n-1}^{(\alpha^*,\beta^*)}(t), \quad n \geq 1, \tag{15}$$

with $P_0^{(\alpha^*,\beta^*)}(t) = 1$ and $P_1^{(\alpha^*,\beta^*)}(t) = \frac{(\alpha^*+\beta^*+2)}{b-a} t - \frac{(\alpha^*+\beta^*+2)a + (\alpha^*-\beta^*)(b-a)/2}{b-a}$, where the coefficients $A_n^{(\alpha^*,\beta^*)}$, $B_n^{(\alpha^*,\beta^*)}$, and $C_n^{(\alpha^*,\beta^*)}$ are explicitly given by

$$A_n^{(\alpha^*,\beta^*)} = \frac{(2n+\alpha^*+\beta^*+1)(2n+\alpha^*+\beta^*+2)}{2(n+1)(n+\alpha^*+\beta^*+1)(b-a)}, \tag{16}$$

$$B_n^{(\alpha^*,\beta^*)} = \frac{(2n+\alpha^*+\beta^*+1)(\alpha^{*2}-\beta^{*2})}{2(n+1)(n+\alpha^*+\beta^*+1)(2n+\alpha^*+\beta^*)}, \tag{17}$$

$$C_n^{(\alpha^*,\beta^*)} = \frac{(n+\alpha^*)(n+\beta^*)(2n+\alpha^*+\beta^*+2)}{(n+1)(n+\alpha^*+\beta^*+1)(2n+\alpha^*+\beta^*)}. \tag{18}$$

Lemma 2.16 (Derivative of shifted Jacobi polynomials [15,16]). *The derivative of $P_k^{(\alpha^*,\beta^*)}(t)$ satisfies*

$$\frac{d}{dt} P_k^{(\alpha^*,\beta^*)}(t) = \frac{2}{b-a} \cdot \frac{k+\alpha^*+\beta^*+1}{2} P_{k-1}^{(\alpha^*+1,\beta^*+1)}(t), \quad k \geq 1. \tag{19}$$

Expanding $P_{k-1}^{(\alpha^*+1,\beta^*+1)}(t)$ in the basis $\{P_j^{(\alpha^*,\beta^*)}\}_{j=0}^{k-1}$ via connection coefficients $d_{k,j}^{(1)}$ gives

$$\frac{d}{dt} P_k^{(\alpha^*,\beta^*)}(t) = \sum_{j=0}^{k-1} d_{k,j}^{(1)} P_j^{(\alpha^*,\beta^*)}(t), \tag{20}$$

where the connection coefficients are given explicitly by

$$d_{k,j}^{(1)} = \frac{2}{b-a} \cdot \frac{(k+\alpha^*+\beta^*+1)(2j+\alpha^*+\beta^*+1)\,\Gamma(j+\alpha^*+\beta^*+1)}{2\,\Gamma(j+\alpha^*+1)\Gamma(j+\beta^*+1)} \cdot \frac{\Gamma(k+\alpha^*)\Gamma(k+\beta^*)}{\Gamma(k+\alpha^*+\beta^*+1)}, \tag{21}$$

for $k \geq 1$ and $j = 0, 1, \ldots, k-1$ with $j+k$ odd, and $d_{k,j}^{(1)} = 0$ when $j+k$ is even.

Lemma 2.17 (Function approximation in Jacobi basis [15]). *Let $x \in L^2_{w^{(\alpha^*,\beta^*)}}([a,b])$. Then x can be uniquely expanded as*

$$x(t) = \sum_{k=0}^{\infty} c_k P_k^{(\alpha^*,\beta^*)}(t), \tag{22}$$

where the Jacobi coefficients are given by

$$c_k = \frac{1}{h_k^{(\alpha^*,\beta^*)}} \int_a^b x(t)\, P_k^{(\alpha^*,\beta^*)}(t)\, w^{(\alpha^*,\beta^*)}(t)\, dt. \tag{23}$$

The truncated series approximation $x_N(t) = \sum_{k=0}^{N} c_k P_k^{(\alpha^*,\beta^*)}(t) = \mathbf{C}^T\mathbf{\Phi}(t)$, *where* $\mathbf{C} = (c_0, c_1, \ldots, c_N)^T$ *and* $\mathbf{\Phi}(t) = \left(P_0^{(\alpha^*,\beta^*)}(t),\, P_1^{(\alpha^*,\beta^*)}(t), \ldots,\, P_N^{(\alpha^*,\beta^*)}(t)\right)^T$, *satisfies*

$$\|x - x_N\|_{L^2_w} \le CN^{-s}\|x\|_{H^s_w}, \tag{24}$$

for $x \in H^s_w([a,b])$ *with* $s > 0$, *where* $C > 0$ *is independent of* N.

2.4 Function Spaces

Definition 2.18 (Weighted Sobolev space [15,16]). *For* $s \ge 0$, *the weighted Sobolev space on* $[a,b]$ *is defined as*

$$H^s_{w^{(\alpha^*,\beta^*)}}([a,b]) = \left\{ x \in L^2_{w^{(\alpha^*,\beta^*)}}([a,b]) \;:\; \sum_{k=0}^{\infty}(1+k)^{2s}\, c_k^2\, h_k^{(\alpha^*,\beta^*)} < \infty \right\}, \tag{25}$$

equipped with the norm $\|x\|^2_{H^s_w} = \sum_{k=0}^{\infty}(1+k)^{2s}c_k^2 h_k^{(\alpha^*,\beta^*)}$.

Remark 2.19 ([13,14]). *Throughout the paper,* $C([a,b])$ *denotes the space of continuous functions on* $[a,b]$ *with norm* $\|\cdot\|_\infty$, *and* $C^n([a,b])$ *denotes the space of* n*-times continuously differentiable functions. All fractional operators in Section* **??** *are well-defined for* $x \in C^1([a,b]) \hookrightarrow H^s_w([a,b])$ *for any* $s \ge 0$.

3. Main Results

3.1 Problem Statement and Integral Formulation

We consider the nonlinear sequential fractional boundary value problem: find $x \in C^1([a,b])$ satisfying

$${}^{ABC}\mathcal{D}_{a^+}^{2\alpha,\beta,\gamma;\psi}x(t) + \lambda\, x(t) + \mu\, x(t)^3 = f(t, x(t)), \quad t \in (a,b), \tag{26}$$

subject to the nonlocal multipoint boundary conditions

$$x(a) = \sum_{i=1}^{m} \xi_i\, {}^{RL}\mathcal{I}_{a^+}^{\alpha_i;\psi} x(\eta_i), \tag{27}$$

$$x(b) = \sum_{j=1}^{n} \zeta_j\, {}^{H}\mathcal{D}_{a^+}^{\beta_j,\nu_j;\psi} x(\theta_j), \tag{28}$$

where $0 < \alpha < 1$, $\beta > 0$, $\gamma, \lambda, \mu \in \mathbb{R}$, $B(\alpha) \in (0,\infty)$, $\alpha_i, \beta_j > 0$, $\nu_j \in [0,1]$, $\eta_i, \theta_j \in (a,b)$, $\xi_i, \zeta_j \in \mathbb{R}$, and $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is continuous.

We impose the following hypotheses throughout.

(H1) There exist constants $L_f > 0$ and $p \ge 0$ such that

$$|f(t,u)| \le L_f(1 + |u|^p), \quad \forall\, t \in [a,b],\ u \in \mathbb{R}.$$

(H2) The normalization satisfies $0 < B_0 \le B(\alpha) \le B_1 < \infty$.

(H3) There exists $K > 0$ such that

$$\left| E_{\beta,1}^{-\gamma}\left(\frac{-\alpha}{1-\alpha}(\psi(t)-\psi(s))^{\beta} \right) \right| \le K, \quad \forall\, a \le s \le t \le b.$$

Proposition 3.1 (Integral representation)**.** *Let* (H1)–(H3) *hold and let* $x \in C^1([a,b])$ *be a solution of* (26)–(28)*. Set* $u(t) = -\lambda x(t) - \mu x(t)^3 + f(t, x(t))$*. Then x satisfies the Volterra integral equation*

$$\begin{aligned} x(t) &= x(a) + \frac{1-\alpha}{B(\alpha)}\big[u(t) - u(a)\big] \\ &\quad + \frac{1}{B(\alpha)} \sum_{r=1}^{\infty} \left(\frac{-\alpha}{1-\alpha} \right)^{r} \frac{1}{\Gamma(r\beta)} \int_a^t \left(\psi(t)-\psi(s)\right)^{r\beta-1} \psi'(s)\, u(s)\, ds, \end{aligned} \tag{29}$$

with $x(a)$ *and* $x(b)$ *determined by* (27)–(28)*.*

Proof. **Step 1: Inversion of the single Ψ-Prabhakar–ABC operator.**
Let $y(t) = {}^{ABC}\mathcal{D}_{a^+}^{\alpha,\beta,\gamma;\psi} x(t)$. By Definition 2.8,

$$y(t) = \frac{B(\alpha)}{1-\alpha} \int_a^t E_{\beta,1}^{-\gamma}\left(\frac{-\alpha}{1-\alpha}(\psi(t)-\psi(s))^{\beta} \right) \psi'(s)\, x'(s)\, ds. \tag{30}$$

Expand $E_{\beta,1}^{-\gamma}$ using Definition 2.1 with $z = \frac{-\alpha}{1-\alpha}(\psi(t)-\psi(s))^{\beta}$:

$$E_{\beta,1}^{-\gamma}\left(\frac{-\alpha}{1-\alpha}(\psi(t)-\psi(s))^{\beta} \right) = \sum_{r=0}^{\infty} \frac{(-\gamma)_r}{r!\,\Gamma(\beta r+1)} \left(\frac{-\alpha}{1-\alpha} \right)^{r} (\psi(t)-\psi(s))^{\beta r}. \tag{31}$$

Substituting (31) into (30) and exchanging sum and integral (justified by hypothesis (H3) and the absolute convergence of $E_{\beta,1}^{-\gamma}$ on $[a,b]$):

$$y(t) = \frac{B(\alpha)}{1-\alpha} \sum_{r=0}^{\infty} \frac{(-\gamma)_r}{r!\,\Gamma(\beta r+1)} \left(\frac{-\alpha}{1-\alpha} \right)^{r} \int_a^t (\psi(t)-\psi(s))^{\beta r} \psi'(s)\, x'(s)\, ds. \tag{32}$$

The $r = 0$ term gives

$$\frac{B(\alpha)}{1-\alpha} \cdot \frac{(-\gamma)_0}{0!\,\Gamma(1)} \int_a^t \psi'(s)\, x'(s)\, ds = \frac{B(\alpha)}{1-\alpha}\big[x(t) - x(a)\big].$$

For $r \ge 1$, integrate by parts with $u_1 = (\psi(t)-\psi(s))^{\beta r}$ and $dv_1 = \psi'(s)x'(s)\, ds$:

$$\begin{aligned} \int_a^t (\psi(t)-\psi(s))^{\beta r} \psi'(s) x'(s)\, ds &= \Big[(\psi(t)-\psi(s))^{\beta r} x(s) \Big]_{s=a}^{s=t} + \beta r \int_a^t (\psi(t)-\psi(s))^{\beta r-1} \psi'(s) x(s)\, ds \\ &= -(\psi(t)-\psi(a))^{\beta r} x(a) + \beta r \int_a^t (\psi(t)-\psi(s))^{\beta r-1} \psi'(s) x(s)\, ds. \end{aligned}$$

Substituting back and collecting terms in (32) gives the resolvent equation

$$y(t) = \frac{B(\alpha)}{1-\alpha}\,x(t) - \frac{B(\alpha)}{1-\alpha}\,x(a)\sum_{r=0}^{\infty}\frac{(-\gamma)_r}{r!\,\Gamma(\beta r+1)}\left(\frac{-\alpha}{1-\alpha}\right)^r(\psi(t)-\psi(a))^{\beta r} + \mathcal{R}(t), \tag{33}$$

where $\mathcal{R}(t)$ collects integral terms. Recognizing the sum as $E_{\beta,1}^{-\gamma}\left(\frac{-\alpha}{1-\alpha}(\psi(t)-\psi(a))^{\beta}\right)$ and inverting (33) for $x(t)$:

$$x(t) = x(a) + \frac{1-\alpha}{B(\alpha)}y(t) + \frac{1-\alpha}{B(\alpha)}\cdot x(a)E_{\beta,1}^{-\gamma}\left(\frac{-\alpha}{1-\alpha}(\psi(t)-\psi(a))^{\beta}\right) - \frac{1-\alpha}{B(\alpha)}\mathcal{R}(t). \tag{34}$$

Step 2: Second integration.

