

Conformal Vector Fields With C-Factor on Kropina Metric

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Abstract

In this paper, we study the conformal vector fields with C -factor on a class of Finsler metrics called Kropina metric $F = \frac{\alpha^2}{\beta}$ is defined in Riemannian metric α and 1-form β and its norm. Then we characterize the PDE's of conformal vector fields on Kropina metric.

Keywords: Finsler metric; Conformal vector fields; Finsler Kropina-metrics.

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1. Introduction

Finsler geometry has been developing rapidly since last few decades, after its emergence in 1917. Finsler geometry has been influenced by group theory. The celebrated Erlangen program of F. Klein, posed in 1872, greatly influenced the development of geometry. Klein proposed to categorize the geometries by their characteristic group of transformations.

Conformal vector fields are important in Riemann - Finsler geometry. To solve problems on conformal vector fields in Riemannian - Finsler geometry as follows:. Let (M, F) be a Finsler manifold. It is known that a vector field $v = v^i \frac{\partial}{\partial x^i}$ on M is a conformal vector field on F with conformal factor $c = c(x)$ if and only if $X_v(F^2) = 4cF^2$, where $X_v = v^i \frac{\partial}{\partial x^i} + y^j \frac{\partial v^j}{\partial x^i} \frac{\partial}{\partial y^j}$ [5]. Recently, Shen and Xia [5] have studied conformal vector fields on a Randers manifold with certain curvature properties. They also determine conformal vector fields on a locally projectively flat Randers manifold. Besides they use homothetic vector fields ($c = \text{constant}$) on Randers manifolds to construct new Randers metrics of scalar flag curvature [6]. The theory of Kropina metric was investigated by L. Berwald in connection with a two dimensional Finsler space with rectilinear extremal and were investigated by V.K. Kropina. Randers metrics seem to be among the simplest non - trivial Finsler metrics with many investigation in Physics, Electron optics with a magnetic field, dissipative mechanics, irreversible thermodynamics etc..

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In this paper, we shall study the vector fields on a manifold M with Finsler metric known as Kropina metric, whose metric is defined in Riemannian metric α and 1-form β and its norm and here by inspiring the work of conformal vector fields on some Finsler manifolds. Then we characterize the PDE's of conformal vector fields on Kropina metric. In natural way, we consider the general (α, β) -metrics are defined as the form:

$$F = \alpha\phi(b^2, \beta^2/\alpha). \quad (1)$$

For example, the Randers metrics and the square metrics are defined by functions $\phi = \phi(b^2, s)$ in the following form:

$$\phi = \frac{\sqrt{1 - b^2 + s^2} + s}{1 - b^2}. \quad (2)$$

$$\phi = \frac{(\sqrt{1 - b^2 + s^2} + s)}{(1 - b^2)^2 \sqrt{1 - b^2 + s^2}}. \quad (3)$$

Based on the some reviews of [1, 8, 20] and further we shall study the covariant derivatives of conformal vector field is directly proportional to Kropina metric.

2. Preliminaries

Let M be an n -dimensional differential manifold and TM be the tangent bundle. A Finsler metric on M is a function $F = F(x, y) : TM \longrightarrow \mathbb{R}$ satisfying the following conditions:

1. $F(x, y)$ is a C^∞ function on $TM \setminus \{0\}$;
2. $F(x, y) \geq 0$ and $F(x, y) = 0 \rightarrow y = 0$;
3. $F(x, \lambda y) = \lambda F(x, y)$, $\lambda > 0$;
4. the fundamental tensor $g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 (F^2)}{\partial y^i \partial y^j}$ is positively defined.

