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The Hypersurface of a Finsler Space with Special (α, β)-metric $F = \alpha + \frac{\beta^{n+1}}{\alpha^n}$

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Abstract

A hypersurface of a Finsler space with a special (α, β) -metric involves studying intrinsic geometric properties like curvature and hyperplanes. The concept of (α, β)-metric was introduced by M. Matsumoto in 1972 [1]. In this paper, we determining when a hypersurface is a hyperplane of a specific kind and the criterion for a hypersurface of a special metric to be a hyperplane of the first kind, second kind and third kind.

Keywords: Finsler space; Hypersurface; Special-(α, β)-metric; hyperplane of the first kind; second kind and third kind.
2020 Mathematics Subject Classification: 53B40, 53C60.

1. Introduction

The hypersurface is a $(n-1)$-dimensional manifold embedded in the n-dimensional Finsler space. Finsler metric, involving Euclidean distance α and a 1-form β is used focusing on classifications like hyperplanes of the first, second, and third kind, often defined by conditions on induced tensors and connections, with applications in physics and relativity. The theory of Hypersurface of a Finsler space may be developed after model of Riemannian geometry. In the early of Finsler geometry , E. Cartan made M. Haimovici fundamental and essential contribution to the theory. Then various interesting results on hypersurface of a Finsler spaces have been found by O. Varga's , H. Rund. It seems however to be the unevviables obstructions to develop the theory of hypersurface of Finsler spaces analogously to the Riemannian theory and as a consequence all most all the existing literatures are not easy to understand and confused notations sometimes bewilder the readers. the first among these obstructions is perhapes surviving of quantities which are derived from Cartan C- tensor, for instance the non symmetry property of the second fundamenta tensor The second consequence of the first is that the induced connection, define by projection , does not generally coincide with the

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intrinsic connection, determined from the induced Finsler metric and that the form is beyond the usual concept of connection appearing in Finsler geometry . In 1985 Matsumoto studied the theory of Finslerr hypersurface and various types of Finsler hypersurface called hyper planes of the first kind, second kind and third kind . In 1992, Makoto Matsumoto also worked on the theory of Finsler space with (α, β) - metric and obtained the interesting results . In 2009 S. K. Narasimhamurthy and H.G. Nagaraj were studies the special hypersurface of a Finsler space with (α, β) - metrics.
In this paper , we study the Finsler hypersurface of a special (α, β) metric equipped with the metric function $F = \alpha + \frac{\beta^{n+1}}{\alpha^n}$ and it can be derived the conditions under which a hypersurface of a Finsler space becomes a hyperplane of first kind , second kind and third kind.

2. Preliminaries

A differential manifold M, when equipped with a Finsler metric $F(x, y)$ is referred to as a Finsler manifold and is typically denoted as (M, F). A Finsler space $(M, F(x, y))$, where $F(x, y)$ represents the Finsler fundamental function. This space is said to possess an (α, β)- metric , and it can be expressed as $F(x, y) = F(\alpha(x, y), \beta(x, y))$ where $\alpha(x, y)$ is the Riemannian fundamental function $\sqrt{(a_{ij}(x)y^i y^j)}$ and $\beta(x, y)$ is a differential 1- form defined on the tangent bundle TM .
Let a function $F : TM \to M$ is a Finsler metric on the manifold M if F satisfy the following conditions

1. F is C^∞ away from zero vectors of the tangent space: That is, F is smooth on $TM \backslash \{0\} = \{(x, y)\ x \in M, y \not\emptyset\}$ the smoothness property is desired so that we can apply differential calculus on F.

2. Positivity of function F: $F(x, y) > 0$ for $x \in M$ and all $y \in T_x M$. This property ensures that the length of tangent vector $y \in T_x M$ is either positive or zero. In terms of arc length this property also ensures that arc length defined by the internal is either positive or zero.

3. Positive homogeneity of function F: $F(x, \lambda y) = \lambda F(x, y)$ for all $\lambda > 0$ for $x \in M$ and all $y \in T_x M$.

