Int. J. Math. And Appl., 8(1)(2020), 33–39
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Non Linear Contraction Mapping and its Application in Dynamic Programming

Himanshu Tiwari[1,*] and Subhashish Biswas[1]

1 Department of Mathematics, Kalinga University, Raipur, Chhattisgarh, India.

Abstract: We introduce the definition of $(\alpha-\beta)$-tripled fixed point in the space of the bounded functions on a set S and we present a result about the existence and uniqueness of such points. Moreover, as an application of our result, we study the problem of existence and uniqueness of solutions for a class of systems of functional equations arising in dynamic programming.

MSC: Primary 47H10, Secondary 54H25.

Keywords: $(\alpha-\beta)$-tripled fixed point, real valued function, functional equations, dynamic programming.

1. Introduction and Preliminaries

In the two last decades, the theory of fixed points has appeared as a crucial technique in the study of nonlinear functional analysis. In particular, the techniques and tools in fixed point theory have application in many branches of applied mathematics and also in many research fields such as physics, chemistry, biology, economics, computer sciences, and many branches of engineering. The most significant result in fixed point theory, known as the Banach Contraction Mapping Principle (BCMP) is given by Banach in [1]. BCMP states that every contraction (self-mapping) $T : X \to X$ on a complete metric space (X, d) has a unique fixed point, that is, $Tx = x$. Due to its wide application potential, this celebrated principle has been generalized in many ways over the years. Generalizations of the above principle have been a heavily investigated branch of research. Particularly, one of these generalizations uses the so-called comparison functions. These functions are defined as functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ which are increasing and satisfy $\varphi^n(t) = 0$ when $n \to \infty$ for $t > 0$ where denotes the n-iteration of φ. Examples of such functions are $\varphi(t) = \lambda(t)$ with $\lambda \in (0, 1), \varphi(t) = \arctan t$, $\varphi(t) = \ln(1 + t)$ and $\varphi(t) = \frac{t}{1+t}$ among others. The above-mentioned generalization of the Banach contraction mapping principle is the following result and it appears in [5, 6].

Theorem 1.1. *Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping satisfying $d(Tx, Ty) \leq \phi(d(x, y))$, for any $x, y \in X$, where ϕ is a comparison function. Then T has a unique fixed point.*

In this paper, we consider a nonempty set S and by $B(S)$ we denote the set of all bounded real functions defined on S. According to the ordinary addition of functions and scalar multiplication and endowing with the norm $\|u\| = \sup_{x\in S} |u(x)|, B(S)$,

* E-mail: himanshu.tiwari.ht@gmail.com

is a Banach space. Notice that the distance in $B(S)$ is defined as

$$d(u,v) = \sup_{x\in S}\{|u(x) - v(x)|\}, \quad \forall \;\; x, y \in B(S).$$

The aim of this paper is to present a result about the existence and uniqueness of an $(\alpha-\beta)$-tripled fixed point (see Section 2) in $B(S)$ and, as an application of this result, we will study the problem of existence and uniqueness of solutions of the following system of functional equations arising in dynamic programming:

$$\begin{aligned} u(x) &= \sup_{y\in D}\{g(x,y) + F(x,y,u(T(x,y)),v(T(x,y)),w(T(x,y)))\} \\ v(x) &= \sup_{y\in D}\{g(x,y) + F(x,y,u(\alpha(T(x,y))),v(\alpha(T(x,y))),w(\alpha(T(x,y))))\} \\ w(x) &= \sup_{y\in D}\{g(x,y) + F(x,y,u(\beta(T(x,y))),v(\beta(T(x,y))),w(\beta(T(x,y))))\} \end{aligned} \tag{1}$$

under certain assumptions. For further information about the functional equations appearing in dynamic programming, we refer the reader to [7–10].

2. Generalized $(\alpha-\beta)$-Tripled Fixed Point Theorem in $B(S)$

Our starting point in this section is the definition of $(\alpha-\beta)$-tripled fixed point in $B(S)$. For this purpose, suppose that S is a nonempty set and $\alpha : S \to S$ a mapping.

