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Product Order Relatively Prime Graph of a Group

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Abstract

In this paper we introduce the concept of a product order relatively prime graph (Simply called PORP graph) $\Gamma_{porp}(G)$ of a finite group G. The product order relatively prime graph $\Gamma_{porp}(G)$ is a graph with $V(\Gamma_{porp}(G)) = G$ and two vertices a and b are adjacent in $\Gamma_{porp}(G)$ if either $(o(a), o(ab)) = 1$ or $(o(b), o(ab)) = 1$. Also we obtain certain graph parameters such as clique number, chromatic number, independent number and domination number.

Keywords: product order relatively prime graph; complete graph; star graph; planar graph; bipartite graph.
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1. Introduction

The study of algebraic graphs associated with finite groups has become an active area of research in algebraic graph theory. In recent years, several graphs arising from groups such as power graphs, order graphs, co-prime graphs and intersection graphs have been extensively investigated due to their rich interplay between algebraic and graph theoretic properties. Cameron [7] presented a detailed survey on graphs defined on groups and discussed several important directions in this area.

One of the earliest and most studied graphs associated with groups is the power graph. Various structural and combinatorial properties of power graphs have been investigated by many authors. Panda et al. [17] studied the minimum degree of the power graph of finite cyclic groups. Later, Biswas et al. [6] investigated the difference between enhanced power graphs and power graphs of finite groups. Kamble and Rewaskar [2] investigated the structure of order graphs arising from groups. Sattanathan and Kala [19] introduced the order prime graph and discussed its algebraic properties.

Another important direction is the study of intersection type graphs associated with groups. Bera [3] introduced and studied the intersection power graph of finite groups. Further investigations on intersection power graphs were carried out by Aprose and Fathima [1]. Lv and Ma [11] studied the

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intersection power graph associated with finite groups, while Ma et al. [12] investigated perfect codes in proper intersection power graphs. Groups having prime order classes were studied by Al-Hasanat et al. [4].

Order based graphs also received considerable attention in recent years. Al-Hasanat and Al-Hasanat [5] introduced the notion of order graphs of finite groups and studied several structural properties. Sattanathan and Kala [18] introduced the order prime graph and discussed its algebraic properties. Muthulakshmi et al. [15] investigated the independence number of order prime graphs, and further developed the notion of p-order prime graphs in [16]. Recently, Manna et al. [13] studied prime order element graphs of finite groups.

The concept of the product order divisor graph of a group has been introduced by R. Ponraj and T. Sutharson in [18] and also in [18] discussed about POD graph of cyclic group and self-inverse group. Nica [14] studied the independence number of regular graphs associated with matrix rings. Graphs defined using co-primality conditions also play an important role in algebraic graph theory. Sehgal et al. [20] studied co-prime order graphs of finite abelian groups and dihedral groups. Swathi and Sunitha [21] investigated non-intersection power graphs and co-prime graphs of finite groups. Das and Saha [8] studied co-maximal subgroup graphs of groups.

Motivated by these developments, in this paper we introduce a new graph associated with finite groups called the *product order relatively prime graph*. Let G be a finite group. The product order relatively prime graph (simply called PORP graph) of G, denoted by $\Gamma_{porp}(G)$, is the graph whose vertex set is G and two distinct vertices a and b are adjacent if $(o(a), o(ab)) = 1$ or $(o(b), o(ab)) = 1$. This graph combines both product structure and order theoretic properties of finite groups. We mainly investigate the structural and graph theoretic properties of $\Gamma_{porp}(G)$ for finite cyclic groups. In particular, we obtain degree formulas, size, independence number, chromatic number, clique number, domination parameters and several structural characterizations for cyclic groups of order p^α and cyclic groups of order $m^\alpha n^\alpha$.

2. Preliminaries

Definition 2.1 ([10]). *The group G is said to be a self-inverse group if $a^2 = e$ for all $a \neq e \in G$.*

Definition 2.2 ([9]). *Let $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ be two simple graphs with $V_1 \cap V_2 = \emptyset$. The join of Γ_1 and Γ_2, denoted by $\Gamma_1 \vee \Gamma_2$, is the graph obtained from the disjoint union of Γ_1 and Γ_2 by joining every vertex of Γ_1 to every vertex of Γ_2.*

Definition 2.3 ([18]). *Let G be a finite group. The product order divisor graph (Simply called POD graph) $\Gamma_{pod}(G)$ is a graph with $V(\Gamma_{pod}(G)) = G$ and two vertices a and b are adjacent in $\Gamma_{pod}(G)$ if either $o(a) \mid o(ab)$ or $o(b) \mid o(ab)$.*

Definition 2.4 ([10]). *The dihedral group of order $2n$, denoted by D_n, is defined as $D_n =< r, s : r^n = e, s^2 =$*

$e, rs = sr^{-1} >$. The elements of D_n are given by $D_n = \{e, r, r^2, \cdots, r^{n-1}, s, sr, sr^2, \cdots, sr^{n-1}\}$. We denote $a_i = r^i, 1 \leq i \leq n$ and $b_j = sr^j, 1 \leq j \leq n$.