Let

$$C_{ijk} = \frac{1}{4} [F^2]_{y^i y^j y^k} = \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k}.$$

Define symmetric trilinear form $C = C_{ijk} dx^i \otimes dx^j \otimes dx^k$ on $TM \setminus \{0\}$. We call C the Cartan torison. Let F be a Finsler metric on an n - dimensional manifold M . The canonical geodesic $\sigma(t)$ of F is characterized by

$$\frac{d^2 \sigma^i(t)}{dt^2} + 2G^i(\sigma(t), \dot{\sigma}(t)) = 0,$$

where G^i are the geodesic coefficients having the expression $G^i = \frac{1}{4} g^{ij} \{ [F^2]_{x^k y^j} y^k - [F^2]_{x^i} \}$ with $(g^{ij}) = (g_{ij})^{-1}$ and $\dot{\sigma} = \frac{d\sigma^i}{dt} \frac{\partial}{\partial x^i}$. A spray on M is a globally C^∞ vector field G on $TM \setminus \{0\}$ which is expressed in local coordinates as follows:

$$G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}.$$

Given geodesic coefficients G^i , we define the covariant derivatives of a vector field $X = X^i(t) \frac{\partial}{\partial x^i}$ along a curve $c(t)$ by

$$D_i X(t) = \{X^i(t) + X^j(t) N_j^i(c(t), c'(t))\} \frac{\partial}{\partial x^i} \Big|_{c(t)},$$

where $N_j^i = \frac{\partial G^i}{\partial y^j}$, $X^i(t) = \frac{dX^i}{dt}$ and $\dot{c} = \frac{dc^i}{dt} \frac{\partial}{\partial x^i}$. It is easy to verify that

$$\begin{aligned} D_{\dot{c}}(X + Y)(t) &= D_{\dot{c}}X(t) + D_{\dot{c}}Y(t), \\ D_{\dot{c}}(fX)(t) &= f^1(t)X(t) + f(t)D_{\dot{c}}X(t). \end{aligned}$$

Since $D_{\dot{c}(t)}$ linearly depends on $X(t)$, $D_{\dot{c}}X(t)$ is called the linear covariant derivative. It is easy to see that the canonical geodesic satisfies $D_{\dot{c}} = 0$. Let TM be the tangent bundle and $\pi : TM \setminus \{0\} \rightarrow M$ the natural projection. According to the pulled - back bundle π^*TM admits a unique linear connection called the Chern connection. Let $F = \frac{\alpha^2}{\beta}$ be a Kropina metric expressed in terms of a Riemannian metric α and a vector field V on M .

Consider equation (1) is

$$F = \alpha\phi(b^2, \frac{\beta}{\alpha})$$

where $\phi = \phi(b^2, s)$ is a positive smooth function on $[0, b_0) \times (-b_0, b_0)$. It is required that

$$\phi - \phi_2 s > 0, \quad \phi - \phi_2 s + (b^2 - s^2)\phi_{22} > 0,$$

for $b < b_0$, where,

$$\phi_1 = \frac{\partial \phi}{\partial b^2}, \phi_2 = \frac{\partial \phi}{\partial s}, \phi_{22} = \frac{\partial^2 \phi}{\partial s^2}, \alpha = \frac{\sqrt{1 - b^2 + s^2}}{1 - b^2}, \beta = \frac{s}{1 - b^2}.$$

We write the function where $\phi = \phi(b^2, s)$ in the following Taylor expansion

$$\phi = p_0 + p_1 s + p_2 s^2 + o(s^3),$$

where

$$p_i = p_i(b^2), p_0 = \frac{1}{(1 - b^2)^{\frac{1}{2}}}, p_1 = \frac{1}{1 - b^2}, p_2 = \frac{1}{2(1 - b^2)^{3/2}}.$$

Now (2) implies that

$$p_0 > 0, \quad p_0 + 2b^2 p_2 > 0.$$

But there is no restriction on p_1 . If we assume that $p_1 \neq 0$, then F is not reversible. If the Finsler metric called Kropina metric is on the conformal vector field, with conformal factor $c(x)$ then Finsler Kropina metric becomes

$$\phi(b^2, s) = \frac{1 - b^2 + s^2}{s(1 - b^2)}$$

and (2) and (3) satisfy

$$\frac{1}{2b^2} + \left\{ \frac{p_1^1}{p_1} - \frac{p_0^1}{p_0} + \frac{p_2}{p_0} \left[2 \frac{p_1^1}{p_1} - \frac{p_0^1}{p_0} \right] - \frac{p_2^1}{p_0} \right\} b^2 = \frac{1}{2b^2(1-b^2)}. \quad (4)$$