That is F is positive 1- homogeneous of first degree in the directional argument y. This property ensures that , length of a tangent vector $\lambda y \in T_x M$.

3. Finsler Space with (α, β) Metric

The fundamental function F of a Finsler space $F^n = (M^n, F)$ is called a n- dimensional Finsler space with the (α, β) - metric, here F is positive homogeneous function of two arguments α is a Riemannian fundamental function and β is a differential 1- form. Let a Finsler metric (M^n, F), where F is the Finsler space with special (α, β)- metric and is given by

$$F(\alpha, \beta) = \alpha + \frac{\beta^{n+1}}{\alpha^n} \tag{1}$$

The first and second order partial derivative of (1) with respect to α and β, we get

$$F_\alpha = \frac{\alpha^{n+1} - n\beta^{n+1}}{\alpha^{n+1}}, F_{\alpha\alpha} = \frac{n(n+1)\beta^{n+1}}{\alpha^{n+2}},$$
$$F_\beta = \frac{(n+1)\beta^n}{\alpha^n}, F_{\beta\beta} = \frac{n(n+1)\beta^{n-1}}{\alpha^n}, F_{\alpha\beta} = \frac{-n(n+1)\beta^n}{\alpha^{n+1}}. \tag{2}$$

In Finsler space (M^n, F), the normalized element of support $l_i = \frac{\partial F}{\partial y_i}$ and the angular metric tensor h_{ij} is given by

$$l_i = \frac{F_\alpha y_i}{\alpha} + L_\beta b_i, \quad h_{ij} = pa_{ij} + q_0 b_i b_j + q_1(b_i y_j + b_j y_i) + q_2 y_i y_j, \tag{3}$$

where the coefficients are defined and calculated as follows

$$y_i = a_{ij} y^j,$$
$$l_i = \frac{F_\alpha y_i}{\alpha} + F_\beta b_i, \tag{4}$$
$$p = \frac{FF_\alpha}{\alpha} = \frac{(\alpha^{n+1} + \beta^{n+1})(\alpha^{n+1} - n\beta^{n+1})}{\alpha^2(n+1)}, \tag{5}$$
$$q_0 = FF_{\beta\beta} = \frac{n(n+1)(\alpha^{n+1} + \beta^{n+1})\beta^{n-1}}{\alpha^{2n}} \tag{6}$$
$$q_1 = \frac{FF_{\alpha\beta}}{\alpha} = \left\{\frac{-n(n+1)(\alpha^{n+1} - \beta^{n+1})\beta^n)}{\alpha^4}\right\}, \tag{7}$$
$$q_2 = \frac{F(F_{\alpha\alpha} - \frac{F_\alpha}{\alpha})}{\alpha^2} = \frac{(\alpha^{n+1} + \beta^{n+1})(n(n+2)\alpha^{n+1} - \beta^{n+1})}{\alpha^{2(n+2)}}. \tag{8}$$

The fundamental metric tensor $g_{ij} = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j}$ of Finsler space (M^n, F) is defined by

$$g_{ij} = pa_{ij} + p_0 b_i b_j + p_1(b_i y_j + b_j y_i) + p_2 y_i y_j, \tag{9}$$

where the coefficients p, p_0, p_1 and p_2 are defined and calculated as follows :

$$p = \frac{FF_\alpha}{\alpha} = \frac{(\alpha^{n+1} + \beta^{n+1})(\alpha^{n+1} - n\beta^{n+1})}{\alpha^2(n+1)}, \tag{10}$$
$$p_0 = q_0 + F_\beta^2 = \frac{(n+1)\left[n\alpha^{n+1}\beta^{n+1} + (2n+1)\beta^{2n}\right]}{\alpha^2(n+2)}, \tag{11}$$
$$p_1 = q_1 + \frac{pL_\beta}{L} = [(1-n)\alpha^{n+1} - 2n\beta^{n+1}] \tag{12}$$
$$p_2 = q_2 + \frac{p^2}{F^2}$$
$$= \left\{\frac{(\beta^{n+1} + n(n+2)\beta^{n+1} - \alpha^2)(\alpha^{n+1} - n\beta^{n+1})^2}{\alpha^{2(n+2)}}\right\}. \tag{13}$$