Definition 2.5 ([10])**.** *The greatest common divisor of two integers a and b is denoted by $(a, b) = 1$. Two positive integers a and b are said to be relatively prime (or coprime) if $(a, b) = 1$.*

Definition 2.6 ([9])**.** *Let $\Gamma = (V, E)$ be a graph. Then the subset D of V is said to be a dominating set if all vertices in Γ are either in D or adjacent to at least one vertex in D. The minimum cardinality of such a set is the domination number, denoted by $\gamma(\Gamma)$.*

Definition 2.7 ([9])**.** *Let $\Gamma = (V, E)$ be a graph. Then the subset D of V is said to be a global dominating set if D is also a dominating set of Γ. The minimum cardinality of such a set is the global domination number denoted by $\gamma_g(\Gamma)$.*

Definition 2.8 ([9])**.** *Let $\Gamma = (V, E)$ be a graph. Then the subset D of V is said to be a total dominating set if D has no isolated vertex. The minimum cardinality of such a set is the total domination number, denoted as $\gamma_t(\Gamma)$.*

Definition 2.9 ([9])**.** *The vertex connectivity of Γ, denoted by $\kappa(\Gamma)$, is the minimum number of vertices whose removal disconnects Γ or reduces it to a trivial graph.*

Definition 2.10 ([9])**.** *The chromatic number of Γ, denoted by $\chi(\Gamma)$, is the minimum number of colors required to color the vertices of Γ such that no two adjacent vertices receive the same color.*

Definition 2.11 ([9])**.** *A clique in Γ is a complete subgraph of Γ. The clique number of Γ, denoted by $\omega(\Gamma)$, is the maximum cardinality of a clique in G.*

Definition 2.12. *[9]An independent set in Γ is a set of pairwise non-adjacent vertices. The independence number of Γ, denoted by $\alpha(\Gamma)$, is the maximum cardinality of an independent set in Γ.*

Definition 2.13 ([9])**.** *The girth of Γ, denoted by $g(\Gamma)$, is the length of a shortest cycle in Γ.*

Definition 2.14 ([9])**.** *The diameter of a connected graph Γ, denoted by $diam(\Gamma)$, is defined as $diam(\Gamma) = \max_{v \in V(\Gamma)} e_\Gamma(v)$.*

Definition 2.15 ([9])**.** *The radius of a connected graph Γ, denoted by $rad(\Gamma)$, is defined as $rad(\Gamma) = \min_{v \in V(\Gamma)} e_\Gamma(v)$.*

3. Product Order Relatively Prime Graph

Definition 3.1. *Let G be a finite group. The product order relatively prime graph (Simply called PORP graph) $\Gamma_{porp}(G)$ is a graph with $V(\Gamma_{porp}(G)) = G$ and two vertices a and b are adjacent in $\Gamma_{porp}(G)$ if either $(o(a), o(ab)) = 1$ or $(o(b), o(ab)) = 1$.*

Example 3.2. *$\Gamma_{pod}(D_4)$ and $\Gamma_{porp}(D_4)$ are given in Figure 1.*

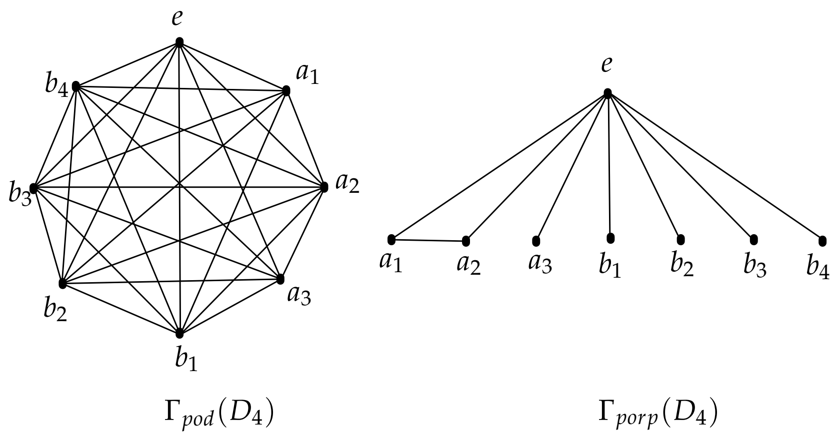

Figure 1.