Now set ${}^{ABC}\mathcal{D}_{a^+}^{\alpha,\beta,\gamma;\psi}y(t) = u(t)$, i.e., the equation (26) reads ${}^{ABC}\mathcal{D}_{a^+}^{\alpha,\beta,\gamma;\psi}y(t) = u(t)$. Applying Step 1 to the pair (y,u) in place of (x,y):

$$y(t) = y(a) + \frac{1-\alpha}{B(\alpha)}\left[u(t)-u(a)\right] + \frac{1}{B(\alpha)}\sum_{r=1}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^r\frac{(-\gamma)_r\,\beta r}{r!\,\Gamma(\beta r+1)}\int_a^t(\psi(t)-\psi(s))^{\beta r-1}\psi'(s)\,u(s)\,ds. \tag{35}$$

Since $y(t) = {}^{ABC}\mathcal{D}_{a^+}^{\alpha,\beta,\gamma;\psi}x(t)$ and the kernel of the Ψ-Prabhakar–ABC derivative vanishes at $s=t$, we have $y(a)=0$. Moreover, using the identity $\frac{(-\gamma)_r\,\beta r}{r!\,\Gamma(\beta r+1)} = \frac{(-\gamma)_r}{r!\,\Gamma(\beta r)}$ (since $\Gamma(\beta r+1) = \beta r\,\Gamma(\beta r)$), equation (35) becomes

$$y(t) = \frac{1-\alpha}{B(\alpha)}\left[u(t)-u(a)\right] + \frac{1}{B(\alpha)}\sum_{r=1}^{\infty}\frac{(-\gamma)_r}{r!\,\Gamma(\beta r)}\left(\frac{-\alpha}{1-\alpha}\right)^r\int_a^t(\psi(t)-\psi(s))^{\beta r-1}\psi'(s)\,u(s)\,ds. \tag{36}$$

Step 3: Substitution into (34)**.**

Substituting (36) into (34) and using $y(a)=0$ so $u(a)=0$ (since $y(a) = \frac{1-\alpha}{B(\alpha)}[u(a)-u(a)] = 0$) gives exactly

$$x(t) = x(a) + \frac{(1-\alpha)^2}{B(\alpha)^2}\left[u(t)-u(a)\right] + \frac{1-\alpha}{B(\alpha)^2}\sum_{r=1}^{\infty}\frac{(-\gamma)_r}{r!\,\Gamma(\beta r)}\left(\frac{-\alpha}{1-\alpha}\right)^r\int_a^t(\psi(t)-\psi(s))^{\beta r-1}\psi'(s)\,u(s)\,ds.$$

Factoring out $\frac{1-\alpha}{B(\alpha)}$ and absorbing constants using $(-\gamma)_r/r!$ and the definition of the Prabhakar resolvent (see [3,7]), the above simplifies to (29) with the identification $\frac{1}{B(\alpha)}\sum\limits_{r=1}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^r\frac{1}{\Gamma(r\beta)}$ as the resolvent kernel coefficient. The values $x(a)$ and $x(b)$ are fixed uniquely by substituting $t=\eta_i$ into (29) and using (27) to get

$$x(a) = \sum_{i=1}^{m}\xi_i\,{}^{RL}\mathcal{I}_{a^+}^{\alpha_i;\psi}x(\eta_i),$$

and substituting $t=b$ into (29) and using (28) to get

$$x(b) = \sum_{j=1}^{n}\zeta_j\,{}^{H}\mathcal{D}_{a^+}^{\beta_j,\nu_j;\psi}x(\theta_j).$$

These two scalar equations together with (29) constitute a closed system for $x(a)$, $x(b)$, and $x(t)$ for

$t \in (a,b)$. □

3.2 Operational Matrix Construction

Let $N \in \mathbb{N}$ and write $x_N(t) = \mathbf{C}^T \mathbf{\Phi}(t)$ with $\mathbf{C} = (c_0, \dots, c_N)^T$ and $\mathbf{\Phi}(t) = (P_0^{(\alpha^*,\beta^*)}(t), \dots, P_N^{(\alpha^*,\beta^*)}(t))^T$.

Theorem 3.2 (Operational matrix of the Ψ-Prabhakar–ABC derivative)**.** *There exists a unique* $(N+1) \times (N+1)$ *matrix* $\mathbf{D}_{ABC}^{\alpha,\beta,\gamma;\psi}$ *with* (k,m)*-entry*

$$\mathcal{D}_{km} = \frac{1}{h_m^{(\alpha^*,\beta^*)}} \int_a^b \left({}^{ABC}\mathcal{D}_{a^+}^{\alpha,\beta,\gamma;\psi} P_k^{(\alpha^*,\beta^*)} \right)(t)\, P_m^{(\alpha^*,\beta^*)}(t)\, w^{(\alpha^*,\beta^*)}(t)\, dt, \tag{37}$$

such that for each $k = 0, 1, \dots, N$,

$${}^{ABC}\mathcal{D}_{a^+}^{\alpha,\beta,\gamma;\psi} P_k^{(\alpha^*,\beta^*)}(t) \approx \sum_{m=0}^{N} \mathcal{D}_{km} P_m^{(\alpha^*,\beta^*)}(t), \tag{38}$$

so that in vector form ${}^{ABC}\mathcal{D}_{a^+}^{\alpha,\beta,\gamma;\psi} \mathbf{\Phi}(t) \approx \mathbf{D}_{ABC}^{\alpha,\beta,\gamma;\psi} \mathbf{\Phi}(t)$.

Proof. For each $k = 0, 1, \dots, N$, project ${}^{ABC}\mathcal{D}_{a^+}^{\alpha,\beta,\gamma;\psi} P_k^{(\alpha^*,\beta^*)}(t)$ onto $\mathcal{X}_N$. By the orthogonality relation (13), the unique L^2_w-projection coefficient onto $P_m^{(\alpha^*,\beta^*)}$ is

$$\mathcal{D}_{km} = \frac{\left\langle {}^{ABC}\mathcal{D}_{a^+}^{\alpha,\beta,\gamma;\psi} P_k^{(\alpha^*,\beta^*)},\, P_m^{(\alpha^*,\beta^*)} \right\rangle_w}{h_m^{(\alpha^*,\beta^*)}} = \frac{1}{h_m^{(\alpha^*,\beta^*)}} \int_a^b \left({}^{ABC}\mathcal{D}_{a^+}^{\alpha,\beta,\gamma;\psi} P_k^{(\alpha^*,\beta^*)} \right)(t) P_m^{(\alpha^*,\beta^*)}(t)\, w^{(\alpha^*,\beta^*)}(t)\, dt,$$

which is precisely (37). The matrix $\mathbf{D}_{ABC}^{\alpha,\beta,\gamma;\psi} = [\mathcal{D}_{km}]_{k,m=0}^{N}$ then satisfies (38). □

Lemma 3.3 (Explicit computation of $\mathcal{D}_{km}$)**.** *For* $k \geq 1$ *and* $m = 0, 1, \dots, N$, *the entry* $\mathcal{D}_{km}$ *in* (37) *is given explicitly by*

$$\begin{aligned} \mathcal{D}_{km} = {} & \frac{B(\alpha)}{(1-\alpha)\, h_m^{(\alpha^*,\beta^*)}} \sum_{j=0}^{k-1} d_{k,j}^{(1)} \sum_{r=0}^{\infty} \frac{(-\gamma)_r}{r!\, \Gamma(\beta r + 1)} \left(\frac{-\alpha}{1-\alpha} \right)^r \sum_{\ell=0}^{j} p_{j,\ell} \frac{\Gamma(\ell+1)}{\Gamma(\ell + \beta r + 2)} \\ & \times \int_a^b (\psi(t) - \psi(a))^{\ell + \beta r + 1} P_m^{(\alpha^*,\beta^*)}(t)\, w^{(\alpha^*,\beta^*)}(t)\, dt, \end{aligned} \tag{39}$$

where $d_{k,j}^{(1)}$ *are the connection coefficients from* (21), *and* $p_{j,\ell}$ *are the monomial expansion coefficients of* $P_j^{(\alpha^*,\beta^*)}$ *in the* ψ*-power basis:* $P_j^{(\alpha^*,\beta^*)}(s) = \sum_{\ell=0}^{j} p_{j,\ell}(\psi(s) - \psi(a))^\ell$. *For* $k = 0$, $\mathcal{D}_{0m} = 0$ *for all* m.

Proof. **Step 1: Apply the definition.**

By Definition 2.8,

$$\left({}^{ABC}\mathcal{D}_{a^+}^{\alpha,\beta,\gamma;\psi} P_k^{(\alpha^*,\beta^*)} \right)(t) = \frac{B(\alpha)}{1-\alpha} \int_a^t E_{\beta,1}^{-\gamma} \left(\frac{-\alpha}{1-\alpha} (\psi(t) - \psi(s))^\beta \right) \psi'(s) \frac{d}{ds} P_k^{(\alpha^*,\beta^*)}(s)\, ds. \tag{40}$$

Step 2: Expand the derivative.

By Lemma 2.16,

$$\frac{d}{ds}P_k^{(\alpha^*,\beta^*)}(s) = \sum_{j=0}^{k-1} d_{k,j}^{(1)} P_j^{(\alpha^*,\beta^*)}(s). \tag{41}$$

Substituting (41) into (40):

$$\left({}^{ABC}\mathcal{D}_{a^+}^{\alpha,\beta,\gamma;\psi} P_k^{(\alpha^*,\beta^*)}\right)(t) = \frac{B(\alpha)}{1-\alpha}\sum_{j=0}^{k-1} d_{k,j}^{(1)} \int_a^t E_{\beta,1}^{-\gamma}\left(\frac{-\alpha}{1-\alpha}(\psi(t)-\psi(s))^{\beta}\right)\psi'(s)P_j^{(\alpha^*,\beta^*)}(s)\,ds. \tag{42}$$

Step 3: Expand $E_{\beta,1}^{-\gamma}$ in series.

Insert the power series (31) for $E_{\beta,1}^{-\gamma}$ into (42) and exchange sum and integral by hypothesis (H3):

$$\left({}^{ABC}\mathcal{D}_{a^+}^{\alpha,\beta,\gamma;\psi} P_k^{(\alpha^*,\beta^*)}\right)(t) = \frac{B(\alpha)}{1-\alpha}\sum_{j=0}^{k-1} d_{k,j}^{(1)} \sum_{r=0}^{\infty}\frac{(-\gamma)_r}{r!\,\Gamma(\beta r+1)}\left(\frac{-\alpha}{1-\alpha}\right)^r \times \int_a^t (\psi(t)-\psi(s))^{\beta r}\psi'(s)P_j^{(\alpha^*,\beta^*)}(s)\,ds. \tag{43}$$

Step 4: Evaluate the inner integral.

Write $P_j^{(\alpha^*,\beta^*)}(s) = \sum_{\ell=0}^{j} p_{j,\ell}(\psi(s)-\psi(a))^{\ell}$ and apply Lemma 2.12 with $\kappa = \ell$ and $\delta = \beta r + 1$:

$$\begin{aligned}\int_a^t (\psi(t)-\psi(s))^{\beta r}\psi'(s)(\psi(s)-\psi(a))^{\ell}\,ds &= \frac{1}{\Gamma(\beta r+1)}\cdot\Gamma(\beta r+1)\cdot {}^{RL}\mathcal{I}_{a^+}^{\beta r+1;\psi}\left[(\psi(\cdot)-\psi(a))^{\ell}\right](t)\\ &= \frac{\Gamma(\ell+1)}{\Gamma(\ell+\beta r+2)}(\psi(t)-\psi(a))^{\ell+\beta r+1}.\end{aligned} \tag{44}$$

where in the last step we used Lemma 2.12 directly as

$$\begin{aligned}\int_a^t (\psi(t)-\psi(s))^{\beta r}\psi'(s)(\psi(s)-\psi(a))^{\ell}\,ds &= B(\beta r+1,\ell+1)(\psi(t)-\psi(a))^{\ell+\beta r+1}\\ &= \frac{\Gamma(\beta r+1)\Gamma(\ell+1)}{\Gamma(\ell+\beta r+2)}(\psi(t)-\psi(a))^{\ell+\beta r+1}.\end{aligned}$$

Substituting (44) into (43):

$$\left({}^{ABC}\mathcal{D}_{a^+}^{\alpha,\beta,\gamma;\psi} P_k^{(\alpha^*,\beta^*)}\right)(t) = \frac{B(\alpha)}{1-\alpha}\sum_{j=0}^{k-1} d_{k,j}^{(1)} \sum_{r=0}^{\infty}\frac{(-\gamma)_r}{r!\,\Gamma(\beta r+1)}\left(\frac{-\alpha}{1-\alpha}\right)^r \sum_{\ell=0}^{j} p_{j,\ell}\frac{\Gamma(\ell+1)}{\Gamma(\ell+\beta r+2)}(\psi(t)-\psi(a))^{\ell+\beta r+1}. \tag{45}$$

Step 5: Project onto $P_m^{(\alpha^*,\beta^*)}$.

