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Mean Labeling of Graphs Associated with Paths and Cycles

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Abstract

In mean labeling, unique integer labels are assigned to the vertices such that the each edge receives a distinct label equal to the ceiling of the arithmetic mean of the labels of its incident vertices. In this paper, we prove that the tortoise graph, PC_n graph, k-triangular snake graph, and alternate k-triangular snake graph admit mean labeling.

Keywords: Mean Labeling; k-Triangular Snake Graph; Alternate k-Triangular Snake Graph.
2020 Mathematics Subject Classification: 05C78.

1. Introduction

Throughout this paper, we consider a finite, simple, connected, and undirected graph $G = (V(G), E(G))$, where $p = |V(G)|$ and $q = |E(G)|$. We follow the comprehensive survey on graph labeling given by J. A. Gallian [3]. We use the terminology and notation introduced by F. Harary [4]. Graph labeling refers to the assignment of integers to the vertices or edges of a graph under prescribed conditions. Over time, numerous labeling patterns have been explored due to their structural significance in graph theory. Mean-based labeling frameworks such as harmonic mean, geometric mean, root square mean, centroidal mean, and Heronian mean have been extensively examined for several graph classes including paths, cycles, trees, and snake graphs.

In 2003 Somasundaram [8] introduced mean labeling defined as the ceiling of the arithmetic mean of the adjacent vertex labels. Ambica [1] proved that the tortoise graph T_n ($n \geq 5$ and $n \equiv 1, 3 \pmod 4$) and PC_n graph for $n \geq 5$ are F-centroidal mean graphs. Arockiaraj [2] proved that the graph $P(1, 2, \cdots, n-1)$ is an F-centroidal mean graph for $n \geq 2$. P. Jeyanthi [5] introduced the term extra mean labeling. Sandhya [6] proved that the triple triangular snake, alternate triple triangular snake, triple quadrilateral snake, and alternate triple quadrilateral snake graphs are Heronian mean graphs. Sangeetha [7] introduced relaxed mean labeling. In this paper, we establish the existence of mean labeling for the tortoise graph, PC_n graph, k-triangular snake graph, and alternate k-triangular snake graph.

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2. Preliminaries

Definition 2.1 ([8]). *A graph G is called a mean graph if there exists an injective mapping $f : V(G) \to \{0, 1, 2, 3, \ldots, q\}$ such that for every edge $e = uv \in E(G)$, the edge e is labeled with $\left\lceil \frac{f(u) + f(v)}{2} \right\rceil$. Then the resulting edge labels are distinct and belong to the set $\{1, 2, 3, \ldots, q\}$.*

Definition 2.2 ([1]). *A tortoise graph T_n ($n \geq 5$ and $n \equiv 1, 3 \pmod 4$) is obtained from a path P_n by attaching an edge between v_i and v_{n-i+1} for $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$.*

Definition 2.3 ([1]). *A graph PC_n ($n \geq 5$) is obtained from $C_n = v_1v_2, v_2v_3, \ldots, v_nv_1$ by adding chords between v_i and v_{n-i+2} for $2 \leq i \leq l$, where $l = \frac{n}{2}$ if n is even and $l = \frac{n-1}{2}$ if n is odd.*

Definition 2.4. *Let P_n be a path with the vertex set $V(P_n) = \{u_1, u_2, \ldots, u_n\}$ and the edge set $E(P_n) = \{u_iu_{i+1} \mid 1 \leq i \leq n-1\}$. The k-triangular snake graph is denoted by $T_k(n)$, where $k \in \mathbb{N}$, and it is obtained from P_n by introducing $(n-1)k$ new vertices $v_{j,i}$ ($1 \leq j \leq k$ and $1 \leq i \leq n-1$) corresponding to each edge u_iu_{i+1}, where $1 \leq i \leq n-1$. Each vertex $v_{j,i}$ ($1 \leq j \leq k$ and $1 \leq i \leq n-1$) is joined to both u_i and u_{i+1}.*

Definition 2.5. *Let P_n be a path. The alternate k-triangular snake graph, denoted by $AT_k(n)$, where $k \in \mathbb{N}$, is obtained from P_n by attaching k triangles to every alternate edge of the path P_n. For each alternate edge of P_n, k new vertices $v_{1,i}, v_{2,i}, \ldots, v_{k,i}$ are introduced, and each of these vertices is joined to the end vertices of that edge. In this way, k triangles are formed on each alternate edge of the path, while the other edges remain unchanged.*