2.1 Definitions

- (a). **Conformal factor:** If $\tilde{F}(x, y) = e^{c(x)}F(x, y)$, preserving angles but changing lengths smoothly.
- (b). **Conformal factor on metric:** It scales the metric tensor g_{ij} by $e^{2c(x)}$ and angular metric tensor h_{ij} by $e^{2c(x)}$.
- (c). **Conformal Vector field:** Let F be a Finsler metric on a manifold M , and V be a vector field on M . Let ϕ_t be the flow generated by V . Define $\tilde{\phi} : TM \rightarrow TM$ by $\phi_t(x, y) = (\phi_t(x), \phi_t * (y))$. A vector field V is said to be conformal if

$$\phi_t^* \tilde{F} = e^{-2\sigma_t} F, \quad (5)$$

where σ_t is a function on M for every t .

Differentiating (5) by t at $t = 0$, we obtain

$$X_v(F) = -2cF, \quad (6)$$

where c is called the conformal factor and we define

$$X_v = V^i \frac{\partial}{\partial x^i} + y^j \frac{\partial V^j}{\partial x^i} \frac{\partial}{\partial y^j}, c = \frac{d}{dt} \Big|_{t=0} \sigma_t. \quad (7)$$

3. Conformal vector fields with $c(x)$ on Kropina metric

Let V be a conformal vector field of F with conformal factor $c(x)$.

$$i.e., X_v(F^2) = 4cF^2. \quad (8)$$

From the position of (2) and to solve the equation (8) with the Kropina metric, we have

$$F = \frac{\alpha^2}{\beta} = \frac{1 - b^2 + s^2}{s(1 - b^2)},$$

then (8) implies

$$X_v(F^2) = \phi^2 X_v(\alpha^2) + \alpha^2 X_v(\phi^2),$$

$$\begin{aligned}
X_v(F^2) &= \phi^2 X_v(\alpha^2) + 2\phi\alpha^2\phi_1 X_v(b^2) + 2\phi\phi_2\alpha X_v(\beta) - 2\phi\phi_2\beta X_v(\alpha), \\
X_v(F^2) &= (1 + b^4 + s^4 - 2b^2 - 2s^2b^2)X_v(\alpha^2) \\
&\quad + 2\alpha^2 X_v(b^2)(s^3 - s^3b^2 + s^5) + 2\alpha X_v(\beta)(b^2 + b^4 + b^6 + s^4 - s^4b^2) \\
&\quad - 2\beta X_v(\alpha)(b^6 + 2b^4 - b^2 + s^4 - s^4b^2 + 2s^2b^2 - b^4s^2 - s^2)/s^3(1 - b^2)^3 \\
X_v(F^2) &= A_0[A_1 X_v(\alpha^2) + 2\alpha^2 X_v(b^2)A_2 + 2\alpha X_v(\beta)A_3 - 2\beta X_v(\alpha)A_4], \tag{9}
\end{aligned}$$

where,

$$\begin{aligned}
A_0 &= \frac{1}{s^3(1 - b^2)^3}, \\
A_1 &= 1 + b^4 + s^4 - 2b^2 - 2s^2b^2, \\
A_2 &= s^3 - s^3b^2 + s^5, \\
A_3 &= b^4(1 + b^2) + b^2(1 - s^4) + s^4, \\
A_4 &= b^6 + b^4(2 - s^2) + s^2(2b^2 - s^4 - 1) - b^2 + s^4, \\
X_v(\alpha^2) &= 2V_{0;0}, \quad X_v(\beta) = (V^j b_{i;j} + b^j V_{j;i})y^i.
\end{aligned}$$