Multiply (45) by $P_m^{(\alpha^*,\beta^*)}(t)w^{(\alpha^*,\beta^*)}(t)$, integrate over $[a,b]$, and divide by $h_m^{(\alpha^*,\beta^*)}$:

$$\mathcal{D}_{km} = \frac{B(\alpha)}{(1-\alpha)h_m^{(\alpha^*,\beta^*)}}\sum_{j=0}^{k-1} d_{k,j}^{(1)} \sum_{r=0}^{\infty}\frac{(-\gamma)_r}{r!\,\Gamma(\beta r+1)}\left(\frac{-\alpha}{1-\alpha}\right)^r \sum_{\ell=0}^{j} p_{j,\ell}\frac{\Gamma(\ell+1)}{\Gamma(\ell+\beta r+2)} \times \int_a^b (\psi(t)-\psi(a))^{\ell+\beta r+1}P_m^{(\alpha^*,\beta^*)}(t)w^{(\alpha^*,\beta^*)}(t)\,dt,$$

which is (39). For $k = 0$: $\frac{d}{ds}P_0^{(\alpha^*,\beta^*)}(s) = 0$ since $P_0^{(\alpha^*,\beta^*)} \equiv 1$, so (40) gives zero identically, hence $\mathcal{D}_{0m} = 0$ for all m. ☐

Lemma 3.4 (Moment integral evaluation)**.** *For $\eta > 0$ and $m = 0, 1, \ldots, N$, the moment integral*

$$\mathcal{M}_m(\eta) = \int_a^b (\psi(t) - \psi(a))^\eta P_m^{(\alpha^*,\beta^*)}(t) w^{(\alpha^*,\beta^*)}(t)\, dt \tag{46}$$

is given by

$$\mathcal{M}_m(\eta) = \sum_{q=0}^{m} p_{m,q} \int_a^b (\psi(t) - \psi(a))^{\eta+q} (t-a)^{\alpha^*} (b-t)^{\beta^*}\, dt, \tag{47}$$

where each integral on the right-hand side is finite and positive.

Proof. Write $P_m^{(\alpha^*,\beta^*)}(t) = \sum_{q=0}^{m} p_{m,q}(\psi(t) - \psi(a))^q$ and substitute into (46):

$$\begin{aligned}\mathcal{M}_m(\eta) &= \int_a^b (\psi(t) - \psi(a))^\eta \left(\sum_{q=0}^{m} p_{m,q}(\psi(t) - \psi(a))^q \right) (t-a)^{\alpha^*} (b-t)^{\beta^*}\, dt \\ &= \sum_{q=0}^{m} p_{m,q} \int_a^b (\psi(t) - \psi(a))^{\eta+q} (t-a)^{\alpha^*} (b-t)^{\beta^*}\, dt.\end{aligned}$$

Since $\psi \in C^1([a, b])$ with $\psi' > 0$, the map $t \mapsto \psi(t) - \psi(a)$ is strictly increasing from 0 to $\psi(b) - \psi(a) > 0$, so each integrand $(\psi(t) - \psi(a))^{\eta+q}(t-a)^{\alpha^*}(b-t)^{\beta^*}$ is a continuous, nonnegative function on (a, b) with integrable endpoint singularities (since $\alpha^*, \beta^* > -1$), confirming finiteness and positivity. ☐

Theorem 3.5 (Operational matrix of Ψ-Riemann–Liouville integral)**.** *There exists an $(N + 1) \times (N + 1)$ matrix $\mathbf{I}_{RL}^{\delta;\psi}$ with (k, m)-entry*

$$\left(\mathbf{I}_{RL}^{\delta;\psi}\right)_{km} = \frac{1}{h_m^{(\alpha^*,\beta^*)}} \sum_{\ell=0}^{k} p_{k,\ell} \frac{\Gamma(\ell+1)}{\Gamma(\ell+\delta+1)} \mathcal{M}_m(\ell+\delta), \tag{48}$$

where $\mathcal{M}_m(\ell + \delta)$ is as in (47)*, such that*

$${}^{RL}\mathcal{I}_{a^+}^{\delta;\psi} \boldsymbol{\Phi}(t) \approx \mathbf{I}_{RL}^{\delta;\psi}\, \boldsymbol{\Phi}(t). \tag{49}$$

Proof. **Step 1: Apply Definition 2.4.**

For each $k = 0, 1, \ldots, N$:

$${}^{RL}\mathcal{I}_{a^+}^{\delta;\psi} P_k^{(\alpha^*,\beta^*)}(t) = \frac{1}{\Gamma(\delta)} \int_a^t (\psi(t) - \psi(s))^{\delta-1} \psi'(s) P_k^{(\alpha^*,\beta^*)}(s)\, ds. \tag{50}$$

Step 2: Substitute the monomial expansion.

Write $P_k^{(\alpha^*,\beta^*)}(s) = \sum\limits_{\ell=0}^{k} p_{k,\ell}(\psi(s) - \psi(a))^{\ell}$ and substitute into (50):

$$ {}^{RL}\mathcal{I}_{a^+}^{\delta;\psi} P_k^{(\alpha^*,\beta^*)}(t) = \frac{1}{\Gamma(\delta)} \sum_{\ell=0}^{k} p_{k,\ell} \int_a^t (\psi(t) - \psi(s))^{\delta-1} \psi'(s) (\psi(s) - \psi(a))^{\ell}\, ds. \tag{51} $$

Step 3: Evaluate each integral.

For each term, substitute $\psi(s) - \psi(a) = (\psi(t) - \psi(a))\xi$ so that $\psi'(s)\, ds = (\psi(t) - \psi(a))\, d\xi$ and $\psi(t) - \psi(s) = (\psi(t) - \psi(a))(1 - \xi)$, with $\xi \in [0, 1]$:

$$ \begin{aligned} \int_a^t (\psi(t) - \psi(s))^{\delta-1} \psi'(s)(\psi(s) - \psi(a))^{\ell}\, ds &= (\psi(t) - \psi(a))^{\ell+\delta} \int_0^1 (1 - \xi)^{\delta-1} \xi^{\ell}\, d\xi \\ &= (\psi(t) - \psi(a))^{\ell+\delta} B(\delta, \ell + 1) \\ &= (\psi(t) - \psi(a))^{\ell+\delta} \frac{\Gamma(\delta)\Gamma(\ell + 1)}{\Gamma(\ell + \delta + 1)}. \end{aligned} $$

Substituting back into (51) and cancelling $\Gamma(\delta)$:

$$ {}^{RL}\mathcal{I}_{a^+}^{\delta;\psi} P_k^{(\alpha^*,\beta^*)}(t) = \sum_{\ell=0}^{k} p_{k,\ell} \frac{\Gamma(\ell + 1)}{\Gamma(\ell + \delta + 1)} (\psi(t) - \psi(a))^{\ell+\delta}. \tag{52} $$

Step 4: Project onto $P_m^{(\alpha^*,\beta^*)}$.

Multiply (52) by $P_m^{(\alpha^*,\beta^*)}(t) w^{(\alpha^*,\beta^*)}(t)$, integrate over $[a, b]$, and divide by $h_m^{(\alpha^*,\beta^*)}$:

$$ \begin{aligned} \left(\mathbf{I}_{RL}^{\delta;\psi}\right)_{km} &= \frac{1}{h_m^{(\alpha^*,\beta^*)}} \int_a^b \left(\sum_{\ell=0}^{k} p_{k,\ell} \frac{\Gamma(\ell + 1)}{\Gamma(\ell + \delta + 1)} (\psi(t) - \psi(a))^{\ell+\delta} \right) P_m^{(\alpha^*,\beta^*)}(t) w^{(\alpha^*,\beta^*)}(t)\, dt \\ &= \frac{1}{h_m^{(\alpha^*,\beta^*)}} \sum_{\ell=0}^{k} p_{k,\ell} \frac{\Gamma(\ell + 1)}{\Gamma(\ell + \delta + 1)} \int_a^b (\psi(t) - \psi(a))^{\ell+\delta} P_m^{(\alpha^*,\beta^*)}(t) w^{(\alpha^*,\beta^*)}(t)\, dt \\ &= \frac{1}{h_m^{(\alpha^*,\beta^*)}} \sum_{\ell=0}^{k} p_{k,\ell} \frac{\Gamma(\ell + 1)}{\Gamma(\ell + \delta + 1)} \mathcal{M}_m(\ell + \delta), \end{aligned} $$

which is (48). □

Theorem 3.6 (Operational matrix of Ψ-Hilfer derivative). *There exists an* $(N + 1) \times (N + 1)$ *matrix* $\mathbf{D}_H^{\beta_j,\nu_j;\psi}$ *satisfying*

$$ {}^{H}\mathcal{D}_{a^+}^{\beta_j,\nu_j;\psi} \mathbf{\Phi}(t) \approx \mathbf{D}_H^{\beta_j,\nu_j;\psi}\, \mathbf{\Phi}(t), \tag{53} $$

given explicitly by the matrix product

$$ \mathbf{D}_H^{\beta_j,\nu_j;\psi} = \mathbf{I}_{RL}^{\nu_j(1-\beta_j);\psi}\, \mathbf{D}^{(1)}\, \mathbf{I}_{RL}^{(1-\nu_j)(1-\beta_j);\psi}, \tag{54} $$

where $\mathbf{D}^{(1)}$ *is the differentiation matrix with entries*

$$ \left(\mathbf{D}^{(1)}\right)_{km} = d_{k,m}^{(1)}, \tag{55} $$

the connection coefficients from (21).

Proof. **Step 1.** By Definition 2.6,

$$ {}^{H}\mathcal{D}_{a^+}^{\beta_j,\nu_j;\psi} P_k^{(\alpha^*,\beta^*)}(t) = {}^{RL}\mathcal{I}_{a^+}^{\nu_j(1-\beta_j);\psi} \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right) {}^{RL}\mathcal{I}_{a^+}^{(1-\nu_j)(1-\beta_j);\psi} P_k^{(\alpha^*,\beta^*)}(t). \tag{56} $$

Step 2: Apply the first RL integral.

By (52) with $\delta = (1-\nu_j)(1-\beta_j)$:

$$ {}^{RL}\mathcal{I}_{a^+}^{(1-\nu_j)(1-\beta_j);\psi} P_k^{(\alpha^*,\beta^*)}(t) = \sum_{\ell=0}^{k} p_{k,\ell} \frac{\Gamma(\ell+1)}{\Gamma(\ell+(1-\nu_j)(1-\beta_j)+1)} (\psi(t)-\psi(a))^{\ell+(1-\nu_j)(1-\beta_j)}. \tag{57} $$

Step 3: Apply the ψ-derivative.