Theorem 3.3. *Let G_1 and G_2 be two groups such that $G_1 \cong G_2$. Then $\Gamma_{porp}(G_1) \cong \Gamma_{porp}(G_2)$.*

Proof. Let f be an isomorphism of G_1 onto G_2. Let $a, b \in V(\Gamma_{pod}(G_1))$. Then a and b are adjacent in $\Gamma_{porp}(G_1)$ if and only if $(o(a), o(ab)) = 1$ or $(o(b), o(ab)) = 1$ if and only if $(o(f(a)), o(f(ab))) = 1$ or $(o(f(b)), o(f(ab))) = 1$ if and only if $(o(f(a)), o(f(a)f(b))) = 1$ or $(o(f(b)), o(f(a)f(b))) = 1$ if and only if $f(a)$ and $f(b)$ are adjacent in $\Gamma_{porp}(G_2)$. □

Theorem 3.4. *Let G be a finite group. Then $\Gamma_{porp}(G)$ is star if and only if $G \cong \mathbb{Z}_2^n$.*

Proof. Suppose $G \cong \mathbb{Z}_2^n$. Hence G is a self-inverse group. Let $a, b \in G$, $a \neq e, b \neq e$, $a, b \in V(\Gamma_{porp}(G))$. This implies $o(a) = o(b) = o(ab) = 2$. Implies a and b are non-adjacent vertices. Consider a and e, where $a \neq e$, $a \in \Gamma_{porp}(G)$. Then $(o(e), o(ea)) = (1, o(a)) = 1$. This implies e and a are adjacent vertices. That is e is adjacent to all vertices $a \neq e$. Therefore $\Gamma_{porp}(G)$ is a star graph.
Conversely, suppose $G \ncong \mathbb{Z}_2^n$. Then there exist an element a such that $a \neq a^{-1}$. Then $(o(a), o(aa^{-1})) = (o(a), o(e)) = (o(a), 1) = 1$. Therefore e is adjacent to a and a^{-1}. Hence $\Gamma_{porp}(G)$ is not a star, a contradiction. □

Theorem 3.5. *Let G be a cyclic group of order n. Then $\Gamma_{porp}(G)$ is complete if and only if $G \cong Z_1$ or $G \cong Z_2$ or $G \cong Z_3$.*

Proof. Suppose $G \cong Z_1$ or $G \cong Z_2$ or $G \cong Z_3$. Clearly $\Gamma_{porp}(G)$ is complete. Conversely, suppose $G \cong \mathbb{Z}_n$, $n > 3$. Consider the vertices 1 and $n-2$. Then $(o(1), o(1+n-2)) = (o(1), o(n-1)) \neq 1$. Therefore 1 and $n-2$ are not adjacent, a contradiction. □

4. Cyclic Group of Order p^α (p is Prime and $\alpha \geq 1$)

In this section, we consider the cyclic group with order p^α where p is prime and $\alpha \geq 1$. Clearly $G \cong (Z_{p^\alpha}, \oplus)$.

Theorem 4.1. *Let $G \cong \mathbb{Z}_{p^\alpha}$, where p is prime and $\alpha \geq 1$. Then*

$$\Gamma_{porp}(G) \cong \begin{cases} K_1 \vee \frac{n-1}{2}K_2, & \text{if } p \text{ is odd prime} \\ K_1 \vee (\frac{n-1}{2}K_2 \cup K_1), & \text{if } p = 2. \end{cases}$$

Proof. Let $G \cong \mathbb{Z}_{p^\alpha}$.

Case 1: p is an odd prime.

Consider the vertices a and a^{-1}. Now, $(o(a), o(aa^{-1})) = (o(a), o(e)) = (o(a), 1) = 1$. Therefore a and a and a^{-1} are adjacent. Let $b \neq a^{-1} \in V(\Gamma_{porp}(G))$. Consider the vertices a and b. Let $o(a) = p^r$, $o(b) = p^s$ and $o(ab) = p^t$ for some positive integers $r, s, t \geq 1$. Clearly $(o(a), o(ab)) = (p^r, p^t) \neq 1$ and $(o(b), o(ab)) = (p^s, p^t) \neq 1$. Therefore a and b are not adjacent vertices. Consider the vertices a and e. Now $(o(a), o(e)) = (o(a), 1) = 1$. Therefore e is adjacent to $V(\Gamma_{porp}(G))$. Thus

$$\Gamma_{porp}(G) \cong K_1 \vee \frac{n-1}{2}K_2.$$

Case 2: $p = 2$.