Definition 2.1. *An element $(u,v) \in B(S)\times B(S)$ is called an $(\alpha-\beta)$-tripled fixed point of a mapping $G : B(S)\times B(S) \to B(S)$ if*

$$G(u,v,w) = u,$$
$$G(u\circ\alpha, v\circ\alpha, w\circ\alpha) = v,$$
$$G(u\circ\beta, v\circ\beta, w\circ\beta) = w$$

The following theorem is the main result of the paper and it gives us a sufficient condition for the existence and uniqueness of an $(\alpha-\beta)$-tripled fixed point.

Theorem 2.2. *Suppose that $\alpha, \beta : S \to S$ and $G : B(S)\times B(S)\times B(S) \to B(S)$ are two mappings. If G satisfies*

$$d(G(x,y,z), G(u,v,w)) \le \phi(\max\{d(x,u), d(y,v), d(z,w)\}) \tag{2}$$

for any $x,y,z,u,v,w \in B(S)$, where ϕ is a comparison function, then G has a unique $(\alpha-\beta)$-tripled fixed point.

Proof. Consider the Cartesian product $B(S)\times B(S)\times B(S)$ endowed with the distance defined by

$$\overline{d}((x,y,z),(u,v,w)) = \max\{d(x,u), d(y,v), d(z,w)\} \tag{3}$$

It is easily seen that $(B(S)\times B(S)\times B(S), d)$ is a complete metric space. Now, we consider the mapping $\overline{G} : B(S)\times B(S)\times B(S) \to B(S)$ defined by

$$\overline{G}(x,y,z) = (G(x,y,z), G(x\circ\alpha, y\circ\alpha, z\circ\alpha), G(x\circ\beta, y\circ\beta, z\circ\beta)) = G_{(x,y,z)} \tag{4}$$

Notice that if $x \in B(S)$ then $x \circ \alpha \in B(S)$ and $x \circ \beta \in B(S)$. Next, we check that $\overline{G}$ satisfies assumptions of Theorem 1.1. In fact, according to (2), we have that for any $x, y, z, u, v, w \in B(S)$

$$\begin{aligned}
\overline{d}(\overline{G}(x,y,z), \overline{G}(u,v,w)) &= \overline{d}(G_{(x,y,z)}, G_{(u,v,w)}) \text{ (from (4))} \\
&= \max \begin{Bmatrix} d((G(x,y,z), G(u,v,w)), \\ d(G(x\circ\alpha, y\circ\alpha, z\circ\alpha), G(u\circ\alpha, v\circ\alpha, w\circ\alpha)) \\ d(G(x\circ\beta, y\circ\beta, z\circ\beta), G(u\circ\beta, v\circ\beta, w\circ\beta)) \end{Bmatrix} \\
&\leq \max \begin{Bmatrix} \phi(\max\{d(x,u), d(y,v), d(z,w))\}), \\ \phi(\max\{d(x\circ\alpha, u\circ\alpha), d(y\circ\alpha, v\circ\alpha), d(z\circ\alpha, w\circ\alpha)\}), \\ \phi(\max\{d(x\circ\beta, u\circ\beta), d(y\circ\beta, v\circ\beta), d(z\circ\beta, w\circ\beta)\}) \end{Bmatrix}. \qquad (5)
\end{aligned}$$

Now, taking into account the definition of the distance on $B(S)$, we have

$$\begin{aligned}
d(x\circ\alpha, u\circ\alpha) &= \sup_{s\in S}\{|(x\circ\alpha)(s) - (u\circ\alpha)(s)|\} \\
&= \sup_{s\in S}\{|x(\alpha(s)) - u(\alpha(s))|\} \\
&\leq \sup_{s\in S}\{|x(s) - u(s)|\} = d(x,u) \qquad (6)
\end{aligned}$$

and, by a similar argument, we have

$$d(y\circ\alpha, v\circ\alpha) \leq d(y,v) \quad \textit{and}$$

$$d(z\circ\alpha, w\circ\alpha) \leq d(z,w)$$

Therefore, from ((5) and (6), we get

$$\overline{d}(\overline{G}(x,y,z), \overline{G}(u,v,w)) \leq \phi(\max\{d(x,u), d(y,v), d(z,w))\} = \phi(\overline{d}((x,y,z),(u,v,w))).$$