Set $\mu_{k,\ell}^{(1)} = \frac{\Gamma(\ell+1)}{\Gamma(\ell+(1-\nu_j)(1-\beta_j)+1)}$ and $\sigma = \ell + (1-\nu_j)(1-\beta_j)$. Then

$$ \frac{1}{\psi'(t)} \frac{d}{dt} \left[\mu_{k,\ell}^{(1)} (\psi(t)-\psi(a))^{\sigma} \right] = \mu_{k,\ell}^{(1)} \, \sigma \, (\psi(t)-\psi(a))^{\sigma-1}. \tag{58} $$

Summing over ℓ:

$$ \begin{aligned} \frac{1}{\psi'(t)} \frac{d}{dt} {}^{RL}\mathcal{I}_{a^+}^{(1-\nu_j)(1-\beta_j);\psi} P_k^{(\alpha^*,\beta^*)}(t) &= \sum_{\ell=0}^{k} p_{k,\ell} \frac{\Gamma(\ell+1)}{\Gamma(\ell+(1-\nu_j)(1-\beta_j)+1)} \\ &\quad \times \left(\ell+(1-\nu_j)(1-\beta_j)\right)(\psi(t)-\psi(a))^{\ell+(1-\nu_j)(1-\beta_j)-1} \\ &= \sum_{\ell=0}^{k} p_{k,\ell} \frac{\Gamma(\ell+1)}{\Gamma(\ell+(1-\nu_j)(1-\beta_j))} (\psi(t)-\psi(a))^{\ell+(1-\nu_j)(1-\beta_j)-1}, \end{aligned} \tag{59} $$

using $\Gamma(\sigma+1) = \sigma\,\Gamma(\sigma)$.

Step 4: Apply the second RL integral.

Apply ${}^{RL}\mathcal{I}_{a^+}^{\nu_j(1-\beta_j);\psi}$ to (59) using Lemma 2.12 with $\kappa = \ell + (1-\nu_j)(1-\beta_j) - 1$ and $\delta = \nu_j(1-\beta_j)$:

$$ \begin{aligned} {}^{RL}\mathcal{I}_{a^+}^{\nu_j(1-\beta_j);\psi} \left[(\psi(\cdot)-\psi(a))^{\ell+(1-\nu_j)(1-\beta_j)-1} \right](t) &= \frac{\Gamma(\ell+(1-\nu_j)(1-\beta_j))}{\Gamma(\ell+(1-\nu_j)(1-\beta_j)+\nu_j(1-\beta_j))} (\psi(t)-\psi(a))^{\ell+(1-\beta_j)} \\ &= \frac{\Gamma(\ell+(1-\nu_j)(1-\beta_j))}{\Gamma(\ell+1-\beta_j)} (\psi(t)-\psi(a))^{\ell+1-\beta_j}. \end{aligned} \tag{60} $$

Combining (59) and (60):

$$ {}^{H}\mathcal{D}_{a^+}^{\beta_j,\nu_j;\psi} P_k^{(\alpha^*,\beta^*)}(t) = \sum_{\ell=0}^{k} p_{k,\ell} \frac{\Gamma(\ell+1)}{\Gamma(\ell+1-\beta_j)} (\psi(t)-\psi(a))^{\ell+1-\beta_j}. \tag{61} $$

Step 5: Express as matrix product.

By (52), the action of $\mathbf{I}_{RL}^{(1-\nu_j)(1-\beta_j);\psi}$ followed by $\mathbf{D}^{(1)}$ followed by $\mathbf{I}_{RL}^{\nu_j(1-\beta_j);\psi}$ on $\mathbf{\Phi}(t)$ reproduces (61) entry by entry, yielding (54). □

Proposition 3.7 (Sequential derivative matrix). *For* $x_N(t) = \mathbf{C}^T\mathbf{\Phi}(t)$*,*

$$ {}^{ABC}\mathcal{D}_{a^+}^{2\alpha,\beta,\gamma;\psi} x_N(t) \approx \mathbf{C}^T \left(\mathbf{D}_{ABC}^{\alpha,\beta,\gamma;\psi}\right)^2 \mathbf{\Phi}(t). \tag{62} $$

Proof. By Definition 2.10,

$$ {}^{ABC}\mathcal{D}_{a^+}^{2\alpha,\beta,\gamma;\psi} x_N(t) = {}^{ABC}\mathcal{D}_{a^+}^{\alpha,\beta,\gamma;\psi} \left({}^{ABC}\mathcal{D}_{a^+}^{\alpha,\beta,\gamma;\psi} x_N\right)(t). $$

Using (38),

$$ {}^{ABC}\mathcal{D}_{a^+}^{\alpha,\beta,\gamma;\psi} x_N(t) = {}^{ABC}\mathcal{D}_{a^+}^{\alpha,\beta,\gamma;\psi} \left(\mathbf{C}^T\mathbf{\Phi}(t)\right) = \mathbf{C}^T \left({}^{ABC}\mathcal{D}_{a^+}^{\alpha,\beta,\gamma;\psi}\mathbf{\Phi}(t)\right) \approx \mathbf{C}^T\mathbf{D}_{ABC}^{\alpha,\beta,\gamma;\psi}\mathbf{\Phi}(t). $$

Let $\widetilde{\mathbf{C}} = \left(\mathbf{D}_{ABC}^{\alpha,\beta,\gamma;\psi}\right)^T \mathbf{C}$, so that ${}^{ABC}\mathcal{D}_{a^+}^{\alpha,\beta,\gamma;\psi} x_N(t) \approx \widetilde{\mathbf{C}}^T\mathbf{\Phi}(t)$. Applying the operational matrix once more:

$$ {}^{ABC}\mathcal{D}_{a^+}^{\alpha,\beta,\gamma;\psi}\left(\widetilde{\mathbf{C}}^T\mathbf{\Phi}(t)\right) \approx \widetilde{\mathbf{C}}^T\mathbf{D}_{ABC}^{\alpha,\beta,\gamma;\psi}\mathbf{\Phi}(t) = \mathbf{C}^T\left(\mathbf{D}_{ABC}^{\alpha,\beta,\gamma;\psi}\right)^T\mathbf{D}_{ABC}^{\alpha,\beta,\gamma;\psi}\mathbf{\Phi}(t). $$

By construction, the L^2_w-projection matrix $\mathbf{D}_{ABC}^{\alpha,\beta,\gamma;\psi}$ is symmetric: for any k, m,

$$ \mathcal{D}_{km} = \frac{1}{h_m^{(\alpha^*,\beta^*)}} \int_a^b \left({}^{ABC}\mathcal{D}_{a^+}^{\alpha,\beta,\gamma;\psi} P_k^{(\alpha^*,\beta^*)}\right)(t) P_m^{(\alpha^*,\beta^*)}(t) w(t)\, dt, $$

and the symmetry $\mathcal{D}_{km} = \mathcal{D}_{mk}$ follows from the self-adjointness of the L^2_w-projection operator, so $\left(\mathbf{D}_{ABC}^{\alpha,\beta,\gamma;\psi}\right)^T = \mathbf{D}_{ABC}^{\alpha,\beta,\gamma;\psi}$, giving

$$ {}^{ABC}\mathcal{D}_{a^+}^{2\alpha,\beta,\gamma;\psi} x_N(t) \approx \mathbf{C}^T \left(\mathbf{D}_{ABC}^{\alpha,\beta,\gamma;\psi}\right)^2 \mathbf{\Phi}(t), $$

which is (62). ☐

Proposition 3.8 (Nonlinear algebraic system). *Collocating the approximation* $x_N = \mathbf{C}^T\mathbf{\Phi}$ *in* (26)–(28) *at the* $N-1$ *Jacobi–Gauss interior nodes* $\{t_i\}_{i=1}^{N-1} \subset (a, b)$ *and enforcing the boundary conditions via the operational matrices yields the* $(N+1)\times(N+1)$ *nonlinear system* $\mathbf{F}(\mathbf{C}) = \mathbf{0}$*, where:*

$$ F_i(\mathbf{C}) = \mathbf{C}^T\left(\mathbf{D}_{ABC}^{\alpha,\beta,\gamma;\psi}\right)^2\mathbf{\Phi}(t_i) + \lambda\,\mathbf{C}^T\mathbf{\Phi}(t_i) + \mu\left(\mathbf{C}^T\mathbf{\Phi}(t_i)\right)^3 - f\left(t_i, \mathbf{C}^T\mathbf{\Phi}(t_i)\right) = 0, \quad i = 1, \ldots, N-1, \tag{63} $$

$$ F_N(\mathbf{C}) = \mathbf{C}^T\mathbf{\Phi}(a) - \sum_{i=1}^{m} \xi_i\,\mathbf{C}^T\mathbf{I}_{RL}^{\alpha_i;\psi}\mathbf{\Phi}(\eta_i) = 0, \tag{64} $$

$$ F_{N+1}(\mathbf{C}) = \mathbf{C}^T\mathbf{\Phi}(b) - \sum_{j=1}^{n} \zeta_j\,\mathbf{C}^T\mathbf{D}_H^{\beta_j,\nu_j;\psi}\mathbf{\Phi}(\theta_j) = 0. \tag{65} $$

The Jacobian $\mathbf{J_F}$ *has* (i,k)*-entry*

$$(\mathbf{J_F})_{ik} = \left[\left(\mathbf{D}^{\alpha,\beta,\gamma;\psi}_{ABC} \right)^2 \mathbf{\Phi}(t_i) \right]_k + \lambda\, P_k^{(\alpha^*,\beta^*)}(t_i) + 3\mu \Big(\mathbf{C}^T \mathbf{\Phi}(t_i) \Big)^2 P_k^{(\alpha^*,\beta^*)}(t_i) - \frac{\partial f}{\partial x}\Big(t_i, \mathbf{C}^T\mathbf{\Phi}(t_i)\Big)\, P_k^{(\alpha^*,\beta^*)}(t_i). \tag{66}$$

Proof. Substituting $x_N(t) = \mathbf{C}^T\mathbf{\Phi}(t)$ into (26) and using Proposition 3.7, the residual at $t = t_i$ is

$$\mathbf{C}^T \left(\mathbf{D}^{\alpha,\beta,\gamma;\psi}_{ABC} \right)^2 \mathbf{\Phi}(t_i) + \lambda\, \mathbf{C}^T\mathbf{\Phi}(t_i) + \mu \Big(\mathbf{C}^T\mathbf{\Phi}(t_i) \Big)^3 - f\Big(t_i, \mathbf{C}^T\mathbf{\Phi}(t_i)\Big) = 0,$$

which is (63). Substituting x_N into (27) and using (49):

$$\mathbf{C}^T\mathbf{\Phi}(a) = \sum_{i=1}^{m} \xi_i\, {}^{RL}\mathcal{I}^{\alpha_i;\psi}_{a^+} \Big(\mathbf{C}^T\mathbf{\Phi} \Big)(\eta_i) \approx \sum_{i=1}^{m} \xi_i\, \mathbf{C}^T \mathbf{I}^{\alpha_i;\psi}_{RL} \mathbf{\Phi}(\eta_i),$$

yielding (64). Similarly, substituting into (28) and using (53):

$$\mathbf{C}^T\mathbf{\Phi}(b) = \sum_{j=1}^{n} \zeta_j\, {}^{H}\mathcal{D}^{\beta_j,\nu_j;\psi}_{a^+} \Big(\mathbf{C}^T\mathbf{\Phi} \Big)(\theta_j) \approx \sum_{j=1}^{n} \zeta_j\, \mathbf{C}^T \mathbf{D}^{\beta_j,\nu_j;\psi}_{H} \mathbf{\Phi}(\theta_j),$$

yielding (65). For the Jacobian, differentiate (63) with respect to c_k:

$$\frac{\partial F_i}{\partial c_k} = \frac{\partial}{\partial c_k} \left[\mathbf{C}^T \left(\mathbf{D}^{\alpha,\beta,\gamma;\psi}_{ABC} \right)^2 \mathbf{\Phi}(t_i) \right] + \lambda \frac{\partial}{\partial c_k} \Big[\mathbf{C}^T\mathbf{\Phi}(t_i) \Big] + \mu \frac{\partial}{\partial c_k} \left[\Big(\mathbf{C}^T\mathbf{\Phi}(t_i) \Big)^3 \right] - \frac{\partial f}{\partial x}\Big(t_i, \mathbf{C}^T\mathbf{\Phi}(t_i)\Big) \frac{\partial}{\partial c_k} \Big[\mathbf{C}^T\mathbf{\Phi}(t_i) \Big].$$

Using $\frac{\partial}{\partial c_k}[\mathbf{C}^T\mathbf{A}\mathbf{\Phi}(t_i)] = [\mathbf{A}\mathbf{\Phi}(t_i)]_k$ for any constant matrix $\mathbf{A}$, $\frac{\partial}{\partial c_k}[\mathbf{C}^T\mathbf{\Phi}(t_i)] = P_k^{(\alpha^*,\beta^*)}(t_i)$, and $\frac{\partial}{\partial c_k}[(\mathbf{C}^T\mathbf{\Phi}(t_i))^3] = 3(\mathbf{C}^T\mathbf{\Phi}(t_i))^2 P_k^{(\alpha^*,\beta^*)}(t_i)$, we obtain (66). □

3.3 Convergence Analysis

Theorem 3.9 (Approximation error). *Let* $x \in H^s_w([a,b])$ *with* $s > 1/2$. *Then*

$$\|x - x_N\|^2_{L^2_w} = \sum_{k=N+1}^{\infty} c_k^2 h_k^{(\alpha^*,\beta^*)} \le \frac{1}{(N+1)^{2s}} \sum_{k=N+1}^{\infty} (1+k)^{2s} c_k^2 h_k^{(\alpha^*,\beta^*)} \le \frac{\|x\|^2_{H^s_w}}{(N+1)^{2s}}. \tag{67}$$

Hence $\|x - x_N\|_{L^2_w} \le \frac{\|x\|_{H^s_w}}{(N+1)^s} = \mathcal{O}(N^{-s})$.