In this case G has a self-inverse element $2^{\alpha-1}$. Clearly it is adjacent to the identity element e only. Thus as in case 1, we have

$$\Gamma_{porp}(G) \cong K_1 \vee (\frac{n-1}{2}K_2 \cup K_1).$$

□

Theorem 4.2. *The degree sequence of $\Gamma_{porp}(G)$ is $\left(p^\alpha - 1, \underbrace{2, 2, \cdots, 2}_{(p^\alpha-2)\text{ times}}, 1\right)$ or $\left(p^\alpha - 1, \underbrace{2, 2, \cdots, 2}_{(p^\alpha-1)\text{ times}}\right)$ according as $p = 2$ or p is an odd prime.*

Proof. Let $a \in G$. First consider the identity element e. Since $o(e) = 1$, we have $(o(e), o(eb)) = (1, o(b)) = 1$ for every $b \in G \setminus \{e\}$. Hence the identity vertex is adjacent to every other vertices. Therefore $d(e) = p^\alpha - 1$. Now let $a \neq e$. Since $G \cong \mathbb{Z}_{p^\alpha}$, the order of every non-zero element is a power of p. Suppose a is adjacent to $b \neq e$. Then by definition, $(o(a), o(ab)) = 1$ or $(o(b), o(ab)) = 1$. If $ab \neq e$, then $o(a) = p^r$, $o(b) = p^s$, $o(ab) = p^t$ for some positive integers $r, s, t \geq 1$. Consequently, $(o(a), o(ab)) = (p^r, p^t) \neq 1$ and similarly, $(o(b), o(ab)) \neq 1$, which is impossible. Hence $ab = e$. Therefore a non-zero vertex a is adjacent precisely to its inverse a^{-1} together with the identity element. If p is odd, then $a \neq a^{-1}$ for every non-zero element a. Hence every non-identity vertex has exactly two adjacent vertices. Therefore $d(a) = 2$. If $p = 2$, then the equation $a = a^{-1}$ holds for exactly one non-zero element, namely $a = 2^{\alpha-1}$. Hence this vertex is adjacent only to the identity element and therefore has degree 1. Every remaining non-identity vertex has degree 2. Therefore the degree sequence of $\Gamma_{porp}(G)$ is $\left(p^\alpha - 1, \underbrace{2, 2, \cdots, 2}_{(p^\alpha-2)\text{ times}}, 1\right)$ or $\left(p^\alpha - 1, \underbrace{2, 2, \cdots, 2}_{(p^\alpha-1)\text{ times}}\right)$ according as $p = 2$ or p is an odd prime. □

Theorem 4.3. *Let $G = \mathbb{Z}_{p^\alpha}$, where $\alpha \geq 1$. Then the size of $\Gamma_{porp}(G)$ is*

$$|E(\Gamma_{porp}(G))| = \begin{cases} 3.2^{\alpha-1} - 2, & \text{if } p = 2, \\ \dfrac{3(p^\alpha - 1)}{2}, & \text{if } p \text{ is odd.} \end{cases}$$

Proof. Let $G \cong Z_{p^\alpha}$ be a cyclic group. Since the identity vertex is adjacent to every other vertex, the graph contains $p^\alpha - 1$ edges incident with the identity element. Now consider the remaining edges among non-identity vertices. A non-identity vertex a is adjacent to another non-identity vertex b if and only if $ab = e$ that is, $b = a^{-1}$. First suppose that p is odd. Then $a \neq a^{-1}$ for every non-zero element a. Hence the non-zero elements form $\frac{p^\alpha-1}{2}$ distinct pairs $[a, a^{-1}]$. Each these pairs contributes exactly one edge. Therefore

$$|E(\Gamma_{porp}(G))| = (p^\alpha - 1) + \frac{p^\alpha - 1}{2}.$$

Hence

$$|E(\Gamma_{porp}(G))| = \frac{3(p^\alpha - 1)}{2}.$$

Now suppose that $p = 2$. Then there exists exactly one non-zero element satisfying $a = a^{-1}$, namely $a = 2^{\alpha-1}$. The remaining $2^\alpha - 2$ non-zero elements form $\frac{2^\alpha-2}{2} = 2^{\alpha-1} - 1$ distinct pairs $[a, a^{-1}]$. Hence

$$|E(\Gamma_{porp}(G))| = (2^\alpha - 1) + (2^{\alpha-1} - 1).$$

Since the $[a, a^{-1}]$ pair edge corresponding to the self-inverse element is already counted through the identity adjacency, we obtain

$$|E(\Gamma_{porp}(G))| = 2^\alpha - 1.$$

Therefore,

$$|E(\Gamma_{porp}(G))| = \begin{cases} 3.2^{\alpha-1} - 2, & \text{if } p = 2, \\ \dfrac{3(p^\alpha - 1)}{2}, & \text{if } p \text{ is odd.} \end{cases}$$