Therefore, Theorem 1.1 gives us the existence of a unique $(x_0, y_0, z_0) \in B(S) \times B(S) \times B(S)$ such that $\overline{G}(x_0, y_0, z_0) = (x_0, y_0, z_0)$ or equivalently $G(x_0, y_0, z_0) = x_0$, $G(x_0\circ\alpha, y_0\circ\alpha, z_0\circ\alpha) = y_0$ and $G(x_0\circ\beta, y_0\circ\beta, z_0\circ\beta) = z_0$. This completes the proof. □

3. Application to Dynamic Programming

The following types of systems of functional equations

$$\begin{aligned}
u(x) &= \sup_{y\in D}\{g(x,y) + F(x,y,u(T(x,y)), v(T(x,y)), w(T(x,y)))\} \\
v(x) &= \sup_{y\in D}\{g(x,y) + F(x,y,u(\alpha(T(x,y))), v(\alpha(T(x,y))), w(\alpha(T(x,y))))\} \\
w(x) &= \sup_{y\in D}\{g(x,y) + F(x,y,u(\beta(T(x,y))), v(\beta(T(x,y))), w(\beta(T(x,y))))\} \qquad (7)
\end{aligned}$$

appear in the study of dynamic programming (see [11]), where $x \in S$ and S is a state space, D is a decision space, $T : S \times D \to S$, $g : S \times D \to \mathbb{R}$, $F : S \times D \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\alpha, \beta : S \to S$ are given mappings. The following theorem gives us a sufficient condition for the existence and uniqueness of solutions to problem (7).

Theorem 3.1. *Suppose the following assumptions:*

(i). $g : S \times D \to \mathbb{R}$ and $F(-, -0, 0, 0) : S \times D \to \mathbb{R}$ are bounded functions,

(ii). there exists a comparison function φ such that for any $x \in S, y \in D$, $t, s, r, t_1, s_1, r_1\mathbb{R}$,

$$|F(x, y, t, s, r) - F(x, y, t_1, s_1, r_1)| \leq \varphi(\max\{|t - t_1|, |s - s_1|, |r - r_1|\})$$

Then, problem (7) has a unique solution $(u_0, v_0, r_0) \in B(S) \times B(S) \times B(S)$.

As a previous result for the proof of Theorem 3.1, we need the next lemma.

Lemma 3.2. *Suppose that $H, G : S \to \mathbb{R}$ are two bounded functions.Then*

$$|\sup_{y\in S} H(y) - \sup_{y\in S} G(y)| \leq \sup_{y\in S} |H(y) - G(y)|. \tag{8}$$

Proof. Obviously, this result is true when $\sup_{y\in S} |H(y)| = \sup_{y\in S} |G(y)|$. If we suppose that $\sup_{y\in S} |H(y)| > \sup_{y\in S} |G(y)|$ (same argument works if we suppose that $\sup_{y\in S} |H(y)| < \sup_{y\in S} |G(y)|$ then for any $y_0 \in S$

$$H(y_0) - \sup_{y\in S} |G(y)| \leq H(y_0) - G(y_0) \leq |(y_0) - G(y_0)| \tag{9}$$

and, consequently,

$$\sup_{y\in S}\{H(y) - \sup_{y\in S} |G(y)|\} \leq \sup_{y\in S}\{|H(y) - G(y)|\} \tag{10}$$