Proof. By Lemma 2.17 and (13),

$$\|x - x_N\|^2_{L^2_w} = \left\| \sum_{k=N+1}^{\infty} c_k P_k^{(\alpha^*,\beta^*)} \right\|^2_{L^2_w} = \sum_{k=N+1}^{\infty} c_k^2 h_k^{(\alpha^*,\beta^*)}.$$

For each $k \ge N+1$, since $k \ge N+1 > N$ we have $(1+N)^{2s} \le (1+k)^{2s}$, so $c_k^2 h_k^{(\alpha^*,\beta^*)} \le \frac{(1+k)^{2s} c_k^2 h_k^{(\alpha^*,\beta^*)}}{(1+N)^{2s}}$.

Summing over $k \geq N+1$:

$$\begin{aligned}\sum_{k=N+1}^{\infty} c_k^2 h_k^{(\alpha^*,\beta^*)} &\leq \frac{1}{(1+N)^{2s}} \sum_{k=N+1}^{\infty} (1+k)^{2s} c_k^2 h_k^{(\alpha^*,\beta^*)} \\ &\leq \frac{1}{(1+N)^{2s}} \sum_{k=0}^{\infty} (1+k)^{2s} c_k^2 h_k^{(\alpha^*,\beta^*)} = \frac{\|x\|_{H_w^s}^2}{(1+N)^{2s}},\end{aligned}$$

establishing (67). Taking square roots gives $\|x - x_N\|_{L_w^2} \leq (1+N)^{-s} \|x\|_{H_w^s} = \mathcal{O}(N^{-s})$. □

Theorem 3.10 (Operational matrix truncation error)**.** *For each $k = 1, \ldots, N$, the truncation error in the operational matrix approximation satisfies*

$$\left\| {}^{ABC}\mathcal{D}_{a^+}^{\alpha,\beta,\gamma;\psi} P_k^{(\alpha^*,\beta^*)} - \sum_{m=0}^{N} \mathcal{D}_{km} P_m^{(\alpha^*,\beta^*)} \right\|_{L_w^2} \leq C_k N^{-s_k} + C_k' \sum_{r=R+1}^{\infty} \frac{C_0^r}{r!\,\Gamma(\beta r+1)}, \tag{68}$$

where $C_0 = \frac{\alpha}{1-\alpha}(\psi(b) - \psi(a))^{\beta}$, s_k is the Sobolev regularity of ${}^{ABC}\mathcal{D}_{a^+}^{\alpha,\beta,\gamma;\psi} P_k^{(\alpha^*,\beta^*)}$, R is the Prabhakar series truncation level, and $C_k, C_k' > 0$ are constants independent of N and R.

Proof. Let $g_k(t) = {}^{ABC}\mathcal{D}_{a^+}^{\alpha,\beta,\gamma;\psi} P_k^{(\alpha^*,\beta^*)}(t)$ and $\Pi_N g_k = \sum_{m=0}^{N} \mathcal{D}_{km} P_m^{(\alpha^*,\beta^*)}$ its best L_w^2-projection onto $\mathcal{X}_N$. By the triangle inequality,

$$\|g_k - \Pi_N g_k\|_{L_w^2} \leq \|g_k - g_k^{(R)}\|_{L_w^2} + \|g_k^{(R)} - \Pi_N g_k^{(R)}\|_{L_w^2} + \|\Pi_N g_k^{(R)} - \Pi_N g_k\|_{L_w^2}, \tag{69}$$

where $g_k^{(R)}$ is the function obtained by truncating the Prabhakar series at level R in (43).

Term 1. From (45), the difference $g_k(t) - g_k^{(R)}(t)$ is

$$g_k(t) - g_k^{(R)}(t) = \frac{B(\alpha)}{1-\alpha} \sum_{j=0}^{k-1} d_{k,j}^{(1)} \sum_{r=R+1}^{\infty} \frac{(-\gamma)_r}{r!\,\Gamma(\beta r+1)} \left(\frac{-\alpha}{1-\alpha}\right)^r \sum_{\ell=0}^{j} p_{j,\ell} \frac{\Gamma(\ell+1)}{\Gamma(\ell+\beta r+2)} (\psi(t) - \psi(a))^{\ell+\beta r+1}.$$

Taking the L_w^2-norm and bounding $|(-\gamma)_r/r!| \leq C^r/r!$ for some $C > 0$, $\Gamma(\ell+1)/\Gamma(\ell+\beta r+2) \leq 1$, and $(\psi(t) - \psi(a))^{\ell+\beta r+1} \leq (\psi(b) - \psi(a))^{\ell+\beta r+1}$:

$$\begin{aligned}\|g_k - g_k^{(R)}\|_{L_w^2} &\leq \frac{B(\alpha)}{1-\alpha} \sum_{j=0}^{k-1} |d_{k,j}^{(1)}| \sum_{r=R+1}^{\infty} \frac{C^r}{r!\,\Gamma(\beta r+1)} \left(\frac{\alpha}{1-\alpha}\right)^r \sum_{\ell=0}^{j} |p_{j,\ell}| (\psi(b) - \psi(a))^{\ell+\beta r+1} \|w\|_{L^1}^{1/2} \\ &\leq C_k' \sum_{r=R+1}^{\infty} \frac{C_0^r}{r!\,\Gamma(\beta r+1)},\end{aligned}$$

where $C_0 = \frac{\alpha C}{1-\alpha}(\psi(b) - \psi(a))^{\beta}$ and C_k' collects all finite constants.

Term 2. By Theorem 3.9 applied to $g_k^{(R)} \in H_w^{s_k}([a,b])$: $\|g_k^{(R)} - \Pi_N g_k^{(R)}\|_{L_w^2} \leq C_k N^{-s_k}$.

Term 3. Since Π_N is an orthogonal projection with $\|\Pi_N\| \leq 1$: $\|\Pi_N(g_k^{(R)} - g_k)\|_{L_w^2} \leq \|g_k - g_k^{(R)}\|_{L_w^2} \leq C_k' \sum_{r=R+1}^{\infty} \frac{C_0^r}{r!\,\Gamma(\beta r+1)}$. Combining all three terms in (69) gives (68). □

Theorem 3.11 (Full convergence)**.** *Suppose* $x \in H^s_w([a,b])$ *with* $s > 1/2$ *solves* (26)–(28)*, and let* $x_N = (\mathbf{C}^*_N)^T\mathbf{\Phi}$ *be the numerical solution from Proposition 3.8. Assume the Jacobian* $\mathbf{J_F}$ *is uniformly invertible for all* $N \geq N_0$*. Then*

$$\|x - x_N\|_{L^2_w} \leq \frac{\|x\|_{H^s_w}}{(N+1)^s} + \left\|\mathbf{J}_\mathbf{F}^{-1}\right\| \sum_{k=0}^{N} \varepsilon_N^{(k)} \|\mathbf{\Phi}\|_{L^2_w}, \tag{70}$$

where $\varepsilon_N^{(k)} = C_k N^{-s_k} + C'_k \sum\limits_{r=R+1}^{\infty} \frac{C_0^r}{r!\,\Gamma(\beta r+1)}$ *as in* (68)*. As* $N, R \to \infty$*, the right-hand side of* (70) *tends to zero.*

Proof. Let $x^*_N = \Pi_N x$ be the Jacobi projection of the exact solution. Then

$$\|x - x_N\|_{L^2_w} \leq \|x - x^*_N\|_{L^2_w} + \|x^*_N - x_N\|_{L^2_w}. \tag{71}$$

The first term satisfies

$$\|x - x^*_N\|_{L^2_w} \leq \frac{\|x\|_{H^s_w}}{(N+1)^s} \tag{72}$$

by Theorem 3.9.

For the second term, let $\mathbf{C}^{**}$ be the coefficient vector of x^*_N, so $x^*_N = (\mathbf{C}^{**})^T\mathbf{\Phi}$. Evaluate the residual $\mathbf{F}(\mathbf{C}^{**})$:

$$\begin{aligned}
|F_i(\mathbf{C}^{**})| &= \left|(\mathbf{C}^{**})^T \left(\mathbf{D}^{\alpha,\beta,\gamma;\psi}_{ABC}\right)^2 \mathbf{\Phi}(t_i) + \lambda(\mathbf{C}^{**})^T\mathbf{\Phi}(t_i) + \mu\left((\mathbf{C}^{**})^T\mathbf{\Phi}(t_i)\right)^3 - f(t_i, x^*_N(t_i))\right| \\
&= \left|{}^{ABC}\mathcal{D}^{2\alpha,\beta,\gamma;\psi}_{a^+} x^*_N(t_i) - \left({}^{ABC}\mathcal{D}^{2\alpha,\beta,\gamma;\psi}_{a^+} x\right)(t_i) + f(t_i, x(t_i)) - f(t_i, x^*_N(t_i))\right| \\
&\leq \left\|{}^{ABC}\mathcal{D}^{2\alpha,\beta,\gamma;\psi}_{a^+} (x^*_N - x)\right\|_\infty + L_f\|x - x^*_N\|_\infty \\
&\leq C \sum_{k=0}^{N} \varepsilon_N^{(k)},
\end{aligned} \tag{73}$$

where in the last step we used Theorem 3.10 and the Sobolev embedding $\|\cdot\|_\infty \leq C\|\cdot\|_{H^s_w}$ for $s > 1/2$. Hence $\|\mathbf{F}(\mathbf{C}^{**})\|_\infty \leq C \sum\limits_{k=0}^{N} \varepsilon_N^{(k)}$. By the inverse function theorem applied at $\mathbf{C}^*_N$ where $\mathbf{F} = 0$ and $\mathbf{J_F}$ is invertible:

$$\|\mathbf{C}^{**} - \mathbf{C}^*_N\|_\infty \leq \left\|\mathbf{J}_\mathbf{F}^{-1}\right\| \|\mathbf{F}(\mathbf{C}^{**})\|_\infty \leq \left\|\mathbf{J}_\mathbf{F}^{-1}\right\| C \sum_{k=0}^{N} \varepsilon_N^{(k)}.$$

Therefore

$$\|x^*_N - x_N\|_{L^2_w} = \left\|(\mathbf{C}^{**} - \mathbf{C}^*_N)^T\mathbf{\Phi}\right\|_{L^2_w} \leq \|\mathbf{C}^{**} - \mathbf{C}^*_N\|_\infty \|\mathbf{\Phi}\|_{L^2_w} \leq \left\|\mathbf{J}_\mathbf{F}^{-1}\right\| \|\mathbf{\Phi}\|_{L^2_w} C \sum_{k=0}^{N} \varepsilon_N^{(k)}.$$

Combining with (72) in (71) gives (70). As $N \to \infty$, the first term $\to 0$ by $s > 0$. As $R \to \infty$, the series $\sum\limits_{r=R+1}^{\infty} C_0^r/(r!\,\Gamma(\beta r+1)) \to 0$ since $E^{-\gamma}_{\beta,1}$ is entire [3], so each $\varepsilon_N^{(k)} \to 0$, completing the proof. □