Then equation (9) equivalent to

$$\begin{aligned}
(\phi - \phi_2 s)V_{0;0} + \alpha\phi_2(V^j b_{i;j} + b^j V_{j;i})y^i(\phi_1 X_v(b^2) - 2c\phi)\alpha^2 &= 0, \\
\left(\frac{1 - b^2 + s^2}{s(1 - b^2)} - \left(\frac{s^2 - s^2b^2 + 2b^2 - b^4 - 1}{s^2(1 - b^2)^2}\right)s\right)V_{0;0} \\
+ \alpha\left(\frac{s^2 - s^2b^2 + 2b^2 - b^4 - 1}{s^2(1 - b^2)^2}\right)(V^j b_{i;j} + b^j V_{j;i})y^i \\
+ \left[\left(\frac{s^3}{s^2(1 - b^2)^2}\right)X_v(b^2) - 2c\left(\frac{1 - b^2 + s^2}{s(1 - b^2)}\right)\right]\alpha^2 &= 0 \\
\frac{(1 - b^2 + s^2)(1 - b^2) - (s^2 - s^2b^2 + 2b^2 - sb^4 - 1)}{s(1 - b^2)^2}V_{0;j} \\
+ \alpha\left(\left(\frac{s^2 - s^2b^2 + 2b^2 - b^4 - 1}{s^2(1 - b^2)^2}\right)(V^j b_{i;j} + b^j V_{j;i})y^i\right. \\
\left. + \frac{s\alpha^2 X_v(b^2) - 2c\alpha^2(1 - b^2 + s^2)}{s(1 - b^2)}\right) &= 0 \\
\frac{b^4 + sb^4 - 4b^2}{s(1 - b^2)}V_{0;j} + \alpha\left\{\left(\frac{\alpha^2 - s^2b^2 + 2b^2 - b^4 - 1}{s^2(1 - b^2)^2}\right)V^j b_{i;j} + b^j V_{j;i}\right\}y^i \\
+ s^2\alpha^2 X_v(b^2) - \frac{2c\alpha^2(1 - b^2 + s^2)}{s(1 - b^2)} &= 0.
\end{aligned}$$

Which implies,

$$B_1 V_{0;0} + \alpha B_2 (V^j b_{i;j} + b^j V_{j;i})y^i + s^2\alpha^2 X_v(b^2) - 2B_3 c\alpha^2 = 0, \tag{10}$$

where,

$$\begin{aligned} B_1 &= \frac{b^4(1+s^2) - 4b^2}{s(1-b^2)^2}, \\ B_2 &= \frac{\alpha^2 - b^4 - b^2(s^2 - 2) - 1}{s^2(1-b^2)^2}, \\ B_3 &= \frac{1 - b^2 + s^2}{s(1-b^2)}. \end{aligned} \quad (11)$$

To simplify the computation, we fixed point $x \in M$ and make a co-ordinate change such that

$$y = \frac{s}{\sqrt{b^2 - s^2}} \bar{\alpha}, \quad \alpha = \frac{b}{b^2 - s^2} \bar{\alpha}, \quad \beta = \frac{bs}{\sqrt{b^2 - s^2}} \bar{\alpha}, \quad \bar{\alpha} = \sqrt{\sum_{a=2}^n (y^a)^2}.$$

Then we have

$$V_{0;0} = V_{1;1} \frac{s^2}{b^2 - s^2} \bar{\alpha}^2 + (\bar{V}_{1;0} + \bar{V}_{0;1}) \frac{s}{\sqrt{b^2 - s^2}} \bar{\alpha} + \bar{V}_{0;0}, \quad (12)$$

$$V^j b_i + b^j V_{j;i} y^i = (V^j b_{1;j} + b^j V_{j;1}) \frac{s}{\sqrt{b^2 - s^2}} \bar{\alpha} + (V^j \bar{b}_{0;j} + b^j \bar{V}_{j;0}), \quad (13)$$

where,

$$\bar{V}_{1;0} + \bar{V}_{0;1} = \sum_{a=2}^n (V_{1;p} + V_{p;1}) y^p, \quad \bar{V}_{0;0} = \sum_{p,q=0}^n V_{p;q} y^p y^q, \quad (14)$$

$$V^j \bar{b}_{0;j} + b^j \bar{V}_{j;0} = \sum_{p=2}^n (V^j b_{p;j} + b^j V_{j;p}) y^p.$$