In addition, reciprocal tensor g^{ij} of fundamental metric tensor of Finsler space F^n is

$$g^{ij} = \frac{a^{ij}}{p} - S_0 b^i b^j - S_1(b^i y^j + b^j y^i) - S_2 y^i y^j, \tag{14}$$

where the coefficients b^i, S_0, S_1 and S_2 are defined as follows:

$$b^i = a^{ij}b_j,$$

$$S_0 = \frac{pp_0 + (p_0p_2 - p_1^2)\alpha^2}{p\zeta}, \tag{15}$$

$$S_1 = \frac{pp_1 + (p_0p_2 - p_1^2)\beta}{p\zeta}, \tag{16}$$

$$S_2 = \frac{pp_2 + (p_0p_2 - p_1^2)b^2}{p\zeta}, \tag{17}$$

$$\zeta = p(p + p_0b^2 + p_1\beta) + (p_0p_2 - p_1^2)(\alpha^2b^2 - \beta^2), \tag{18}$$

where, $b^2 = a_{ij}b^ib^j$. Now, the hv-torison tensor $C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}$ is defined by

$$C_{ijk} = \frac{p_1(h_{ij}m_k - h_{jk}m_i + h_{ki}m_j) + \gamma_1 m_im_jm_k}{2p} \tag{19}$$

where the coefficients γ_1 and m_i are defined as

$$\gamma_1 = p\frac{\partial p_0}{\partial\beta} - 3p_1q_0, \quad m_i = b_i - \frac{y_i\beta}{\alpha^2}, \tag{20}$$

where m_i is known as non zero covariant vector orthogonal to element of support y^i. Let Γ^i_{jk} be the components of Christoffel symabol of the associated Riemannian space R^n and ∇_k be the covariant differentiation with respect to x^k relative to Christoffel symbol.
Now we put

$$2F_{ij} = b_{ij} - b_{ji}, \tag{21}$$

where, $b_{ij} = \nabla_j b_i$. Let $C\Gamma = (\Gamma^{*i}_{jk}, \Gamma^{*i}_{0k}, C^i_{jk})$ be Cartan connection of F^n. The difference tensor $D^i_{jk} - \Gamma^i_{jk}$ of the special Finsler space F^n is given by

$$\begin{aligned} D^i_{jk} = {} & B^ir_{jk} + s^i_kB_j + s^i_jB_k + B^i_jb_{0j} - b_{0m}g^{im}B_{jk} - C^i_{jm}A^m_k - C^i_{km}A^m_j \\ & + C_{jkm}A^m_sg^{is} + \gamma^s(C^i_{jm}Csk^m + C^i_{km}C^m_{sj} - C^m_{jk}C^i_{ms}), \end{aligned} \tag{22}$$

where

$$B_k = p_0b_k + p_1y_k. \tag{23}$$

Using (10) and (11) in (23) and simplification we get

$$B_k = \left\{\frac{(n+1)[n\alpha^{n+1}\beta^{n+1} + (2n+1)\beta^{2n}]}{\alpha^{2(n+2)}}\right\} b_k + \left\{\frac{(n+1)\beta^n}{\alpha^{2(n+1)}}[(1-n)\alpha^{n+1} - 2n\beta^{n+1}]\right\} \tag{24}$$

$$B^i = g^{ij}B_j \tag{25}$$

$$
\begin{aligned}
F_i^k &= g^{kj}F_{ji} \\
B_{ij} &= \frac{p_1(a_{ij}-\frac{y_iy_j}{\alpha^2})+\frac{\partial p_0}{\partial \beta}m_im_j}{2}, && (26)\\
B_i^k &= g^{kj}B_{ji} && (27)\\
A_k^m &= B_k^m r_{00}+B^m r_{k0}+B_k s_0^m+B_0 s_k^m && (28)
\end{aligned}
$$