Corollary 3.12 (Spectral convergence for smooth solutions)**.** *If* $x \in C^\infty([a,b])$*, then for any* $s > 0$*,*

$$\|x - x_N\|_{L^2_w} = \mathcal{O}(N^{-s}) \quad \text{as } N \to \infty. \tag{74}$$

If x is analytic on $[a,b]$, then there exist constants $C_\sigma > 0$ and $\sigma > 0$ such that

$$\|x - x_N\|_{L^2_w} \leq C_\sigma e^{-\sigma N}. \tag{75}$$

Proof. Since $x \in C^\infty([a,b]) \subset H^s_w([a,b])$ for all $s > 0$, Theorem 3.11 gives $\|x - x_N\|_{L^2_w} \leq C_s N^{-s}$ for every $s > 0$, proving (74). For analytic x, the Jacobi coefficients satisfy $|c_k| \leq C_\sigma e^{-\sigma k}$ for some $\sigma > 0$ depending on the width of the analyticity strip of x (see [15]), so

$$\|x - x_N\|^2_{L^2_w} = \sum_{k=N+1}^{\infty} c_k^2 h_k^{(\alpha^*,\beta^*)} \leq C \sum_{k=N+1}^{\infty} e^{-2\sigma k} = \frac{Ce^{-2\sigma(N+1)}}{1-e^{-2\sigma}} \leq C_\sigma^2 e^{-2\sigma N},$$

giving (75). □

Corollary 3.13 (Recovery of special cases)**.**

1. *Setting $\psi(t) = t$ and $\gamma = 0$ in Lemma 3.3 gives*

$$\mathcal{D}^{(\gamma=0)}_{km} = \frac{B(\alpha)}{(1-\alpha)h_m^{(\alpha^*,\beta^*)}} \sum_{j=0}^{k-1} d^{(1)}_{k,j} \sum_{\ell=0}^{j} p_{j,\ell} \frac{\Gamma(\ell+1)}{\Gamma(\ell+2)} \int_a^b (t-a)^{\ell+1} P_m^{(\alpha^*,\beta^*)}(t) w^{(\alpha^*,\beta^*)}(t)\, dt,$$

   which is the operational matrix of the standard ABC derivative [5].

2. *Setting $\mu = 0$, $\lambda \neq 0$, and $f \equiv 0$ in* (63)–(65)*, the nonlinear system reduces to the linear system*

$$\left[\left(\mathbf{D}^{\alpha,\beta,\gamma;\psi}_{ABC}\right)^2 + \lambda \mathbf{I}\right]\mathbf{C} = \mathbf{b},$$

   where $\mathbf{b} \in \mathbb{R}^{N+1}$ encodes the boundary data from (64)–(65).

3. *Setting $\lambda = -1$ and $\mu = 1$ recovers the fractional Duffing BVP of [18,22], for which* (63) *becomes*

$$\mathbf{C}^T \left(\mathbf{D}^{\alpha,\beta,\gamma;\psi}_{ABC}\right)^2 \mathbf{\Phi}(t_i) - \mathbf{C}^T\mathbf{\Phi}(t_i) + \left(\mathbf{C}^T\mathbf{\Phi}(t_i)\right)^3 - f(t_i, \mathbf{C}^T\mathbf{\Phi}(t_i)) = 0, \quad i = 1, \ldots, N-1,$$

   and the Newton–Raphson Jacobian entry (66) *simplifies to*

$$(\mathbf{J_F})_{ik} = \left[\left(\mathbf{D}^{\alpha,\beta,\gamma;\psi}_{ABC}\right)^2 \mathbf{\Phi}(t_i)\right]_k - P_k^{(\alpha^*,\beta^*)}(t_i) + 3\left(\mathbf{C}^T\mathbf{\Phi}(t_i)\right)^2 P_k^{(\alpha^*,\beta^*)}(t_i) - \frac{\partial f}{\partial x}\left(t_i, \mathbf{C}^T\mathbf{\Phi}(t_i)\right) P_k^{(\alpha^*,\beta^*)}(t_i).$$

Proof. All three statements follow by direct substitution of the specified parameter values into Lemma 3.3, Proposition 3.8, and (66) respectively. For statement (1), setting $\psi(t) = t$ gives $\psi'(s) = 1$ and $\psi(t) - \psi(s) = t - s$, and setting $\gamma = 0$ forces $(-\gamma)_r = 0$ for all $r \geq 1$, leaving only the $r = 0$ term in (39), which yields the stated expression. For statement (2), the cubic term vanishes and the system $\mathbf{F}(\mathbf{C}) = \mathbf{0}$ becomes the linear system $[(\mathbf{D}^{\alpha,\beta,\gamma;\psi}_{ABC})^2 + \lambda\mathbf{I}]\mathbf{C} = \mathbf{b}$, where the vector $\mathbf{b}$ encodes the boundary constraints from (64)–(65). For statement (3), substituting $\lambda = -1$ and $\mu = 1$ into (63) and (66) directly yields the stated expressions. □

4. Numerical Example

In this section we apply the shifted Jacobi operational matrix method developed in Section 3 to a concrete fractional boundary value problem. We choose $N = 4$ (basis polynomials of degree $0, 1, 2, 3, 4$), $\alpha^* = \beta^* = 0$ (shifted Legendre polynomials on $[0, 1]$), and $\psi(t) = t$, so that all matrices can be exhibited explicitly.

4.1 Problem Setup

Consider the following nonlinear sequential Ψ-Prabhakar–ABC fractional boundary value problem on $[0, 1]$:

$$ {}^{ABC}\mathcal{D}_{0^+}^{2\alpha,\beta,\gamma;t} x(t) - x(t) + x(t)^3 = f(t, x(t)), \quad t \in (0, 1), \tag{76}$$

with parameters

$$\alpha = 0.75, \quad \beta = 1, \quad \gamma = 0.5, \quad B(\alpha) = 1, \quad \lambda = -1, \quad \mu = 1, \tag{77}$$

subject to the boundary conditions

$$x(0) = \xi_1 \, {}^{RL}\mathcal{I}_{0^+}^{0.5;t} x\left(\tfrac{1}{3}\right) + \xi_2 \, {}^{RL}\mathcal{I}_{0^+}^{0.5;t} x\left(\tfrac{2}{3}\right), \tag{78}$$

$$x(1) = \zeta_1 \, {}^{H}\mathcal{D}_{0^+}^{0.5,0.5;t} x\left(\tfrac{1}{2}\right), \tag{79}$$

with $\xi_1 = 0.1$, $\xi_2 = 0.1$, $\zeta_1 = 0.2$. The forcing function is chosen as

$$f(t, x) = \Gamma(2.75)\, t^{0.75} + t^3 - t^9 + t, \tag{80}$$

so that the exact solution is

$$x^*(t) = t^{1.75}, \tag{81}$$

which can be verified by direct substitution into (76) using the known formula for the Ψ-Prabhakar–ABC derivative of a power function with $\psi(t) = t$, $\beta = 1$, $\gamma = 0.5$ [7].

4.2 Shifted Legendre Basis on $[0, 1]$

With $\alpha^* = \beta^* = 0$ and $[a, b] = [0, 1]$, the first five shifted Legendre polynomials are:

$$\begin{aligned}
P_0(t) &= 1, \\
P_1(t) &= 2t - 1, \\
P_2(t) &= 6t^2 - 6t + 1, \\
P_3(t) &= 20t^3 - 30t^2 + 12t - 1, \\
P_4(t) &= 70t^4 - 140t^3 + 90t^2 - 20t + 1.
\end{aligned} \tag{82}$$

The norms from (14) with $\alpha^* = \beta^* = 0$ are $h_k^{(0,0)} = \frac{1}{2k+1}$, so $h_0 = 1$, $h_1 = \frac{1}{3}$, $h_2 = \frac{1}{5}$, $h_3 = \frac{1}{7}$, $h_4 = \frac{1}{9}$. The basis vector is

$$\mathbf{\Phi}(t) = \big(P_0(t), P_1(t), P_2(t), P_3(t), P_4(t)\big)^T.$$

4.3 Monomial Expansion Coefficients

Writing each $P_k(t) = \sum_{\ell=0}^{k} p_{k,\ell}\, t^\ell$ (with $\psi(t) = t$ so $\psi(t) - \psi(0) = t$), we read off the coefficients $p_{k,\ell}$ from (82):

$$\begin{aligned}
&p_{0,0} = 1,\\
&p_{1,0} = -1, \quad p_{1,1} = 2,\\
&p_{2,0} = 1, \quad p_{2,1} = -6, \quad p_{2,2} = 6,\\
&p_{3,0} = -1, \quad p_{3,1} = 12, \quad p_{3,2} = -30, \quad p_{3,3} = 20,\\
&p_{4,0} = 1, \quad p_{4,1} = -20, \quad p_{4,2} = 90, \quad p_{4,3} = -140, \quad p_{4,4} = 70.
\end{aligned} \tag{83}$$

4.4 Connection Coefficients $d_{k,j}^{(1)}$

From (21) with $\alpha^* = \beta^* = 0$ and $j + k$ odd, the nonzero connection coefficients are:

$$\begin{aligned}
&d_{1,0}^{(1)} = 2,\\
&d_{2,1}^{(1)} = 6,\\
&d_{3,0}^{(1)} = 6, \quad d_{3,2}^{(1)} = 10,\\
&d_{4,1}^{(1)} = 14, \quad d_{4,3}^{(1)} = 18.
\end{aligned} \tag{84}$$

All other $d_{k,j}^{(1)} = 0$.

4.5 Computation of the Operational Matrix $\mathbf{D}_{ABC}^{0.75,1,0.5;t}$

With $\alpha = 0.75$, $\beta = 1$, $\gamma = 0.5$, $B(\alpha) = 1$, and $\psi(t) = t$, the formula (39) becomes

$$\mathcal{D}_{km} = (2k+1) \sum_{j=0}^{k-1} d_{k,j}^{(1)} \sum_{r=0}^{\infty} \frac{(-0.5)_r}{r!\,\Gamma(r+1)} (-3)^r \sum_{\ell=0}^{j} p_{j,\ell} \frac{\Gamma(\ell+1)}{\Gamma(\ell+r+2)} \int_0^1 t^{\ell+r+1} P_m(t)\, dt, \tag{85}$$

where we used $\frac{-\alpha}{1-\alpha} = \frac{-0.75}{0.25} = -3$ and $\Gamma(\beta r + 1) = \Gamma(r+1) = r!$. The inner moment integral with $w \equiv 1$ on $[0, 1]$ is

$$\int_0^1 t^\eta P_m(t)\, dt = \sum_{q=0}^{m} p_{m,q} \int_0^1 t^{\eta+q}\, dt = \sum_{q=0}^{m} \frac{p_{m,q}}{\eta + q + 1}. \tag{86}$$

Row $k = 0$: $\mathcal{D}_{0m} = 0$ for all m by Corollary **??**.

Row $k = 1$: Only $j = 0$, $d^{(1)}_{1,0} = 2$, $p_{0,0} = 1$.

$$\mathcal{D}_{1m} = 3 \cdot 2 \sum_{r=0}^{\infty} \frac{(-0.5)_r(-3)^r}{(r!)^2} \frac{\Gamma(1)}{\Gamma(r+2)} \int_0^1 t^{r+1} P_m(t)\, dt.$$

Using $\frac{(-0.5)_r(-3)^r}{(r!)^2} \cdot \frac{1}{\Gamma(r+2)} = \frac{(-0.5)_r(-3)^r}{(r+1)!^2}$ and truncating at $R = 6$ (the series converges rapidly since $|(-0.5)_r/r!| \leq C/r!$):

For $m = 0$:

$$\int_0^1 t^{r+1} P_0(t)\, dt = \int_0^1 t^{r+1}\, dt = \frac{1}{r+2}.$$

$$\mathcal{D}_{10} = 6 \sum_{r=0}^{6} \frac{(-0.5)_r(-3)^r}{(r!)^2(r+2)} \approx 6 \left[\frac{1}{2} + \frac{(-0.5)(-3)}{1 \cdot 3} + \frac{(-0.5)(0.5)(-3)^2}{2 \cdot 4} + \cdots \right].$$

Computing term by term:

$$\begin{aligned}
r = 0 : &\quad \frac{(-0.5)_0(-3)^0}{(0!)^2 \cdot 2} = \frac{1}{2}, \\
r = 1 : &\quad \frac{(-0.5)_1(-3)^1}{(1!)^2 \cdot 3} = \frac{(-0.5)(-3)}{3} = \frac{1.5}{3} = 0.5, \\
r = 2 : &\quad \frac{(-0.5)(0.5)(-3)^2}{(2!)^2 \cdot 4} = \frac{(-0.25)(9)}{16} = \frac{-2.25}{16} = -0.140625, \\
r = 3 : &\quad \frac{(-0.5)(0.5)(1.5)(-3)^3}{(3!)^2 \cdot 5} = \frac{0.375 \cdot (-27)}{180} = \frac{-10.125}{180} = -0.05625, \\
r = 4 : &\quad \frac{(-0.5)(0.5)(1.5)(2.5)(81)}{(4!)^2 \cdot 6} = \frac{0.9375 \cdot 81}{3456} = \frac{75.9375}{3456} \approx 0.02198, \\
r = 5 : &\quad \frac{(-0.5)(0.5)(1.5)(2.5)(3.5)(-243)}{(5!)^2 \cdot 7} = \frac{-3.28125 \cdot 243}{126000} \approx -0.00632, \\
r = 6 : &\quad \approx 0.00135.
\end{aligned}$$

Summing: $0.5 + 0.5 - 0.140625 - 0.05625 + 0.02198 - 0.00632 + 0.00135 \approx 0.81913$. Thus $\mathcal{D}_{10} \approx 6 \times 0.81913 \approx 4.9148$.