3. Main Results

Theorem 3.1. *A tortoise graph T_n ($n \geq 5$ and $n \equiv 1, 3 \pmod 4$) is a mean graph.*

Proof. Let $V(T_n) = \{v_i \; ; \; 1 \leq i \leq n\}$ and $E(T_n) = \left\{e_i \; ; \; 1 \leq i \leq \frac{3(n-1)}{2}\right\}$, where $e_i = v_iv_{i+1}$; $1 \leq i \leq n-1$ and $e_{n-1+i} = v_iv_j$; $j = n-i+1$ and $1 \leq i \leq \frac{n-1}{2}$. Then T_n has $p = n$ vertices and $q = \frac{3(n-1)}{2}$ edges.
Define $f : V(T_n) \to \{0, 1, 2, \ldots, q\}$ as follows:

$$f(v_i) = \frac{3n+1}{2} - 3i, \quad \text{for } 1 \leq i \leq \frac{n-1}{2};$$
$$f(v_i) = 3i - \frac{3n+1}{2} - 1, \quad \text{for } \frac{n+1}{2} \leq i \leq n.$$

Accordingly, the edge labeling f^* induced by f is described as follows:

$$f^*(e_i) = \frac{3(n-1)}{2} - 3i + 1, \quad \text{for } 1 \leq i \leq \frac{n-1}{2};$$
$$f^*(e_i) = 3i + 2 - \frac{3(n+1)}{2}, \quad \text{for } \frac{n+1}{2} \leq i \leq n-1;$$
$$f^*(e_i) = \frac{3(n-1)}{2} - 3i + 3n, \quad \text{for } n \leq i \leq \frac{3(n-1)}{2}.$$

It can be verified that f is a mean labeling on T_n ($n \geq 5$ and $n \equiv 1,3 \pmod 4$). Hence, T_n ($n \geq 5$ and $n \equiv 1,3 \pmod 4$) is a mean graph. □

Example 3.2. *Mean labeling of the graph T_9 is shown in Figure 1.*

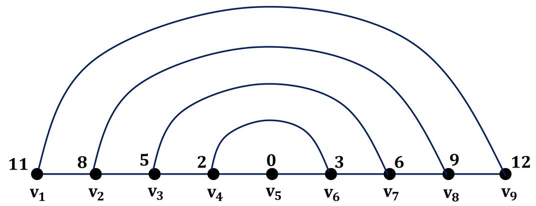

Figure 1: Mean Labeling of T_9

Theorem 3.3. *A graph PC_n ($n \geq 5$) is a mean graph.*

Proof. **Case 1.** n is even.
Let $V(PC_n) = \{v_1, v_2, \dots, v_n\}$ and $E(PC_n) = \{e_1, e_2, \dots, e_n\} \cup \left\{e'_1, e'_2, \dots, e'_{\frac{n-2}{2}}\right\}$, where $e_i = v_i v_{i+1}$; $1 \leq i \leq n-1$, $e_n = v_n v_1$, and $e'_i = v_{i+1} v_j$; $j = n - i + 1$ and $1 \leq i \leq \frac{n-2}{2}$. Then PC_n has $p = n$ vertices and $q = \frac{3n-2}{2}$ edges for even n.
Subcase 1. $n \equiv 2 \pmod 4$.
Define $f : V(PC_n) \to \{0, 1, 2, \dots, q\}$ as follows:

$$f(v_1) = 0;$$
$$f(v_i) = 3i - 4, \quad \text{for } 2 \leq i \leq \frac{n+2}{2};$$
$$f(v_i) = \begin{cases} 3n - 3i + 3, & \text{if } i \text{ is even} \\ 3n - 3i + 4, & \text{if } i \text{ is odd} \end{cases}, \text{ for } \frac{n+4}{2} \leq i \leq n.$$

Subcase 2. $n \equiv 0 \pmod 4$.
Define $f : V(PC_n) \to \{0, 1, 2, \dots, q\}$ as follows:

$$f(v_1) = 0;$$
$$f(v_i) = 3i - 4, \quad \text{for } 2 \leq i \leq \frac{n+2}{2};$$
$$f(v_i) = \begin{cases} 3n - 3i + 3, & \text{if } i \text{ is odd} \\ 3n - 3i + 4, & \text{if } i \text{ is even} \end{cases}, \text{ for } \frac{n+4}{2} \leq i \leq n-1;$$
$$f(v_n) = 3.$$

Accordingly, the edge labeling f^* induced by f for both the subcases is commonly described as follows:

$$f^*(e_i) = 3i - 2, \quad \text{for } 1 \leq i \leq \frac{n}{2};$$
$$f^*(e_i) = 3n - 3i + 2, \quad \text{for } \frac{n+2}{2} \leq i \leq n;$$

$$f^*(e'_i) = 3i, \quad \text{for } 1 \le i \le \frac{n-2}{2}.$$

Case 2. n is odd.