Since $\sup_{y\in S}\{H(y) - a\} = \sup_{y\in S}\{H(y)\} - a$ for any $a \in \mathbb{R}$ it follows

$$\sup_{y\in S}\{H(y)\} - \sup_{y\in S}\{G(y)\} \leq \sup_{y\in S}\{|H(y) - G(y)|\} \tag{11}$$

and this proves our claim. □

Proof. To proof of Theorem 3.1. Consider the operator G defined on $B(S) \times B(S) \times B(S)$ as

$$G(u, v, w)(x) = \sup_{y\in D}\{g(x, y) + F(x, y, u(T(x, y)), v(T(x, y)), w(T(x, y)))\} \tag{12}$$

for $(u, v, w) \in B(S) \times B(S) \times B(S)$ and $x \in S$. By assumptions (i) and (ii), we have

$$\begin{aligned} |G(u, v, w)(x)| &\leq \sup_{y\in D} |g(x, y) + F(x, y, u(T(x, y)), v(T(x, y)), w(T(x, y)))| \\ &\leq \sup_{y\in D} |g(x, y)| + \sup_{y\in D} |F(x, y, u(T(x, y)), v(T(x, y)), w(T(x, y)))| \\ &\leq \sup_{y\in D} |g(x, y)| + \sup_{y\in D} |F(x, y, u(T(x, y)), v(T(x, y)), w(T(x, y))) - F(x, y, 0, 0, 0) + F(x, y, 0, 0, 0)| \\ &\leq \sup_{y\in D} |g(x, y)| + \sup_{y\in D}\{\varphi(\max\{u(T(x, y)), v(T(x, y)), w(T(x, y))\} + |F(x, y, 00, 0)|\} \end{aligned} \tag{13}$$

According to assumption (i) and, since $(u, v, w) \in B(S) \times B(S) \times B(S)$, we obtain that $G(u, v, w) \in B(S)$. Therefore, $G : B(S)\times B(S)\times B(S) \to B(S)$. Now, we check that G satisfies condition (2) of Theorem 2.2. In fact, for any $u, v, w, u_1, v_1, w_1 \in B(S)$, we have

$$d(G(u, v, w), G(u_1, v_1, w_1)) = \sup x \in S|G(u, v, w) - G(u_1, v_1, w_1)|. \tag{14}$$

Then, from assumption (ii) and Lemma 3.2 and using the fact that φ is an increasing function, for any $x \in S$, we have

$$\begin{aligned}
|G(u,v,w) - G(u_1,v_1,w_1)| &= |\sup_{y\in D}\{g(x,y) + F(x,y,u(T(x,y)),v(T(x,y)),w(T(x,y)))\} \\
&\quad - \sup_{y\in D}\{g(x,y) + F(x,y,u_1(T(x,y)),v_1(T(x,y)),w_1(T(x,y)))\}| \\
&= |\sup_{y\in D}\{F(x,y,u(T(x,y)),v(T(x,y)),w(T(x,y))) - F(x,y,u_1(T(x,y)),v_1(T(x,y)),w_1(T(x,y)))\}| \\
&\leq \varphi(\max\{|u(T(x,y)) - u_1(T(x,y))|, |v(T(x,y)) - v_1(T(x,y))|, |w(T(x,y)) - w_1(T(x,y))|\} \\
&\leq \varphi(\max\{\|u-u_1\|, \|v-v_1\|, \|w-w_1\|\} \\
&\leq \varphi(\max\{d(u,u_1), d(v,v_1), d(w,w_1)\}. \qquad (15)
\end{aligned}$$

Therefore, condition (2) of Theorem 2.2 is satisfied and, consequently, G has a unique $(\alpha-\beta)$-tripled fixed point $(u_0,v_0,w_0) \in B(S)\times B(S)\times B(S)$. This means that $G(u_0,v_0,w_0) = u_0$, $G(u_0\circ\alpha, v_0\circ\alpha, w_0\circ\alpha) = v_0$ and $G(u_0\circ\beta, v_0\circ\beta, w_0\circ\beta) = w_0$ or, equivalently, for $x \in S$,

$$\begin{aligned}
u_0(x) &= \sup_{y\in D}\{g(x,y) + F(x,y,u_0(T(x,y)),v_0(T(x,y)),w_0(T(x,y)))\} \\
v_0(x) &= \sup_{y\in D}\{g(x,y) + F(x,y,u_0(\alpha(T(x,y))),v_0(\alpha(T(x,y))),w_0(\alpha(T(x,y))))\} \\
w_0(x) &= \sup_{y\in D}\{g(x,y) + F(x,y,u_0(\beta(T(x,y))),v_0(\beta(T(x,y))),w_0(\beta(T(x,y))))\} \qquad (16)
\end{aligned}$$