□

Theorem 4.4. *Let $G \cong \mathbb{Z}_{p^\alpha}$, where $\alpha \geq 1$. Then the independence number of $\Gamma_{porp}(G)$ is*

$$\alpha(\Gamma_{porp}(G)) = \begin{cases} 2^{\alpha-1}, & \text{if } p = 2, \\ \frac{p^\alpha-1}{2}, & \text{if } p \text{ is odd.} \end{cases}$$

Proof. Every non-zero element a is adjacent precisely to its inverse a^{-1}. If p is odd, then every inverse pair consists of two distinct elements. Hence the non-zero elements split into $\frac{p^\alpha-1}{2}$ $[a, a^{-1}]$ pairs. From each pair, at most one element can belong to an independent set. Choosing one element from each pair gives an independent set of cardinality $\frac{p^\alpha-1}{2}$. Therefore

$$\alpha(\Gamma_{porp}(G)) = \frac{p^\alpha - 1}{2}$$

. Now suppose $p = 2$. Then there exists exactly one non-zero self-inverse element, namely $2^{\alpha-1}$. The remaining $2^\alpha - 2$ non-zero elements form $\frac{2^\alpha-2}{2} = 2^{\alpha-1} - 1$ $[a, a^{-1}]$ pairs. Choosing one element from each $[a, a^{-1}]$ pair together with the self-inverse element gives an independent set of cardinality $2^{\alpha-1}$. Hence

$$\alpha(\Gamma_{porp}(G)) = 2^{\alpha-1}.$$

□

Theorem 4.5. *Let $G \cong \mathbb{Z}_{p^\alpha}$, where $\alpha \geq 1$. Then the chromatic number of $\Gamma_{porp}(G)$ is*

$$\chi(\Gamma_{porp}(G)) = \begin{cases} 2, & \text{if } G \cong \mathbb{Z}_2, \\ 3, & \text{otherwise.} \end{cases}$$

Proof. If $G \cong \mathbb{Z}_2$, then the graph is K_2, and therefore $\chi(\Gamma_{porp}(G)) = 2$. Now assume $|G| > 2$. Let $a \neq e$ such that $a \neq a^{-1}$. Then the vertices $\{e, a, a^{-1}\}$ are mutually adjacent. Hence $\{e, a, a^{-1}\}$ forms a triangle. Therefore $\chi(\Gamma_{porp}(G)) \geq 3$. Now assign the color C_1 to the identity vertex. Consider the pair $[a, a^{-1}]$, $a \in V(\Gamma_{porp}(G))$. Assign the color C_2 to a vertex a and assign the color C_3 to a^{-1}. This gives a proper coloring using three colors. Hence $\chi(\Gamma_{porp}(G)) \leq 3$. Therefore $\chi(\Gamma_{porp}(G)) = 3$. □

Theorem 4.6. *Let $G \cong \mathbb{Z}_{p^\alpha}$, where $\alpha \geq 1$. Then the clique number of $\Gamma_{porp}(G)$ is*

$$\omega(\Gamma_{porp}(G)) = \begin{cases} 2, & \text{if } G \cong \mathbb{Z}_2, \\ 3, & \text{otherwise.} \end{cases}$$

Proof. If $G \cong \mathbb{Z}_2$, then the graph is K_2. Hence $\omega(\Gamma_{porp}(G)) = 2$. Now assume $|G| > 2$. For any non-zero vertex a, e is adjacent to a, e is adjacent to a^{-1}, a is adjacent to a^{-1}. Hence $\{e, a, a^{-1}\}$ forms a clique of order 3. Therefore $\omega(\Gamma_{porp}(G)) \geq 3$. Since adjacency among non-identity vertices occurs only between $[a, a^{-1}]$ pairs, no four vertices can form a complete subgraph. Hence $\omega(\Gamma_{porp}(G)) \leq 3$. Therefore

$$\omega(\Gamma_{porp}(G)) = 3.$$

□

Theorem 4.7. *Let $G \cong \mathbb{Z}_{p^\alpha}$, where $\alpha \geq 1$. Then*

(1). $$\operatorname{diam}(\Gamma_{porp}(G)) = \begin{cases} 1, & \text{if } |G| = 2 \\ 2, & \text{otherwise.} \end{cases}$$

(2). $\text{rad}(\Gamma_{porp}(G)) = 1$,

(3). $\text{girth}(\Gamma_{porp}(G)) = \begin{cases} 3, & \textit{if } |G| > 2, \\ \infty, & \textit{if } |G| = 2, \end{cases}$

(4). the vertex connectivity of $\Gamma_{porp}(G)$ *is* $\kappa(\Gamma_{porp}(G)) = 1$.

Proof.