Let $V(PC_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(PC_n) = \{e_1, e_2, \ldots, e_n\} \cup \left\{e'_1, e'_2, \ldots, e'_{\frac{n-3}{2}}\right\}$, where $e_i = v_i v_{i+1}$; $1 \le i \le n-1$, $e_n = v_n v_1$, and $e'_i = v_{i+1} v_j$; $j = n - i + 1$ and $1 \le i \le \frac{n-3}{2}$. Then PC_n has $p = n$ vertices and $q = \frac{3n-3}{2}$ edges for odd n.

Subcase 1. $n \equiv 1 \pmod 4$.

Define $f : V(PC_n) \to \{0, 1, 2, \ldots, q\}$ as follows:

$$f(v_1) = 0;$$
$$f(v_i) = 3i - 4, \quad \text{for } 2 \le i \le \frac{n+1}{2};$$
$$f(v_i) = q, \quad \text{for } i = \frac{n+3}{2};$$
$$f(v_i) = \begin{cases} 3n - 3i + 3, & \text{if } i \text{ is odd} \\ 3n - 3i + 4, & \text{if } i \text{ is even} \end{cases}, \text{ for } \frac{n+5}{2} \le i \le n.$$

Subcase 2. $n \equiv 3 \pmod 4$.

Define $f : V(PC_n) \to \{0, 1, 2, \ldots, q\}$ as follows:

$$f(v_1) = 0;$$
$$f(v_i) = 3i - 4, \quad \text{for } 2 \le i \le \frac{n+1}{2};$$
$$f(v_i) = q, \quad \text{for } i = \frac{n+3}{2};$$
$$f(v_i) = \begin{cases} 3n - 3i + 3, & \text{if } i \text{ is even} \\ 3n - 3i + 4, & \text{if } i \text{ is odd} \end{cases}, \text{ for } \frac{n+5}{2} \le i \le n-1;$$
$$f(v_n) = 3.$$

Accordingly, the edge labeling f^* induced by f for both the subcases is commonly described as follows:

$$f^*(e_i) = 3i - 2, \quad \text{for } 1 \le i \le \frac{n-1}{2};$$
$$f^*(e_i) = 3(i-1), \quad \text{for } i = \frac{n+1}{2};$$
$$f^*(e_i) = 3n - 3i + 2, \quad \text{for } \frac{n+3}{2} \le i \le n;$$
$$f^*(e'_i) = 3i, \quad \text{for } 1 \le i \le \frac{n-3}{2}.$$

From both the cases and their subcases, it can be verified that f is a mean labeling on PC_n. Hence, PC_n $(n \ge 5)$ is a mean graph. □

Example 3.4. *Mean labeling of the graph PC_{11} is shown in Figure 2.*

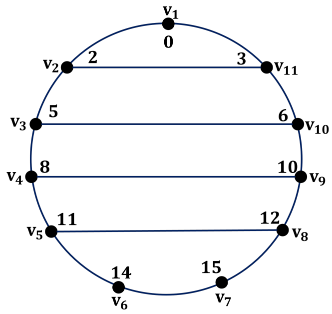

Figure 2: Mean Labeling of PC_{11}

Theorem 3.5. *A k-triangular snake graph $T_k(n)$, where $k \in \mathbb{N}$ is a mean graph.*