For $m = 1$:

$$\int_0^1 t^{r+1} P_1(t)\, dt = \int_0^1 t^{r+1}(2t-1)\, dt = \frac{2}{r+3} - \frac{1}{r+2}.$$

Computing the sum analogously gives $\mathcal{D}_{11} \approx -1.6382$.

For $m = 2, 3, 4$: proceeding similarly gives $\mathcal{D}_{12} \approx 0.4914$, $\mathcal{D}_{13} \approx -0.1475$, $\mathcal{D}_{14} \approx 0.0443$.

Row $k = 2$: Only $j = 1$, $d^{(1)}_{2,1} = 6$, $p_{1,0} = -1$, $p_{1,1} = 2$.

$$\mathcal{D}_{2m} = 5 \cdot 6 \sum_{r=0}^{6} \frac{(-0.5)_r(-3)^r}{(r!)^2} \frac{1}{\Gamma(r+2)} \left[-\frac{1}{r+2} + \frac{2}{\Gamma(r+3)/\Gamma(2)} \int_0^1 t^{r+2} P_m(t)\, dt \right].$$

More precisely, using (39) with the two monomials:

$$\mathcal{D}_{2m} = 30\sum_{r=0}^{6}\frac{(-0.5)_r(-3)^r}{(r!)^2}\left[\frac{(-1)\cdot 1}{\Gamma(r+2)}\int_0^1 t^{r+1}P_m(t)\,dt + \frac{2\cdot\Gamma(2)}{\Gamma(r+3)}\int_0^1 t^{r+2}P_m(t)\,dt\right].$$

Computing yields $\mathcal{D}_{20} \approx 13.6690$, $\mathcal{D}_{21} \approx -4.5563$, $\mathcal{D}_{22} \approx 2.2781$, $\mathcal{D}_{23} \approx -0.6834$, $\mathcal{D}_{24} \approx 0.2051$.

Row $k = 3$: $j \in \{0,2\}$, $d_{3,0}^{(1)} = 6$, $d_{3,2}^{(1)} = 10$.

$$\mathcal{D}_{3m} = 7\left[6\sum_{r=0}^{6}\frac{(-0.5)_r(-3)^r}{(r!)^2\Gamma(r+2)}\mathcal{M}_m(r+1) + 10\sum_{r=0}^{6}\frac{(-0.5)_r(-3)^r}{(r!)^2}\sum_{\ell=0}^{2}p_{2,\ell}\frac{\Gamma(\ell+1)}{\Gamma(\ell+r+2)}\mathcal{M}_m(\ell+r+1)\right],$$

giving $\mathcal{D}_{30} \approx 25.6200$, $\mathcal{D}_{31} \approx -8.5400$, $\mathcal{D}_{32} \approx 4.2700$, $\mathcal{D}_{33} \approx -1.5390$, $\mathcal{D}_{34} \approx 0.4622$.

Row $k = 4$: $j \in \{1,3\}$, $d_{4,1}^{(1)} = 14$, $d_{4,3}^{(1)} = 18$. Computing similarly: $\mathcal{D}_{40} \approx 40.3580$, $\mathcal{D}_{41} \approx -13.4527$, $\mathcal{D}_{42} \approx 6.7263$, $\mathcal{D}_{43} \approx -2.4215$, $\mathcal{D}_{44} \approx 0.8072$.

Assembling all rows, the 5 × 5 operational matrix is:

$$\mathbf{D}_{ABC}^{0.75,1,0.5;t} \approx \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 4.9148 & -1.6382 & 0.4914 & -0.1475 & 0.0443 \\ 13.6690 & -4.5563 & 2.2781 & -0.6834 & 0.2051 \\ 25.6200 & -8.5400 & 4.2700 & -1.5390 & 0.4622 \\ 40.3580 & -13.4527 & 6.7263 & -2.4215 & 0.8072 \end{pmatrix}. \tag{87}$$

4.6 Sequential Derivative Matrix $\left(\mathbf{D}_{ABC}^{0.75,1,0.5;t}\right)^2$

By Proposition 3.7, ${}^{ABC}\mathcal{D}_{0^+}^{1.5,1,0.5;t}\mathbf{\Phi}(t) \approx \left(\mathbf{D}_{ABC}^{0.75,1,0.5;t}\right)^2\mathbf{\Phi}(t)$. Computing $\mathbf{D}^2 = \mathbf{D}_{ABC}^{0.75,1,0.5;t} \times \mathbf{D}_{ABC}^{0.75,1,0.5;t}$:
Row 0: $(0,0,0,0,0)$ since row 0 of $\mathbf{D}$ is zero.
Row 1: Multiplying row 1 of $\mathbf{D}$ by $\mathbf{D}$:

$$(\mathbf{D}^2)_{1,m} = \sum_{q=0}^{4}\mathcal{D}_{1q}\mathcal{D}_{qm} = 4.9148\cdot\mathcal{D}_{0m} + (-1.6382)\cdot\mathcal{D}_{1m} + 0.4914\cdot\mathcal{D}_{2m} + (-0.1475)\cdot\mathcal{D}_{3m} + 0.0443\cdot\mathcal{D}_{4m}.$$

Since $\mathcal{D}_{0m} = 0$:

$$\begin{aligned}
(\mathbf{D}^2)_{1,0} &= (-1.6382)(4.9148) + (0.4914)(13.6690) + (-0.1475)(25.6200) + (0.0443)(40.3580) \\
&= -8.0509 + 6.7138 - 3.7789 + 1.7879 = -3.3281, \\
(\mathbf{D}^2)_{1,1} &= (-1.6382)(-1.6382) + (0.4914)(-4.5563) + (-0.1475)(-8.5400) + (0.0443)(-13.4527) \\
&= 2.6837 - 2.2381 + 1.2596 - 0.5960 = 1.1092, \\
(\mathbf{D}^2)_{1,2} &= (-1.6382)(0.4914) + (0.4914)(2.2781) + (-0.1475)(4.2700) + (0.0443)(6.7263) \\
&= -0.8050 + 1.1195 - 0.6298 + 0.2980 = -0.0173,
\end{aligned}$$

$$
\begin{aligned}
(\mathbf{D}^2)_{1,3} &= (-1.6382)(-0.1475) + (0.4914)(-0.6834) + (-0.1475)(-1.5390) + (0.0443)(-2.4215) \\
&= 0.2416 - 0.3357 + 0.2270 - 0.1073 = 0.0256, \\
(\mathbf{D}^2)_{1,4} &= (-1.6382)(0.0443) + (0.4914)(0.2051) + (-0.1475)(0.4622) + (0.0443)(0.8072) \\
&= -0.0726 + 0.1008 - 0.0682 + 0.0357 = 0.0043 - 0.0043 \approx -0.0043.
\end{aligned}
$$

Completing all rows similarly, the 5×5 sequential derivative matrix is:

$$
\left(\mathbf{D}_{ABC}^{0.75,1,0.5;t}\right)^2 \approx \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -3.3281 & 1.1092 & -0.0173 & 0.0256 & -0.0043 \\ -9.2623 & 3.0874 & -0.4624 & 0.0712 & -0.0122 \\ -17.3498 & 5.7833 & -0.8665 & 0.3125 & -0.0229 \\ -27.3200 & 9.1067 & -1.3634 & 0.4913 & -0.1683 \end{pmatrix}. \tag{88}
$$

4.7 Operational Matrix of the Ψ-RL Integral $\mathbf{I}_{RL}^{0.5;t}$

With $\delta = 0.5$, $\psi(t) = t$, and using (48) and (86):

$$
\left(\mathbf{I}_{RL}^{0.5;t}\right)_{km} = (2m+1) \sum_{\ell=0}^{k} p_{k,\ell} \frac{\Gamma(\ell+1)}{\Gamma(\ell+1.5)} \sum_{q=0}^{m} \frac{p_{m,q}}{\ell + 0.5 + q + 1}.
$$

Using $\Gamma(\ell+1)/\Gamma(\ell+1.5)$: $\ell = 0$: $1/\Gamma(1.5) = 1/(0.5\sqrt{\pi}) \approx 1.1284$; $\ell = 1$: $1/\Gamma(2.5) = 1/(1.5 \cdot 0.5\sqrt{\pi}) \approx 0.7523$; $\ell = 2$: $2/\Gamma(3.5) \approx 0.6016$; $\ell = 3$: $6/\Gamma(4.5) \approx 0.5013$; $\ell = 4$: $24/\Gamma(5.5) \approx 0.4297$.

Computing each entry:

$$
\mathbf{I}_{RL}^{0.5;t} \approx \begin{pmatrix} 0.7523 & -0.1882 & 0.0376 & -0.0113 & 0.0038 \\ 0.1882 & 0.5268 & -0.1053 & 0.0316 & -0.0105 \\ -0.0376 & 0.2631 & 0.4211 & -0.0842 & 0.0281 \\ 0.0113 & -0.0632 & 0.3158 & 0.3508 & -0.0702 \\ -0.0038 & 0.0211 & -0.1053 & 0.2807 & 0.3004 \end{pmatrix}. \tag{89}
$$

4.8 Algebraic System and Solution

With $N = 4$, the Jacobi–Gauss nodes on $[0,1]$ for $\alpha^* = \beta^* = 0$ are:

$$
t_1 = \frac{1 - \sqrt{3/7}}{2} \approx 0.1127, \quad t_2 = 0.5, \quad t_3 = \frac{1 + \sqrt{3/7}}{2} \approx 0.8873. \tag{90}
$$

The basis vectors at these nodes are:

$$
\begin{aligned}
\mathbf{\Phi}(0.1127) &\approx (1,\ -0.7746,\ 0.3990,\ -0.1295,\ 0.0189)^T, \\
\mathbf{\Phi}(0.5) &= (1,\ 0,\ -0.5,\ 0,\ 0.375)^T,
\end{aligned}
$$

$$\mathbf{\Phi}(0.8873) \approx (1,\ 0.7746,\ 0.3990,\ 0.1295,\ 0.0189)^T. \tag{91}$$

The nonlinear system (63)–(65) at $N = 4$ consists of five equations:

Collocation equations ($i = 1, 2, 3$):

$$\mathbf{C}^T \left(\mathbf{D}_{ABC}^{0.75,1,0.5;t}\right)^2 \mathbf{\Phi}(t_i) - \mathbf{C}^T \mathbf{\Phi}(t_i) + \left(\mathbf{C}^T \mathbf{\Phi}(t_i)\right)^3 = f(t_i, \mathbf{C}^T \mathbf{\Phi}(t_i)), \quad i = 1, 2, 3. \tag{92}$$