(1). Since the identity element is adjacent to every vertex, every pair of vertices is connected by a path of length at most 2. If $|G| = 2$, then $\Gamma_{porp}(G) \cong K_2$ and hence $\text{diam}(\Gamma_{porp}(G)) = 1$. Now suppose $|G| > 2$. Since adjacency among non-identity vertices occurs only between $[a, a^{-1}]$ pairs, there exist non-adjacent non-identity vertices. Hence the distance between such vertices is 2 through the identity vertex. Therefore $\text{diam}(\Gamma_{porp}(G)) = 2$.

(2). Next, the identity vertex is adjacent to every other vertex. Hence its eccentricity is 1. Since radius is the minimum eccentricity among all vertices,

$$\text{rad}(\Gamma_{porp}(G)) = 1.$$

(3). Now consider the girth. If $|G| = 2$, then the graph contains no cycle. Hence

$$\text{girth}(\Gamma_{porp}(G)) = \infty.$$

Assume $|G| > 2$. Let $a \neq e$ such that $a \neq a^{-1}$. Then the vertices $\{e, a, a^{-1}\}$ are mutually adjacent. Thus $\{e, a, a^{-1}\}$ forms a cycle of length 3. Therefore $\text{girth}(\Gamma_{porp}(G)) \leq 3$. Since every cycle has length at least 3, $\text{girth}(\Gamma_{porp}(G)) = 3$.

(4). (4) Finally, removing the identity vertex disconnects the graph into isolated vertices and edges of K_2. Hence the graph becomes disconnected after removing a single vertex. Therefore $\kappa(\Gamma_{porp}(G)) \leq 1$. Since the graph is connected, $\kappa(\Gamma_{porp}(G)) \geq 1$. Hence $\kappa(\Gamma_{porp}(G)) = 1$.

□

Theorem 4.8. *Let* $G \cong \mathbb{Z}_{p^\alpha}$, *where* $\alpha \geq 1$. *Then*

(1). the domination number of $\Gamma_{porp}(G)$ *is* $\gamma(\Gamma_{porp}(G)) = 1$,

(2). the total domination number of $\Gamma_{porp}(G)$ *is* $\gamma_t(\Gamma_{porp}(G)) = 2$,

(3). the global domination number of $\Gamma_{porp}(G)$ *is* $\gamma_g(\Gamma_{porp}(G)) = \begin{cases} 2, & \textit{if } G \cong \mathbb{Z}_2, \\ 3, & \textit{otherwise}, \end{cases}$

(4). the matching number of $\Gamma_{porp}(G)$ *is* $\mu(\Gamma_{porp}(G)) = \left\lfloor \frac{p^\alpha}{2} \right\rfloor$.

Proof.

(1). Since the identity element e is adjacent to every vertex of $\Gamma_{porp}(G)$, the singleton set $\{e\}$ forms a dominating set. Hence $\gamma(\Gamma_{porp}(G)) \leq 1$. Since every dominating set contains at least one vertex, $\gamma(\Gamma_{porp}(G)) \geq 1$. Therefore $\gamma(\Gamma_{porp}(G)) = 1$.

(2). Now let $S = \{e, a\}$ where $a \neq e$. Since the identity vertex is adjacent to every nonidentity vertex and $e \sim a$, every vertex of the graph is adjacent to a vertex in S. Hence S is a total dominating set. Therefore $\gamma_t(\Gamma_{porp}(G)) \leq 2$. Since no single vertex can totally dominate a simple graph, $\gamma_t(\Gamma_{porp}(G)) \geq 2$. Hence $\gamma_t(\Gamma_{porp}(G)) = 2$.

(3). Now consider the global domination number. If $G \cong \mathbb{Z}_2$, then $\Gamma_{porp}(G) \cong K_2$, and therefore $\gamma_g(\Gamma_{porp}(G)) = 2$. Now assume $|G| > 2$.Since the identity vertex e is adjacent to every vertex in $\Gamma_{porp}(G)$,it becomes an isolated vertex in $\overline{\Gamma_{porp}(G)}$. Hence every global dominating set must contain e. Let $a \neq e$. In the complement graph, the vertex a is not adjacent to its inverse a^{-1}. Therefore, if a global dominating set contains only one non-zero vertex, then its inverse is not dominated in $\overline{\Gamma_{porp}(G)}$. Hence both a and a^{-1} must belong to the global dominating set. Thus $\{e, a, a^{-1}\}$ is a global dominating set, and hence $\gamma_g(\Gamma_{porp}(G)) \leq 3$. Further, no set of cardinality 2 can globally dominate both the graph and its complement. Hence $\gamma_g(\Gamma_{porp}(G)) \geq 3$. Therefore

$$\gamma_g(\Gamma_{porp}(G)) = \begin{cases} 2, & \text{if } G \cong \mathbb{Z}_2, \\ 3, & \text{otherwise.} \end{cases}$$