Proof. Let $V(T_k(n)) = \{u_i; 1 \le i \le n\} \cup \{v_{j,i}; 1 \le j \le k$ and $1 \le i \le n-1\}$ and $E(T_k(n)) = \{e_i = u_iu_{i+1}; 1 \le i \le n-1\} \cup \{e_{j,i} = v_{j,i}u_i; 1 \le j \le k$ and $1 \le i \le n-1\} \cup \{e'_{j,i} = v_{j,i}u_{i+1}; 1 \le j \le k$ and $1 \le i \le n-1\}$. Then $T_k(n)$ has $p = n + k(n-1)$ vertices and $q = (n-1)(2k+1)$ edges, where $k \in \mathbb{N}$.
Define $f : V(T_k(n)) \to \{0, 1, 2, \ldots, q\}$ as follows:

$$f(u_i) = (2k+1)(i-1), \quad \text{for } 1 \le i \le n;$$

$$f(v_{j,i}) = f(u_i) + 2j, \quad \text{for } 1 \le i \le n-1 \text{ and } 1 \le j \le k.$$

Accordingly, the edge labeling f^* induced by f is described as follows:

$$f^*(e_i) = (2k+1)i - k, \quad \text{for } 1 \le i \le n-1;$$

$$f^*(e_{j,i}) = (2k+1)i - (2k+1) + j, \quad \text{for } 1 \le j \le k \text{ and } 1 \le i \le n-1;$$

$$f^*(e'_{j,i}) = (2k+1)i - k + j, \quad \text{for } 1 \le j \le k \text{ and } 1 \le i \le n-1.$$

It can be verified that f is a mean labeling on $T_k(n)$, where $k \in \mathbb{N}$. Hence, $T_k(n)$, where $k \in \mathbb{N}$ is a mean graph. □

Example 3.6. *Mean labeling of the graph $T_5(4)$ is shown in Figure 3.*

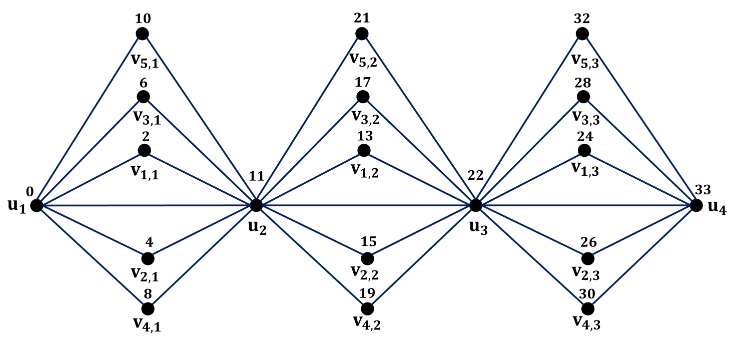

Figure 3: Mean Labeling of $T_5(4)$

Theorem 3.7. *An alternate k-triangular snake graph $AT_k(n)$, where $k \in \mathbb{N}$ is a mean graph.*

Proof. **Case 1.** The k-triangle starts from the 1^{st} vertex of the path P_n.

Subcase 1. n is even.

Let $V(AT_k(n)) = \{u_i; 1 \le i \le n\} \cup \left\{v_{j,i}; 1 \le j \le k \text{ and } 1 \le i \le \frac{n}{2}\right\}$ and $E(AT_k(n)) = \{e_i = u_iu_{i+1}; 1 \le i \le n-1\} \cup \{e_{j,i} = v_{j,i}u_i; 1 \le j \le k \text{ and } 1 \le i \le n\}$. Then $AT_k(n)$ has $p = n + k\left(\frac{n}{2}\right)$ vertices and $q = (n-1) + kn$ edges, where $k \in \mathbb{N}$.

Define $f : V(AT_k(n)) \to \{0, 1, 2, \dots, q\}$ as follows:

$$f(u_i) = \begin{cases} (k+1)(i-1), & \text{if } i \text{ is odd} \\ (k+1)i - 1, & \text{if } i \text{ is even} \end{cases}, \text{ for } 0 \le i \le n;$$

$$f(v_{j,i}) = f(u_{2i-1}) + 2j, \quad \text{for } 1 \le j \le k \text{ and } 1 \le i \le \frac{n}{2}.$$

Accordingly, the edge labeling f^* induced by f is described as follows:

$$f^*(e_i) = (k+1)i, \quad \text{for } 1 \le i \le n-1;$$

$$f^*(e_{j,i}) = j + (i-1)(k+1), \quad \text{for } 1 \le j \le k \text{ and } 1 \le i \le n.$$

Subcase 2. n is odd.