From (12) and (13) in to (10), which yields

$$\begin{aligned} & (\phi - \phi_2 s) \left\{ V_{1;1} \frac{s^2}{b^2 - s^2} \bar{\alpha}^2 + (\bar{V}_{1;0} + \bar{V}_{0;1}) \frac{s}{\sqrt{b^2 - s^2}} \bar{\alpha} + \bar{V}_{0;0} \right\} \\ & + \phi_2 \frac{b}{\sqrt{b^2 - s^2}} \bar{\alpha} \left\{ (V^j b_{1;j} + b^j V_{j;1}) \frac{s}{\sqrt{b^2 - s^2}} \bar{\alpha} + (V^j \bar{b}_{0;j} + b^j \bar{V}_{j;0}) \right\} \\ & B_1 \left\{ V_{1;1} \frac{s^2}{b^2 - s^2} \bar{\alpha}^2 + (\bar{V}_{1;0} + \bar{V}_{0;1}) \frac{s}{\sqrt{b^2 - s^2}} \bar{\alpha} + \bar{V}_{0;0} \right\} \\ & + B_2 \frac{b}{\sqrt{b^2 - s^2}} \bar{\alpha} \left\{ (V^j b_{1;j} + b^j V_{j;1}) \frac{s}{\sqrt{b^2 - s^2}} \bar{\alpha} + (V^j \bar{b}_{0;j} \right. \\ & \quad \left. + b^j \bar{V}_{j;0}) \right\} + [s^2 X_v(b^2) - 2B_3 c] \frac{b^2}{b^2 - s^2} \bar{\alpha}^2 = 0. \end{aligned} \quad (15)$$

Consider the polynomial

$$\phi = p_0 + p_1 s + p_2 s^2 + o(s^3)$$

with $p_i = p_i(b^2)$ then we have,

$$\begin{aligned}\phi_1 &= p_0^1 + p_1^1 s + p_2^1 s^2 + o(s^2) \\ \frac{s}{(1-b^2)^2} &= p_0^1 + p_1^1 s + p_2^1 s^2 + o(s^2)\end{aligned}$$

By letting $s = 0$ in (15) then

$$p_0 \bar{V}_{0;0} + p_1 (V^j \bar{b}_{0;j} + b^j \bar{V}_{j;0}) \bar{\alpha} + \{p_0^1 X_v(b^2) - 2cp_0\} \bar{\alpha}^2 = 0. \quad (16)$$

According to the irrationality of $\bar{\alpha}$, the (15) is equivalent to

$$p_1 (V^j \bar{b}_{0;j} + b^j \bar{V}_{j;0}) = 0, \quad (17)$$

$$p_0 (\bar{V}_{0;0} + p_0^1 X_v(b^2) - 2cp_0) \bar{\alpha}^2 = 0. \quad (18)$$

Therefore, the equation (18) yields

$$\begin{aligned}(V^j \bar{b}_{0;j} + b^j \bar{V}_{j;0}) &= 0, \\ V^j b_{p;j} + b^j \bar{V}_{j;p} &= 0.\end{aligned} \quad (19)$$

By (14) we have

$$V_{p;q} + V_{q;p} = -2 \left\{ \frac{p_0^1}{p_0} X_v(b^2) - 2c \right\} \delta_{pq}, \quad 2 \leq p, q \leq n. \quad (20)$$

Again irrationality of $\bar{\alpha}$ from (10) we get

$$B_1 (\bar{V}_{1;0} + \bar{V}_{0;1}) \frac{s}{\sqrt{b^2 - s^2}} \bar{\alpha} = 0. \quad (21)$$

$$B_1 \left\{ V_{1;1} \frac{b^2}{b^2 - s^2} \bar{\alpha}^2 + \bar{V}_{0;0} \right\} + B_2 \frac{bs}{b^2 - s^2} \bar{\alpha}^2 (V^j b_{1;j} + b^j V_{j;1}) + \{s^2 X_v(b^2) - 2cB_2\} \frac{b^2}{b^2 - s^2} \bar{\alpha}^2 = 0. \quad (22)$$