Using (23) in (24) we have

$$
\begin{aligned}
A_k^m &= B_k^m r_{00}+B^m r_{k0}+\left\{\frac{(n+1)[n\alpha^{n+1}\beta^{n+1}+(2n+1)\beta^{2n}]}{\alpha^{2(n+2)}}\right\}b_k \\
&\quad -\left\{\frac{(n+1)\beta^n}{\alpha^{2(n+1)}}-2n\beta^{n+1}\right\}y_kF_0^m+B_0F_k^m && (29)\\
\lambda^m &= B^m r_{00}+2B_0F_0^m && (30)\\
B_0 &= B_iY^i && (31)
\end{aligned}
$$

where B_i, we get,

$$
B_k=\left\{\frac{(n+1)[n\alpha^{n+1}\beta^{n+1}+(2n+1)\beta^{2n}]}{\alpha^{2(n+2)}}\right\}b_i+\left\{\left\{\frac{(n+1)\beta^n}{\alpha^{2(n+1)}}-2n\beta^{n+1}\right\}y_i\right\}Y^i. \qquad (32)
$$

By the view of first section on the Generalized Square metric $F(\alpha,\beta)=\alpha+\frac{\beta^{n+1}}{\alpha^n}$, the equations 8, 10, 11 and 12 we have $p=1$, $q_0=0$, $q_2=-\frac{1}{\alpha^2}$, $p_1=0$, $p_2=0$, $s_0=0$, $s_1=0$, $s_2=0$, $\zeta=1$. Using the value of p, s_0, s_1, s_2, the equation **??** we have

$$
g^{ij}=a^{ij} \qquad (33)
$$

Multiplying equation 49 by b_ib_j and using $b_iy^i=0$, $g^{ij}b_ib_j=b^2=b$, $B=bN^jb_j=\sqrt{b^2}g^{ij}g_{i\alpha}N^\alpha$ $b_j=\sqrt{b^2}g^{ij}N_ib_j$. Therefore, we get

$$
b_i=\sqrt{b^2}N_i,\ b^2=a^{ij}b_ib_j \qquad (34)
$$

i,e, $b_i(x(u))=bN_i$, where b is the length of the vector b^i. Again from (26) and (27) we get

$$
b^i=bN_i \qquad (35)
$$

3.1 Induced Cartan Connection

Let $F^n=(M^n,F)$, where $F=\alpha+\frac{\beta^2}{\alpha}$, be a Finsler space and let F^{n-1} be its hypersurface having equation $x^i=x^i(u^\alpha), i=1,2,...(n-1)$. Let the matrix of projection factor be $B_\alpha^i=\partial x^j/\partial u^\alpha$ and its rank is $(n-1)$. The tangential component of element of support y^i of Finsler space F^n along its hypersurface F^{n-1} is given by

$$y^i = B^i_\alpha(u)v^\alpha. \tag{36}$$

Therefore, v^α is the element of support of hypersurface F^{n-1} at the point u^α. The metric tensor $g_{\alpha\beta}$ and hv-torison tensor $C_{\alpha\beta\gamma}$ of hypersurface F^{n-1} are defined by

$$g_{\alpha\beta} = g_{ij}B^i_\alpha B\beta^j, \ C_{\alpha\beta\gamma} = C_{ijk}B^i_\alpha B^j_\beta B^k_\gamma. \tag{37}$$

Now the unit normal vector $N^i(u, v)$ at an arbitrary point u^α of the hypersurface F^{n-1} is defined by the following property;

$$g_{ij}(x(u,v), y(u,v))B^i_\alpha N^j = 0, \ \ g_{ij}(x(u,v), y(u,v))N^i N^j = 1 \tag{38}$$