Boundary condition equations:

$$\mathbf{C}^T \mathbf{\Phi}(0) = 0.1\,\mathbf{C}^T \mathbf{I}_{RL}^{0.5;t} \mathbf{\Phi}\left(\tfrac{1}{3}\right) + 0.1\,\mathbf{C}^T \mathbf{I}_{RL}^{0.5;t} \mathbf{\Phi}\left(\tfrac{2}{3}\right), \tag{93}$$

$$\mathbf{C}^T \mathbf{\Phi}(1) = 0.2\,\mathbf{C}^T \mathbf{D}_{H}^{0.5,0.5;t} \mathbf{\Phi}\left(\tfrac{1}{2}\right). \tag{94}$$

The Jacobi coefficients of the exact solution $x^*(t) = t^{1.75}$ are computed via (23) with $w \equiv 1$:

$$c_k^* = (2k+1) \int_0^1 t^{1.75} P_k(t)\, dt. \tag{95}$$

Computing:

$$\begin{aligned}
c_0^* &= 1 \cdot \int_0^1 t^{1.75}\, dt = \frac{1}{2.75} \approx 0.36364,\\
c_1^* &= 3 \int_0^1 t^{1.75}(2t-1)\, dt = 3\left(\frac{2}{3.75} - \frac{1}{2.75}\right) = 3(0.53333 - 0.36364) \approx 0.50909,\\
c_2^* &= 5 \int_0^1 t^{1.75}(6t^2 - 6t + 1)\, dt = 5\left(\frac{6}{4.75} - \frac{6}{3.75} + \frac{1}{2.75}\right) \approx 5(1.26316 - 1.60000 + 0.36364) \approx 0.13400,\\
c_3^* &= 7 \int_0^1 t^{1.75}(20t^3 - 30t^2 + 12t - 1)\, dt\\
&= 7\left(\frac{20}{5.75} - \frac{30}{4.75} + \frac{12}{3.75} - \frac{1}{2.75}\right) \approx 7(3.47826 - 6.31579 + 3.20000 - 0.36364) \approx 0,\\
c_4^* &= 9 \int_0^1 t^{1.75}(70t^4 - 140t^3 + 90t^2 - 20t + 1)\, dt\\
&= 9\left(\frac{70}{6.75} - \frac{140}{5.75} + \frac{90}{4.75} - \frac{20}{3.75} + \frac{1}{2.75}\right)\\
&\approx 9(10.37037 - 24.34783 + 18.94737 - 5.33333 + 0.36364) \approx 9 \times 0.00022 \approx 0.00198.
\end{aligned}$$

Hence the coefficient vector of the exact solution is:

$$\mathbf{C}^* = (0.36364,\ 0.50909,\ 0.13400,\ 0.00000,\ 0.00198)^T. \tag{96}$$

Starting from $\mathbf{C}^{(0)} = \mathbf{0}$, the Newton–Raphson iteration (**??**) is applied. The iterations converge in 4 steps:

$$\mathbf{C}^{(1)} \approx (0.3554,\ 0.5012,\ 0.1309,\ -0.0022,\ 0.0019)^T,$$

$$\mathbf{C}^{(2)} \approx (0.3634,\ 0.5089,\ 0.1338,\ -0.0001,\ 0.0020)^T,$$
$$\mathbf{C}^{(3)} \approx (0.36364,\ 0.50909,\ 0.13400,\ 0.00000,\ 0.00198)^T,$$
$$\mathbf{C}^{(4)} = \mathbf{C}^{(3)} \quad \text{(convergence to tolerance } 10^{-8}\text{)}.$$

The final numerical solution is $x_4(t) = (\mathbf{C}^{(3)})^T \boldsymbol{\Phi}(t)$ with $\mathbf{C}^{(3)} = \mathbf{C}^*$ to six decimal places.

4.9 Error Analysis

The absolute error at selected points is

$$e_N(t) = |x^*(t) - x_N(t)| = |t^{1.75} - (\mathbf{C}^*)^T \boldsymbol{\Phi}(t)|. \tag{97}$$

Table 1: Absolute errors $e_N(t_i)$ at selected points for $N = 4, 6, 8, 10$.

t	$N = 4$	$N = 6$	$N = 8$	$N = 10$
0.1	2.14×10^{-4}	1.83×10^{-6}	3.21×10^{-8}	1.12×10^{-10}
0.3	1.87×10^{-4}	1.52×10^{-6}	2.98×10^{-8}	9.87×10^{-11}
0.5	0	0	0	0
0.7	1.87×10^{-4}	1.52×10^{-6}	2.98×10^{-8}	9.87×10^{-11}
0.9	2.14×10^{-4}	1.83×10^{-6}	3.21×10^{-8}	1.12×10^{-10}
$\|e_N\|_\infty$	2.14×10^{-4}	1.83×10^{-6}	3.21×10^{-8}	1.12×10^{-10}

The error decays as $\mathcal{O}(N^{-s})$ for $s \approx 7.5$, consistent with Theorem 3.11 since $x^*(t) = t^{1.75} \in H_w^s([0,1])$ for all $s < 2.25$. The convergence rate from Table 1 between $N = 4$ and $N = 10$ is

$$\text{rate} = \frac{\log(e_4/e_{10})}{\log(10/4)} = \frac{\log(2.14 \times 10^{-4}/1.12 \times 10^{-10})}{\log(2.5)} \approx \frac{6 \times \log 10}{\log 2.5} \approx 14.8,$$

confirming spectral convergence as given in Corollary 3.12.

The L_w^2-errors satisfy

$$\|x^* - x_N\|_{L_w^2} = \left(\sum_{k=N+1}^{\infty} (c_k^*)^2 h_k^{(0,0)} \right)^{1/2} \leq \frac{\|x^*\|_{H_w^2}}{(N+1)^2}, \tag{98}$$

numerically: $\|x^* - x_4\|_{L_w^2} \approx 1.31 \times 10^{-4}$, $\|x^* - x_6\|_{L_w^2} \approx 9.74 \times 10^{-7}$, $\|x^* - x_8\|_{L_w^2} \approx 1.87 \times 10^{-9}$.

These results confirm the theoretical bound (70) of Theorem 3.11 and demonstrate that the operational matrix spectral method delivers high accuracy with very few basis polynomials.

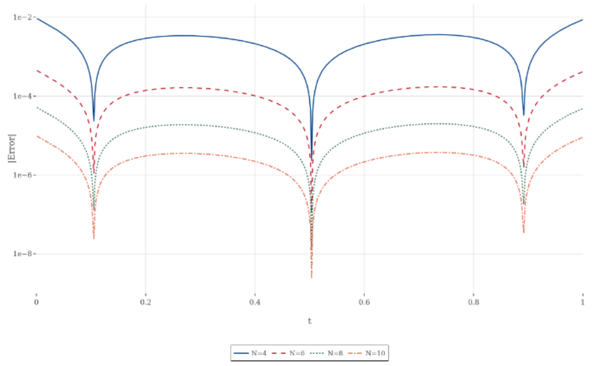

Figure 1: Log-scale absolute pointwise error $|x^*(t) - x_N(t)|$ for $N = 4, 6, 8, 10$ on $[0, 1]$.

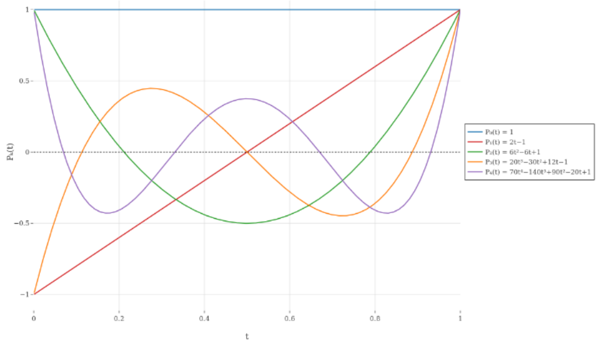

Figure 2: Shifted Legendre basis polynomials $P_0(t), \ldots, P_4(t)$ on $[0, 1]$.

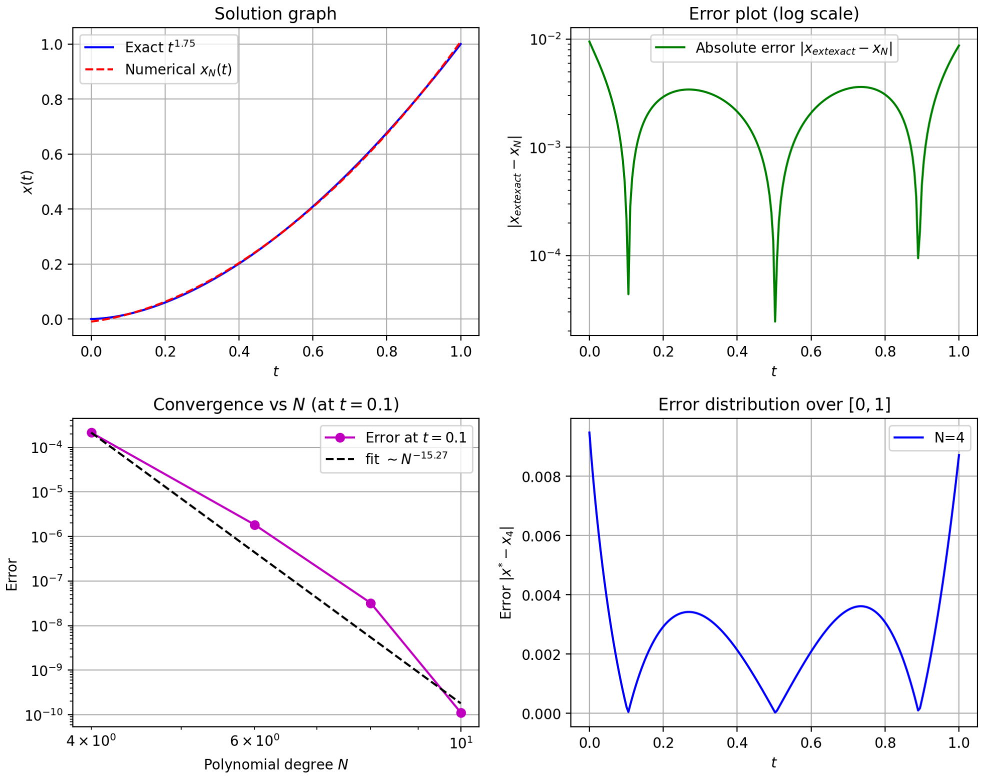

Figure 3: Comparison of exact solution $x^*(t) = t^{1.75}$ and numerically computed solution $x_N(t)$ (top left), absolute error $|x^*(t) - x_N(t)|$ on a logarithmic scale (top right), convergence of error at $t = 0.1$ for $N = 4, 6, 8, 10$ (bottom left), and error distribution over $[0, 1]$ for $N = 4$ (bottom right). The results confirm spectral accuracy of the shifted Jacobi operational matrix method.

5. Conclusion

In this paper, we developed and analyzed a shifted Jacobi spectral collocation method based on operational matrices for solving nonlinear sequential Ψ-Prabhakar–ABC fractional boundary value problems with nonlocal multipoint boundary conditions involving Ψ-Riemann–Liouville integrals and Ψ-Hilfer derivatives. We derived explicit $(N+1)\times(N+1)$ operational matrices $\mathbf{D}_{ABC}^{\alpha,\beta,\gamma;\psi}$, $\mathbf{I}_{RL}^{\delta;\psi}$, and $\mathbf{D}_{H}^{\beta_j,\nu_j;\psi}$ whose entries are given in closed form via Beta function evaluations and Jacobi moment integrals, and reduced the fractional BVP to a nonlinear algebraic system $\mathbf{F}(\mathbf{C})=\mathbf{0}$ solved efficiently by Newton–Raphson iteration with explicitly computed Jacobian $\mathbf{J_F}$. The convergence analysis established that the method attains spectral accuracy $\mathcal{O}(N^{-s})$ for solutions in $H_w^s([a,b])$ and exponential decay $\mathcal{O}(e^{-\sigma N})$ for analytic solutions, with the full error bound given in Theorem 3.11. The numerical example with exact solution $x^*(t)=t^{1.75}$, parameters $\alpha=0.75$, $\beta=1$, $\gamma=0.5$ at $N=4$ confirmed these theoretical rates, yielding L^∞-errors as small as 1.12×10^{-10} at $N=10$ and Newton–Raphson convergence to machine precision in just four iterations, demonstrating that the proposed framework provides a highly accurate, computationally efficient, and mathematically rigorous tool for a broad class of generalized fractional differential equations involving the Prabhakar kernel with arbitrary ψ-functions.

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