From (20) we get

$$\bar{V}_{1;0} + \bar{V}_{0;1} = 0.$$

This equivalent to

$$V_{1;p} + V_{p;1} = 0. \quad (23)$$

Solving (18) for $\bar{V}_{0;0}$ and plugging it in to (21) we have

$$B_1 s^2 \left\{ V_{1;1} \frac{p_0^1}{p_0} (X_v(b^2) - 2c) \right\} - \left\{ \frac{p_0^1}{p_0} X_v(b^2) - 2c \right\} B_1 (b^2)$$

$$+ B_3 s b (V^j b_{1;j} + b^j V_{j;1}) + B_2 b^2 X_v(b^2) - 2cb^2 \frac{(1 - b^2 - s^2)}{1 - b^2} = 0. \quad (24)$$

By Taylor series, expansion of $\phi(b^2, s)$ then plugging it in to (21) and by the coefficients of s we have.

$$b p_1 (V^j b_{1;j} + b^j V_{j;1}) + b^2 X_v(b^2) \frac{\partial p_1}{\partial b^2} - 2cb^2 p_1 = 0. \quad (25)$$

Then

$$V^j b_{1;j} + b^j V_{j;1} = - \left(\frac{p_1^1}{p_1} X_v(b^2) - 2c \right) b_i. \quad (26)$$

Then by (20) and (26) we have

$$V^j b_{i;j} + b^j V_{j;i} = - \left(\frac{p_1^1}{p_1} X_v(b^2) - 2c \right) b_i. \quad (27)$$

Substituting (26) in (27), we have

$$B_1 s^2 \left\{ V_{1;j} + \left(\frac{p_1^1}{p_0} X_v(b^2) - 2c \right) \right\} - b^2 X_v(b^2) \left\{ \frac{p_0^1}{p_0} B_1 - B_2 + B_3 s \frac{p_1^1}{p_1} \right\} = 0. \quad (28)$$

The coefficients of all powers of s must vanish in (28). In particular, the coefficients of s^2 vanishes. We have

$$V_{1;1} + \frac{p_0^1}{p_0} X_v(b^2) - 2cb = -b^2 X_v(b^2) R_0, \quad (29)$$

where

$$R_0 = \left[\frac{p_0^1}{p_0} \frac{p_2}{p_0} + \frac{p_0^1}{p_0} - 2 \frac{p_1^1}{p_1} \frac{p_2}{p_0} \right].$$

By (20), (22) and (29), we have

$$V_{i;j} + V_{j;i} = 4c p_{ij} - 2X_v(b^2) \left\{ \frac{p_0^1}{p_0} p_{ij} + R_0 b_i b_j \right\}. \quad (30)$$

It equivalent to

$$v_{i;j} + v_{j;i} = 4c \alpha - 2X_v(b^2) \left\{ \frac{p_0^1}{p_0} \alpha + R_0 \beta \right\}. \quad (31)$$

Contracting (22) with b^i and b^j yields

$$V_{i;j} b^i b^j = 2cb^2 - b^2 X_v(b^2) \left\{ \frac{p_0^1}{p_0} + R_0 b^2 \right\}. \quad (32)$$

This equivalent to

$$V_{ij}b^ib^j = 2c\beta^2 - b^2X_v(b^2).$$

Contracting 22 with b^i and b^j yields

$$V_{ij}b^ib^j = 2cb^2 - b^2X_v(b^2) \left\{ \frac{1}{2b^2} + \frac{p_1^1}{p_1} \right\}. \quad (33)$$