Let $V(AT_k(n)) = \{u_i; 1 \le i \le n\} \cup \left\{v_{j,i}; 1 \le j \le k \text{ and } 1 \le i \le \frac{n-1}{2}\right\}$ and $E(AT_k(n)) = \{e_i = u_iu_{i+1}; 1 \le i \le n-1\} \cup \{e_{j,i} = v_{j,i}u_i; 1 \le j \le k \text{ and } 1 \le i \le n-1\}$. Then $AT_k(n)$ has $p = n + k\left(\frac{n-1}{2}\right)$ vertices and $q = (n-1) + k(n-1)$ edges, where $k \in \mathbb{N}$.

Define $f : V(AT_k(n)) \to \{0, 1, 2, \dots, q\}$ as follows:

$$f(u_i) = \begin{cases} (k+1)(i-1), & \text{if } i \text{ is odd} \\ (k+1)i - 1, & \text{if } i \text{ is even} \end{cases}, \text{ for } 0 \le i \le n;$$

$$f(v_{j,i}) = f(u_{2i-1}) + 2j, \quad \text{for } 1 \le j \le k \text{ and } 1 \le i \le \frac{n-1}{2}.$$

Accordingly, the edge labeling f^* induced by f is described as follows:

$$f^*(e_i) = (k+1)i, \quad \text{for } 1 \le i \le n-1;$$

$$f^*(e_{j,i}) = j + (i-1)(k+1), \quad \text{for } 1 \le j \le k \text{ and } 1 \le i \le n-1.$$

Case 2. The k-triangle starts from the 2^{nd} vertex of the path P_n.

Subcase 1. n is even.

Let $V(AT_k(n)) = \{u_i; 1 \le i \le n\} \cup \left\{v_{j,i}; 1 \le j \le k \text{ and } 1 \le i \le \frac{n-2}{2}\right\}$ and $E(AT_k(n)) = \{e_i = u_iu_{i+1}; 1 \le i \le n-1\} \cup \{e_{j,i} = v_{j,i}u_{i+1}; 1 \le j \le k \text{ and } 1 \le i \le n-2\}$. Then $AT_k(n)$ has $p = n + k\left(\frac{n-2}{2}\right)$ vertices and $q = (n-1) + k(n-2)$ edges, where $k \in \mathbb{N}$.

Define $f : V(AT_k(n)) \to \{0,1,2,\ldots,q\}$ as follows:

$$f(u_i) = \begin{cases} (k+1)(i-1), & \text{if } i \text{ is odd} \\ (k+1)i-2k-1, & \text{if } i \text{ is even} \end{cases} \text{, for } 0 \le i \le n;$$

$$f(v_{j,i}) = f(u_{2i}) + 2j, \quad \text{for } 1 \le j \le k \text{ and } 1 \le i \le \frac{n-2}{2}.$$

Accordingly, the edge labeling f^* induced by f is described as follows:

$$f^*(e_i) = (k+1)i - k, \quad \text{for } 1 \le i \le n-1;$$

$$f^*(e_{j,i}) = (j+1) + (i-2)(k+1), \quad \text{for } 1 \le j \le k \text{ and } 2 \le i \le n-1.$$

Subcase 2. n is odd.

Let $V(AT_k(n)) = \{u_i; 1 \le i \le n\} \cup \left\{v_{j,i}; 1 \le j \le k \text{ and } 1 \le i \le \frac{n-1}{2}\right\}$ and $E(AT_k(n)) = \{e_i = u_iu_{i+1}; 1 \le i \le n-1\} \cup \{e_{j,i} = v_{j,i}u_{i+1}; 1 \le j \le k \text{ and } 1 \le i \le n-1\}$. Then $AT_k(n)$ has $p = n + k\left(\frac{n-1}{2}\right)$ vertices and $q = (n-1) + k(n-1)$ edges, where $k \in \mathbb{N}$.

Define $f : V(AT_k(n)) \to \{0,1,2,\ldots,q\}$ as follows:

$$f(u_i) = \begin{cases} (k+1)(i-1), & \text{if } i \text{ is odd} \\ (k+1)i-2k-1, & \text{if } i \text{ is even} \end{cases} \text{, for } 0 \le i \le n;$$

$$f(v_{j,i}) = f(u_{2i}) + 2j, \quad \text{for } 1 \le j \le k \text{ and } 1 \le i \le \frac{n-1}{2}.$$

Accordingly, the edge labeling f^* induced by f is described as follows:

$$f^*(e_i) = (k+1)i - k, \quad \text{for } 1 \le i \le n-1;$$

$$f^*(e_{j,i}) = (j+1) + (i-2)(k+1), \quad \text{for } 1 \le j \le k \text{ and } 2 \le i \le n.$$