(4). Finally, consider the matching number. Every non-identity element a is adjacent precisely to its inverse $-a$. If p is odd, then every non-zero element has a distinct inverse. Hence the nonzero vertices split into $\frac{p^\alpha-1}{2}$ $[a, a^{-1}]$ pairs, producing $\frac{p^\alpha-1}{2}$ independent edges. Thus

$$\mu(\Gamma_{prop}(G)) = \frac{p^\alpha - 1}{2} = \left\lfloor \frac{p^\alpha}{2} \right\rfloor.$$

If $p = 2$, then there exists exactly one non-zero self-inverse element, namely $2^{\alpha-1}$. The remaining $2^\alpha - 2$ non-zero vertices form $2^{\alpha-1} - 1$ $[a, a^{-1}]$ pairs. Together with one edge incident to the identity vertex, we obtain a matching of size $2^{\alpha-1} = \left\lfloor \frac{2^\alpha}{2} \right\rfloor$. Therefore

$$\mu(\Gamma_{porp}(G)) = \left\lfloor \frac{p^\alpha}{2} \right\rfloor.$$

□

Theorem 4.9. *Let $G \cong \mathbb{Z}_{p^\alpha}$, where $\alpha \geq 1$. Then:*

(1). $\Gamma_{porp}(G)$ is planar for all $n \geq 1$,

(2). $\Gamma_{porp}(G)$ *is Hamiltonian if and only if* $G \cong \mathbb{Z}_3$*,*

(3). $\Gamma_{porp}(G)$ *is Eulerian if and only if* $G \cong \mathbb{Z}_{2^\alpha}$*,*

(4). $\Gamma_{porp}(G)$ *is bipartite if and only if* $G \cong \mathbb{Z}_2$*.*

Proof.

(1). Since the identity vertex is adjacent to every vertex and every nonidentity vertex is adjacent only to its inverse, the graph consists of a universal vertex together with disjoint $[a, a^{-1}]$ pair edges. Hence the graph can be drawn in the plane without edge crossings. Therefore $\Gamma_{porp}(G)$ is planar.

(2). Now consider Hamiltonicity. If $G \cong \mathbb{Z}_3$, then $\Gamma_{porp}(G) \cong K_3$, which is Hamiltonian. Now assume $|G| > 3$. Then there exists a nonidentity vertex of degree at most 2. Since a Hamiltonian graph with more than two vertices cannot contain a cut vertex, and the identity vertex is a cut vertex of $\Gamma_{porp}(G)$, the graph is not Hamiltonian.

(3). Next consider Eulerian property. A connected graph is Eulerian if and only if every vertex has even degree. For odd prime p, $\deg(e) = p^\alpha - 1$, which is even, but every non-identity vertex has degree 2. Hence all vertex degrees are even, and therefore $\Gamma_{porp}(G)$ is Eulerian. Now suppose $p = 2$. Then the unique self-inverse non-identity vertex has degree 1 while every remaining non-identity vertex has degree 2. Hence the graph contains a vertex of odd degree and therefore is not Eulerian.

(4). Finally, consider bipartiteness. If $G \cong \mathbb{Z}_2$, then $\Gamma_{porp}(G) \cong K_2$, which is bipartite. Now assume $|G| > 2$. Let $a \neq e$ such that $a \neq a^{-1}$. Then the vertices $\{e, a, a^{-1}\}$ are mutually adjacent. Thus $\{e, a, a^{-1}\}$ forms a triangle. Hence $\Gamma_{porp}(G)$ contains an odd cycle and therefore is not bipartite.

□

5. Cyclic Group of Order $m^\alpha n^\alpha$, (m,n are Distinct Odd Primes)

Theorem 5.1. *Let* $G \cong \mathbb{Z}_{m^\alpha n^\alpha}$ *where* m *and* n *are distinct odd primes. Then the degree of a vertex* $a \in G$ *in* $\Gamma_{porp}(G)$ *is given by*

$$d(a) = \begin{cases} m^\alpha n^\alpha - 1, & \text{if } a = e, \\ n^{\alpha-i+1} + 1, & \text{if } o(a) = m^i,\ 1 \leq i \leq \alpha, \\ m^{\alpha-i+1} + 1, & \text{if } o(a) = n^i,\ 1 \leq i \leq \alpha, \\ 3, & \text{if } o(a) = m^i n^j,\ 1 \leq i, j < \alpha, \\ 2, & \text{if } o(a) = m^\alpha n^\alpha. \end{cases}$$

Proof. Let $G \cong \mathbb{Z}_{m^\alpha n^\alpha}$. First consider the identity element e. Since $o(e) = 1$ for every non-zero element $b \in G$, $(o(e), o(eb)) = (1, o(b)) = 1$. Hence the identity vertex is adjacent to every other vertices. Therefore $d(e) = m^\alpha n^\alpha - 1$. Now let $a \neq e$.