This completes the proof. □

In order to illustrate our results, we present the following example. Consider the following system of functional equations, where $x \in [0,1]$,

$$\begin{aligned}
u(x) &= \sup_{y\in\mathbb{R}}\left\{e^{-(x+|y|)} + \arctan\left(\frac{1}{3}(x+|y|+|u(|\sin(x+y)|)|+|v(|\sin(x+y)|)|+|w(|\sin(x+y)|)|)\right)\right\} \\
v(x) &= \sup_{y\in\mathbb{R}}\left\{e^{-(x+|y|)} + \arctan\left(\tfrac{1}{3}\left(x+|y|+\left|u\left(\tfrac{1}{1+|\sin(x+y)|}\right)\right|+\left|v\left(\tfrac{1}{1+|\sin(x+y)|}\right)\right|+\left|w\left(\tfrac{1}{1+|\sin(x+y)|}\right)\right|\right)\right)\right\} \\
w(x) &= \sup_{y\in\mathbb{R}}\left\{e^{-(x+|y|)} + \arctan\left(\tfrac{1}{3}\left(x+|y|+\left|u\left(\tfrac{|\sin(x+y)|}{1+|\sin(x+y)|}\right)\right|+\left|v\left(\tfrac{|\sin(x+y)|}{1+|\sin(x+y)|}\right)\right|+\left|w\left(\tfrac{|\sin(x+y)|}{1+|\sin(x+y)|}\right)\right|\right)\right)\right\} \qquad (17)
\end{aligned}$$

This system appears in dynamic programming, where the state space is $S = [0,1]$ and the decision space is $D = \mathbb{R}$. Notice that the system (17) is a particular case of (7), where $S = [0,1]$, $D = \mathbb{R}.$, $g : [0,1]\times\mathbb{R} \to \mathbb{R}$ is defined as $g(x,y) = e^{-(x+|y|)}$, $\alpha,\beta : [0,1] \to [0,1]$ is given by $\alpha(t) = \frac{1}{1+t}$ and $\beta(t) = \frac{t}{1+t}$, $T : [0,1]\times\mathbb{R} \to [0,1]$ is $T(x,y) = |\sin(x+y)|$, and $F : [0,1]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \times\mathbb{R}$ is defined as

$$F(x,y,t,s,r) = \arctan\left(\frac{1}{3}(x+|y|+|t|+|s|+|r|\right). \qquad (18)$$

Notice that $|g(x,y)| \leq 1$ and $|F(x,y,0,0,0) = \arctan(\frac{1}{3}(x+|y|) \leq \frac{\pi}{3}$. Therefore, assumption (i) of Theorem 3.1 is satisfied. On the other hand, for $x \in [0,1]$ and $y,t,s,r,t_1,s_1,r_1 \in \mathbb{R}$, we have

$$\begin{aligned}
|F(x,y,t,s,r) - F(x,y,t_1,s_1,r_1)| &= \left|\arctan\left(\frac{1}{3}(x+|y|+|t|+|s|+|r|\right) - \arctan\left(\frac{1}{3}(x+|y|+|t_1|+|s_1|+|r_1|\right)\right| \\
&\leq \arctan\left(\frac{1}{3}(|t|+|s|+|r|+|t_1|+|s_1|+|r_1|)\right) \\
&\leq \arctan\left(\frac{1}{3}(\|t\|+\|s\|+\|r\|+\|t_1\|+\|s_1\|+\|r_1\|)\right) \\
&\leq \arctan\left(\frac{1}{3}(|t-t_1|+|s-s_1|+|r-r_1|)\right)
\end{aligned}$$