From both the cases and their subcases, it can be verified that f is a mean labeling on $AT_k(n)$, where $k \in \mathbb{N}$. Hence, $AT_k(n)$, where $k \in \mathbb{N}$ is a mean graph. □

Example 3.8. *Mean labeling of the graph $AT_4(6)$) for both cases is shown in Figure 4 and Figure 5.*

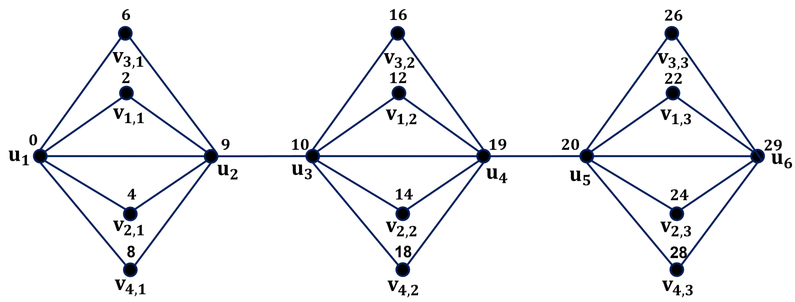

Figure 4: Mean Labeling of $AT_4(6)$ (case 1)

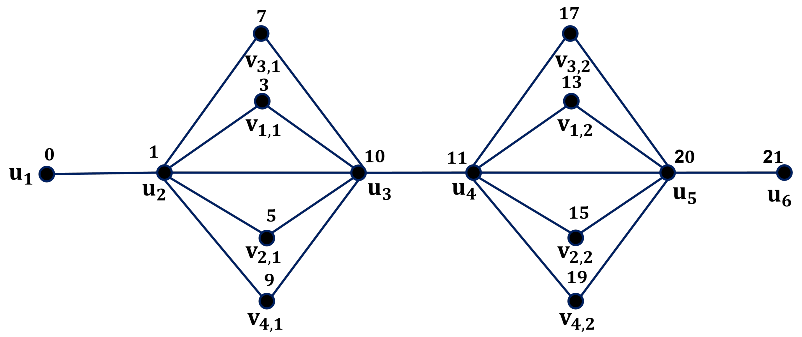

Figure 5: Mean Labeling of $AT_4(6)$ (case 2)

Example 3.9. *Mean labeling of the graph $AT_4(7)$) for both cases is shown in Figure 6 and Figure 7.*

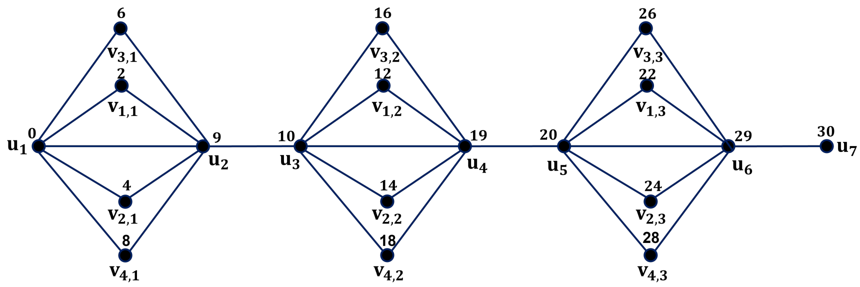

Figure 6: Mean Labeling of $AT_4(7)$ (case 1)

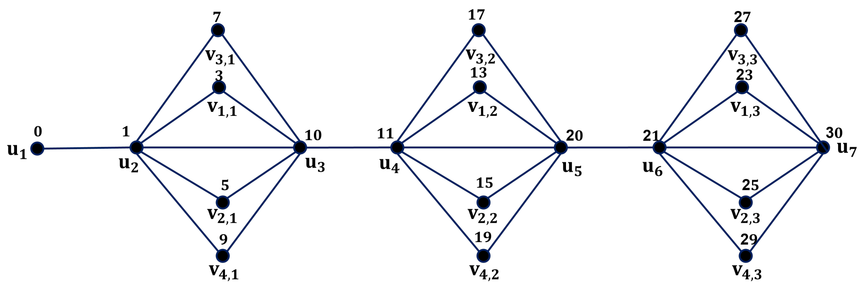

Figure 7: Mean Labeling of $AT_4(7)$ (case 2)

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