Case 1: Suppose $o(a) = m^i$, $1 \leq i \leq \alpha$.

If a is adjacent to b. then $(o(a), o(ab)) = 1$ or $(o(b), o(ab)) = 1$. Since $o(a) = m^i$, the first condition is satisfied precisely when the order of ab is not divisible by m. Hence $o(ab) = n^r$ or $o(ab) = 1$. Therefore a is adjacent to $n^{\alpha-i+1}$ vertices. Together with the inverse vertex a^{-1}, we obtain $d(a) = n^{\alpha-i+1} + 1$.

Case 2: Suppose $o(a) = n^i$, $1 \leq i \leq \alpha$.

By symmetry, adjacency occurs precisely when $o(ab)$ is a power of m or 1. Therefore the number of adjacent vertices $m^{\alpha-i+1}$. Including the inverse vertex a^{-1}, we obtain $d(a) = m^{\alpha-i+1} + 1$.

Case 3: Suppose $o(a) = m^i n^j$, $1 \leq i, j < \alpha$.

If $ab \neq e$, then the order of ab is divisible by either m or n. Hence $(o(a), o(ab)) \neq 1$ and similarly $(o(b), o(ab)) \neq 1$. Therefore adjacency occurs only when $ab = e$. Thus a is adjacent to

- the identity vertex,
- its inverse vertex a^{-1},
- adjacent to the vertex b, $o(a) = m^r$ or $o(a) = n^s$.

Hence $d(a) = 3$.

Case 4: Suppose $o(a) = m^\alpha n^\alpha$.

Then every non-zero element order shares a common prime divisor with $o(a)$. Hence adjacency is possible only when $ab = e$. Therefore a is adjacent only to

- the identity vertex,
- its inverse vertex.

Hence $d(a) = 2$. Therefore the result follows. □

Corollary 5.2. *Let $G \cong \mathbb{Z}_{m^\alpha n^\alpha}$, where m and n are distinct odd primes. Then the size of $\Gamma_{porp}(G)$ is*

$$|E(\Gamma_{porp}(G))| = \frac{1}{2}\left[m^\alpha n^\alpha - 1 + \sum_{i=1}^{\alpha} \phi(m^i)(n^{\alpha-i+1} + 1) + \sum_{i=1}^{\alpha} \phi(n^i)(m^{\alpha-i+1} + 1)\right.$$

$$\left. + 3\sum_{i=1}^{\alpha-1}\sum_{j=1}^{\alpha-1} \phi(m^i n^j) + 2\phi(m^\alpha n^\alpha)\right]$$

where ϕ denotes Euler's phi function.

Proof. By the Handshaking Lemma, $2|E(\Gamma_{porp}(G))| = \sum_{a \in G} \deg(a)$. From Theorem 5.1, the degrees of vertices are determined according to the order of the elements. Since $|G| = m^\alpha n^\alpha$,the identity element contributes $m^\alpha n^\alpha - 1$ to the degree sum. For each $1 \leq i \leq \alpha$, there are $\phi(m^i)$ elements of order m^i, and each such vertex has degree $n^{\alpha-i+1}+1$. Hence their total contribution is $\sum\limits_{i=1}^{\alpha}\phi(m^i)(n^{\alpha-i+1}+1)$. Similarly, the elements of order n^i contribute $\sum\limits_{i=1}^{\alpha}\phi(n^i)(m^{\alpha-i+1}+1)$. Now for $1\leq i,j<\alpha$, there are $\phi(m^in^j)$ elements of order m^in^j, and each such vertex has degree 3. Therefore their total contribution is $3\sum\limits_{i=1}^{\alpha-1}\sum\limits_{j=1}^{\alpha-1}\phi(m^in^j)$. Finally, there are $\phi(m^\alpha n^\alpha)$ elements of order $m^\alpha n^\alpha$, and each has degree 2. Hence their contribution is $2\phi(m^\alpha n^\alpha)$. Therefore

$$|E(\Gamma_{porp}(G))| = \frac{1}{2}\Bigg[m^\alpha n^\alpha - 1 + \sum_{i=1}^{\alpha}\phi(m^i)(n^{\alpha-i+1}+1) + \sum_{i=1}^{\alpha}\phi(n^i)(m^{\alpha-i+1}+1)$$

$$+3\sum_{i=1}^{\alpha-1}\sum_{j=1}^{\alpha-1}\phi(m^in^j) + 2\phi(m^\alpha n^\alpha)\Bigg].$$

□

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