$$\begin{aligned} &\leq \arctan\left(\frac{1}{3}(3\max\{|t-t_1|,|s-s_1|,|r-r_1|\})\right) \\ &\leq \arctan\left(\max\{|t-t_1|,|s-s_1|,|r-r_1|\}\right) \end{aligned} \tag{19}$$

where we have used the nondecreasing character of the function $\varphi(t) = \arctan(t)$ and the fact that $|\arctan t - \arctan s| \leq \arctan(|t-s|)$, for any $t, s \in \mathbb{R}^+$. It is easily seen that $\varphi(t) = \arctan(t)$, for $t \geq 0$, is a comparison function and, therefore, assumption (ii) of Theorem 3.1 is satisfied. By Theorem 3.1, the system (17) has a unique solution $(u_0, v_0, w_0) \in B([0,1]) \times B([0,1]) \times B([0,1])$.

4. Conclusion

In this paper we proof an $(\alpha-\beta)$-tripled fixed point in $B(S)$ and give an application of this result in problem of existence and uniqueness of solutions of the following system of functional equations arising in dynamic programming:

$$\begin{aligned} u(x) &= \sup_{y\in D}\{g(x,y) + F(x,y,u(T(x,y)),v(T(x,y)),w(T(x,y)))\} \\ v(x) &= \sup_{y\in D}\{g(x,y) + F(x,y,u(\alpha(T(x,y))),v(\alpha(T(x,y))),w(\alpha(T(x,y))))\} \\ w(x) &= \sup_{y\in D}\{g(x,y) + F(x,y,u(\beta(T(x,y))),v(\beta(T(x,y))),w(\beta(T(x,y))))\} \end{aligned}$$

under certain assumptions.

Acknowledgements

The authors are grateful to the referees for their suggestions that contributed to the improvement of the paper.

References

[1] S. Banach, *Sur les opérations dans les ensembles abstraits et leurs applications aux équations intégrales*, Fund. Math., 3(1922) 133-181.

[2] A. Gupta, *An Application of Meir Keeler type tripled fixed point*, International Journal Of Advances In Applied Mathematics & Mechanics, 2(2)(2014), 21-38.

[3] A. Gupta, R. Narayan and R. N. Yadava, *Tripled fixed point for compatible mappings in partially ordered fuzzy metric spaces*, The Journal of Fuzzy Mathematics, 22(3)(2014), 565-580.

[4] A. Gupta and R. Kushwaha, *Tripled common fixed point for weak (μ, ψ, Ψ, Φ)-contractions in partially ordered metric spaces*, Mathematical Theory And Modeling, 3(6)(2013), 46-53.

[5] J. Matkowski, *Integrable solutions of functional equations*, Dissertationes Mathematicae, 127(1975), 1-68.

[6] V. Berinde, *Iterative Approximation of Fixed Points*, vol. 1912 of Lecture Notes in Mathematics, Springer, Berlin, Germany, 2nd edition, (2007).

[7] Z. Liu, R. P. Agarwal and S. M. Kang, *On solvability of functional equations and systemof functional equations arising in dynamic programming*, Journal of Mathematical Analysis and Applications, 297(1)(2004), 111-130.

[8] Z. Liu, J. S. Ume and S. M. Kang, *Some existence theorems for functional equations and systemof functional equations arising in dynamic programming*, Taiwanese Journal of Mathematics, 14(4)(2010), 1517-1536.

[9] Z. Liu, J. S. Ume and S. M. Kang, *On properties of solutions for two functional equations arising in dynamic programming*, Fixed Point Theory and Applications, 2010(2010), Article ID 905858, 19 pages.

[10] Deepmala, *Existence theorems for solvability of a functional equation arising in dynamic programming*, International Journal ofMathematics and Mathematical Sciences, 2014(2014), Article ID 706585, 9 pages.

[11] R. Bellman and E. S. Lee, *Functional equations in dynamic programming*, Aequationes Mathematicae, 17(1)(1978), 1-18.