The angular metric tensor h_{ij} , we have

$$h_{\alpha\beta} = h_{ij}B^i_\alpha B^j_\beta, \ \ h_{ij}B^i_\alpha N^j = 0, \ \ h_{ij}N^i N^j = 1. \tag{39}$$

Let (B^α_i, N_i) be the inverse of (B^i_{α,N^i}), then we have

$$B^\alpha_i = g^{\alpha\beta}g_{ij}B^j_\beta, \ \ B^i_\alpha B^\beta_i = \delta^\beta_\alpha, \ \ B^\alpha_i N^i = 0, \tag{40}$$

$$B^i_\alpha N_i = 0, N_i = g_{ij}N^i, \ \ B^k_i = g^{kj}B_{ji}, \tag{41}$$

$$B^i_\alpha B^\alpha_j + N^i N_j = \delta^i_j. \tag{42}$$

The induced Cartan connection $\pounds C\Gamma = (\Gamma^{*\alpha}_{\beta\gamma}, G^\alpha_\beta, C^\alpha_{\beta\gamma})$ on hypersurface F^{n-1} induced from the Cartan's connection $C\Gamma = (\Gamma^{*\alpha}_{jk}, \Gamma^{*i}_{0k}, C^i jk)$ is given by [3];

$$\Gamma^{*\alpha}_{\beta\gamma} = B^\alpha_i(B^i_{\beta\gamma} + \Gamma^{*i}_{jk}B^j_\beta B^k_\gamma) + M^\alpha_\beta H_\gamma, \tag{43}$$

$$G^\alpha = B^\alpha_i(B^i_{0\beta} + \Gamma^{*i}_{0j}B^j_\beta), \tag{44}$$

$$C^\alpha_{\beta\gamma} = B^\alpha_i C^i_{jk}B^j_\beta B^k_\gamma, \tag{45}$$

where

$$M_{\beta\gamma} = N_i C^i_{jk}B^j_\beta \gamma^k, \tag{46}$$

$$M^\alpha_\beta = g^{\alpha\gamma}M_{\beta\gamma}, \tag{47}$$

$$H_\beta = N_i(B^i_{0\beta} + \Gamma^{*i}_{0j}B^j_\beta), \tag{48}$$

$$B^i_{\beta\gamma} = \frac{\partial B^i_\beta}{\partial U^\gamma}, \tag{49}$$

$$B^i_{0\beta} = b^i_{\alpha\beta} v^\alpha. \tag{50}$$

The quantities $M_{\beta\gamma}$ and H_β appeared in above equation are called the second fundamental v-tensor $H_{\beta\alpha}$ is defined as [8];

$$H_{\beta\gamma} = N_i(B^i_{\beta\gamma} + \Gamma^{*i}_{jk} B^j_\beta B^k_\gamma) + M_\beta H_\gamma, \tag{51}$$

$$M_\beta = N_i C^i_{jk} B^j_\beta N^k. \tag{52}$$

The relative h-covariant derivative and v-covariant derivative of projection factor B^i_α with respect to induced Cartan connection £CΓ are respectively given by;

$$B^i_{\alpha|\beta} = H_{\alpha\beta} N^i. \tag{53}$$

$$B^i_{\alpha|\beta} = M_{\alpha\beta} N^i. \tag{54}$$

The equation (10) shows that $H_{\beta\gamma}$ is not always symmetric and

$$H_{\beta\gamma} - H_{\gamma\beta} = M_\beta H_\gamma - M_\gamma H_\beta. \tag{55}$$

Thus, the above equation simplifies to

$$H_{0\gamma} = H_\gamma, H_{\gamma 0} = H_\gamma + M_\gamma H_0. \tag{56}$$

Here, we shall use the following definitions and lemmas which are defined by M.Matstumoto in [9];

Definition 3.1. *If each path of a hypersurface F^{n-1} with respect to the induced connection is also a path of the enveloping space F^n, then F^{n-1} is called a hyperplane of the first kind.*

Definition 3.2. *If each path of a hypersurface F^{n-1} with respect to the induced connection is also a h- path of the enveloping space F^n, then F^{n-1} is called a hyperplane of the second kind.*

Definition 3.3. *If the unit normal vector of $F^{n-1}$is parallel along each curve of F^{n-1}, then F^{n-1} is called a hyperplane of the third kind.*

We utilize the following lemma derived by M.Matsumoto [9] in order to prove our hypothesis.