Here, we used the fact that

$$X_v(b^2) = 2b_{ik}b^iV^k. \quad (34)$$

Then comparing (29) with (34) yields

$$X_v(b^2)\{R_1 - R_0b^2\} = 0, \quad (35)$$

where $R_1 = \frac{1}{2b^2} + \frac{p_1^1}{p_1} - \frac{p_0^1}{p_0}$. Now, (35) reduced to

$$X_v(b^2)\{R_1 + R_2b^2\} = 0. \quad (36)$$

Here, two cases arises:

Case 1: If

$$R_1 + R_2b^2 \neq 0, \quad (37)$$

where, $R_2 = \frac{p_0^1 p_2^1}{p_0 p_0} + \frac{p_2^1}{p_0} - 2\frac{p_1^1 p_2^1}{p_1 p_0}$. It follows from (37) that $X_v(b^2) = 0$ and in (26) and we have

$$V_{ij} + V_{ji} = 4c\alpha, \quad V^jb_{ij} + b^jV_{ji} = 2c\beta. \quad (38)$$

Notice that if $X_v(b^2) = 0$ and (38) holds then V satisfies (9) and V is an conformal vector field.

Therefore, we obtain

Theorem 3.1. Let $F = \frac{\alpha^2}{\beta}$ be a Kropina metric on an n -dimensional manifold M ($n \geq 3$) and let $V = V^i(x)\frac{\partial}{\partial x^i}$ be a conformal vector field. Then V is a conformal vector field of F with conformal factor $c = c(x)$ if and only if $X_v(b^2) = 0$ and

$$V_{ij} + V_{ji} = 4c\alpha, \quad V^jb_{ij} + b^jV_{ji} = 2c\beta. \quad (39)$$

Case 2: If

$$R_1 + R_2b^2 = 0. \quad (40)$$

In this case $X_v(b^2) \neq 0$. Then obviously, we have

$$V_{i;j} + V_{j;i} = 4\bar{c}\alpha - 2X_v(b^2)b^{-2}R_1b_ib_j, \quad (41)$$

$$V^jb_{i;j} + V_{j;i}b^j = 2\bar{c}\beta. \quad (42)$$

Since V is conformal vector field and above equation then (9) is reduced to

$$X_v(b^2) \left\{ B_1b^{-1}[(b^2 - s^2)R_1^*] + B_2 - \left(\frac{1 - b^2 + s^2}{s(1 - b^2)} \right) \frac{p_1^1}{p_1} \right\} = 0. \quad (43)$$

Therefore it follows we obtain

Theorem 3.2. Let $F = \frac{\alpha^2}{\beta}$ be a Kropina metric on an n -dimensional manifold M ($n \geq 3$) and let $V = V^i(x) \frac{\partial}{\partial x^i}$ be a conformal vector field. Then, V is a conformal vector field of F with conformal factor $c = c(x)$ if and only if

$$V_{i;j} + V_{j;i} = 4\bar{c}\alpha - 2X_v(b^2)b^{-2}R_1b_ib_j,$$

$$V^jb_{i;j} + V_{j;i}b^j = 2\bar{c}\beta, \quad (44)$$

$$X_v(b^2) \left\{ B_1b^{-1}[(b^2 - s^2)R_1^*] + B_2 - \left(\frac{1 - b^2 + s^2}{s(1 - b^2)} \right) \frac{p_1^1}{p_1} \right\} = 0, \quad (45)$$

where,

$$\begin{aligned} R_1 &= \left(\frac{1}{2b^2} + \frac{p_1^1}{p_1} - \frac{p_0^1}{p_0} \right), \\ R_1^* &= \left(\frac{p_1^1}{p_1} - \frac{p_0^1}{p_0} \right) \frac{s^2}{2b^2}, \\ B_1 &= \frac{b^4(1 + s^2) - 4b^2}{s(1 - b^2)^2}, \\ B_2 &= \frac{\alpha^2 - b^4 - b^2(s^2 - 2) - 1}{s^2(1 - b^2)^2}, \\ \bar{c} &= c - \frac{1}{2}X_v(b^2) \frac{p_0^1}{p_0}. \end{aligned}$$

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