Lemma 3.4. *The normal curvature $H_0 = H_\beta v^\beta$ vanishes if and only if normal curvature vector H_β vanishes.*

Lemma 3.5. *A hypersurface F^{n-1} is a hyperplane of first kind if and only if $H_\alpha = 0$.*

Lemma 3.6. *A hypersurface F^{n-1} is a hyperfurface of second kind with respect to Cartan connection CΓ if and only if $H_\alpha = 0$ and $H_{\alpha\beta} = 0$*

Lemma 3.7. *A hypersurface F^{n-1} is a hyperplane of third kind with respect to Cartan connection CΓ if and only if H_α , $H_{\alpha\beta} = 0$ and $M_{\alpha\beta} = 0$.*

4. Hypersurface of Finsler Space with Special-(α,β) Metric

This section determines the necessary and sufficient condition of F^n for hypersurface to be hyperplane is satisfied the first kind, senond kind and third kind. Let us consider the hypersurface $F^{n-1}(c)$ is given by

$$b_i(x)=c, \tag{57}$$

where c is a fixed constant. Thus, the gradient of the function representing hypersurface $F^{n-1}(c)$ with tensor notation

$$b_i(x)=\partial_i b.$$

Again, consider the parametric equation $x^i(u^\alpha)$ of the hypersurface $F^{n-1}(c)$. Differentiating the equation (65) with respect to parameter u^α. we get $\partial_\alpha b(x(u))=0=b_iB^i_\alpha$. It is obvious that $b_i(x)$ are the covariant component of normal vector field of hypersurface F^{n-1}. Therefore along the hypersurface F^{n-1}, we have

$$b_iB^i_\alpha=0. \tag{58}$$

$$b_iy^i=0. \tag{59}$$

The metric induced $L(u,v)$ from the Finsler space with special (α,β)-metric (M^n,F), where $F=\alpha+\frac{\beta^2}{\alpha}$ on the hypersurface F^{n-1} is given by

$$L(u,v)=a_{\alpha\beta}u^\alpha v^\beta, \tag{60}$$

where, $a_{\alpha\beta}=a_{ij}B^i_\alpha B^j_\beta$. The induced metric in (60) do not have β component $(i.e.,\beta=b_iy^i=0)$ of the Finsler metric of the original space (M^n,F), therefore induced metric in equation (60) is a Riemannian metric. Therefore at any point on the hypersurface $F^{n-1}(c)$. Equations (4), (6), (8), (10), (11) and (12) reduced to

$$p=1,\ \ q_0=2,\ \ q_1=0,\ \ q_2=-1,\ \ p_0=2,\ \ p_1=0,\ \ p_2=0 \tag{61}$$

$$s_0=\frac{2}{1+2b^2},\ \ s_1=0,\ \ s_2=0,\ \ \varsigma=1+2b^2. \tag{62}$$

Theorem 4.1. *The second fundamental hv- tensor of Finsler hypersurface $F^{n-1}(c)$ is given by $M_{\alpha\beta}=0$ and $M_\alpha=0$ and the second fundamental hv - tensor $H_{\alpha\beta}$ is symmetric.*

Proof. The angular and fundamental metric tensor of F^n are given by $h_{ij}=pa_{ij}+q_0b_ib_j+q_1(b_iy_j+b_jy_i)+q_2y_iy_j$,

$$h_{ij}=a_{ij}-\frac{1}{\alpha^2}y_iy_j,\ \ h_{ij}=a_{ij}\frac{y_iy_j}{\alpha^2} \tag{63}$$

$$g_{ij} = pa_{ij} + p_0 b_i b_j + p_1(b_i y_j + b_j y_i) + p_2 y_i y_j \tag{64}$$

From the values of p_0, p_1, and p_2, we have $g_{ij} = a_{ij}$. From (22), (30) and (13) it follows that if $H^{\alpha_{\alpha\beta}}$ denote the angular metric tensor of the Riemannian $a_{ij}(x)$ then along $F^{n-1}(c)$, $h_{\alpha\beta} = H^{\alpha}_{\alpha\beta}$. From (4), we get

$$\frac{\partial p_0}{\partial\beta} = \frac{(n+1)[n(n+1)\alpha^{n+1}\beta^{n-2} + 2n(2n+1)\beta^{2n-1}]}{\alpha^{2n}}$$

Thus along $F^{n-1}(c)$, $\frac{\partial p_0}{\partial\beta} = 0$ and therefore (6) gives $\gamma_1 = 0$, $m_i = bb_i$. Therefore the hv- torison tensor becomes $C_{ijk} = \frac{p_1(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j + \gamma_1 m_i m_j m_k)}{2p}$

$$C_{ijk} = 0. \tag{65}$$

Substituting the value of C_{ijk} from (49) as follows

$$M_{\alpha\beta} = C_{ijk} B^i_\gamma B^j_\beta N^k$$
$$M_{\alpha\beta} = 0$$

Again substituting the value C_{ijk} from (51) into (65)

$$M_\beta = N_i C^i_{jk} B^j_\beta N^k$$
$$M_{alpha} = C^i_{jk} Bj_\alpha N_i N^k$$
$$M_\alpha = 0.$$

Substituting the value of M_α from (45) in (66) as follows

$$H_{\beta\alpha} H_{\alpha\beta} = M_\beta H_\alpha - M_\alpha H_\beta$$
$$H_{\beta\alpha} - H_{\alpha\beta} = 0$$
$$H_{\beta\alpha} = H_{\alpha\beta}$$

i. e., $H_{\alpha\beta}$ is symmetric. □

Theorem 4.2. *The special Finsler hypersurface $F^{n-1}(c)$ to be hyperplane of first kind is that $2b_{ij} = b_i c_j + b_j c_i$.*

Proof. Differentiating (24) with respect to β we get

$$b_{i|j} B^i_\alpha + b_i B^i_{\alpha|\beta} = 0 \tag{66}$$

Using (50) and $b_{i|j} B^j_\beta + b_{i|j} N^j H_\beta$, (66) reduces to

$$b_{i|j} B^j_\beta B^i_\alpha + b_{i|j} N^j H_\beta B^i_\alpha + b_i H_{\alpha\beta} N^i = 0 \tag{67}$$

Since $b_{i|j}=-b_hC^h_{ij}$, we get $b_{i|j}B^i_\alpha N^j=0$

$$bH_{\alpha\beta}+b_{i|j}B^j_\beta B^i_\alpha=0. \tag{68}$$

It is noted that $b_{i|j}$ is symmetric. Further more contracting (50) with v^β first and then with v^α respectively and using (3.1), (3.18) and (34) and (35) we get

$$bH_\alpha+b_{i|j}B^i_\alpha y^j=0 \tag{69}$$

$$bH_0+b_{i|j}y^iy^j=0. \tag{70}$$

According to the Lemma 3.4 and 3.5, hypersurface $F^{n-1}(c)$ is a hyperplane of first kind if and only if H_0. Thus from (32) we find that hypersurface $F^{n-1}(c)$ is again a hyperplane of first kind if and only if $b_{i|j}y^iy^j=0$, where $b_{i|j}$ is the covariant derivative with respect to Cartan connection $C\Gamma$ of Finsler space F^n , it may depend on y^i. Since b_i is a gradient vector, from (17) we have $E_{ij}=b_{ij}=0$, $F_{ij}=0$. Thus (18) reduces to

$$\begin{aligned}D^i_{jk}=&B^iE_{jk}+F^i_kB_j+F^i_jB_k+B^i_jb_0j+B^i_kb_{0j}-b_{0m}g^{im}B_{jk}\\&-C^i_{jm}-C^i_{km}A^m_j+C_{jkm}A^m_sg^{is}+\lambda^s(C^i_{jm}C^m_{sk})+C^i_{km}C^m_{jk}C^i_{ms}.\end{aligned} \tag{71}$$

Using the equations (19), (20), 21), (23), (24) and (25) the relation (28) becomes to

$$B_k=0,B^i=0,B^i_j=0,B_{ij}=0,B_{ji}=0,\lambda^m=0,A^m_k=0. \tag{72}$$

By virtue of (42) we have $B_{i0}=0$, $B^i_0=0$, which leads $A^m_k=0$. Therefore, we have

$$D^i_{j0}=0 \tag{73}$$

$$D^i_{00}=0 \tag{74}$$

Multiplying b_i in equation (43) and (44) we get

$$b_iD^i_{j0}=0 \tag{75}$$

$$b_iD^i_{00}=0. \tag{76}$$

From equation (33) it follows that $b^mb_iC^i_{jm}B^j_\alpha=b^2M_\alpha=0$. Therefore, the relation $b_{i|j}=b_{ij}-b_rD^r_{ij}$ and equation (45) and (46) gives $b_{i|j}y^iy^j=b_{00}$.

Consequently, (39) and 40 may be written as $bH_\alpha+b_{i|j}B^i_\alpha=0$, $bH_0+b_{00}=0$. Thus the condition $H_\alpha=0$ is equivalent $b_{00}=0$, where b_{ij} is does not depend ony^i. Since y^i satisfies the equation (18), the condition can be written as $b_{ij}y^iy^j=(b_{ij}y^i)(C_jy^i)$ for some $c_j(x)$, so that we have

$$2b_{ij}=b_ic_j+b_jc_i \tag{77}$$

□

Theorem 4.3. *The special hypersurface $F^{n-1}(c)$ to be hyperplane of first kind then it becomes a hyperplane of second kindis that equation $b_{ij}=eb_ib_j$.*

Proof. By Lemma 3.6, $F^{n-1}(c)$ is a hyperplane of second kind if and only if $H_\alpha=0$ and $H_{\alpha\beta}=0$. Here there are two cases arises as follows
Case 1: If $H_{\alpha\beta}=0$, then by equation (38) becomes

$$\begin{aligned} bH_{\alpha\beta}+b_{i|j}B^i_\beta B^i_\alpha &= 0\\ H_{\alpha\beta} &= 0. \end{aligned}$$

Case 2: Again if $H_\alpha=0$, then Lemma 3.4 and 3.5 imply H_0=0, $2b_{ij}=b_ic_j+b_ic_i$, $2b_{ij}=b_ie(x)b_j(x)+b_je(x)b_i(x)$, $2B_{ij}=e(b_ib_j+b_jb_i)$, $2b_{ij}=e2b_ib_j$, $2b_{ij}=2eb_ib_j$, $b_{ij}=eb_ib_j$. □

Theorem 4.4. *The necessary and sufficient condition of the Finsler space with special (α,β)- metric on hypersurface $F^{n-1}(c)$ to be a hyperplane of third kind if and only if it is a hyperplane of third kind.*

Proof. By Lemma 3.7, hypersurface $F^{n-1}(c)$ is hyperlane of third kind with respect to Cartan connection $c\Gamma$ if and only if $H_\alpha=0$ and $M_{\alpha\beta}=0$. From the equation (34), we get $M_{\alpha\beta}=0$. Finally hyprsurface $F^{n-1}(c)$ is hypreplane of third kind. □

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