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Application of Perove Type Fixed Point Theorem in Vector Valued B-Metric Spaces

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Abstract

In this paper, we extended the definition of b-metric spaces to encompass the vectorial scenario, wherein the distance is represented as a vector, and the constant in the triangle inequality axiom is substituted with a matrix. For these spaces, we present findings that are similar to those in the b-metric framework: fixed-point theorems, stability results, and a version of Ekeland's variational principle. Consequently, we also obtain a variation of Caristi's fixed-point theorem.

Keywords: triangle inequality axiom; b-metric space; variational principle; fixed point.
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1. Introduction

Various results from the classical theory of metric spaces have been extended to b-metric spaces, including fixed-point theorems, estimations, stability results, and variational principles. In [21], the metric was allowed to take vector values, and results analogous to those for b-metric spaces were established, with matrices converging to zero replacing the contraction constants, but not the constant b from the triangle inequality axiom. Throughout this paper, we consider $\mathbb{R}^n$-valued vector metrics ($n \geq 1$) on a set X, i.e., mappings $d : X \times X \to \mathbb{R}^n_+$. In the scalar case ($n = 1$), we use the special notation ρ to denote a standard metric or a b-metric. The classical definition of a b-metric reads as follows:

Definition 1.1. *Let X be a set and let $b \geq 1$ be a given real number. A mapping $\rho : X \times X \to \mathbb{R}_+$ is said to be a b-metric if for all $x, y, z \in X$ the following conditions are satisfied: $\rho(x, y) \geq 0$, $\rho(x, y) = 0$ if and only if $x = y$, $\rho(x, y) = \rho(y, x)$ and $\rho(x, z) \leq b\,(\rho(x, y) + \rho(y, z))$. The pair (X, ρ) is called a b-metric space.*

Definition 1.2. *Let X be a set, $n \geq 1$ and let $B \in \mathcal{M}_{n\times n}(\mathbb{R})$ be an arbitrary matrix. A mapping $d = (d_1, d_2, \ldots, d_n)\colon X \times X \to \mathbb{R}^n_+$ is called a vector B-metric if for all $u, v, w \in X$, one has*

(positivity): $d(u, v) \geq 0$ and $d(u, v) = 0$ if and only if $u = v$;

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(symmetry): $d(u,v) = d(v,u)$;

(triangle inequality): $d(u,w) \leq B\left(d(u,v) + d(v,w)\right)$.

The pair (X, d) *is called a vector B-metric space.*

2. Preliminaries

In this section, the vectors in $\mathbb{R}^n$ are looked as column matrices and ordering between them and, more generally, between matrices of the same size is understood by components. Likewise, the convergence of a sequence of vectors or matrices is understood componentwise.
The spaces of square matrices of size n with real number entries and nonnegative entries are denoted by $\mathcal{M}_{n\times n}(\mathbb{R})$ and $\mathcal{M}_{n\times n}(\mathbb{R}_+)$, respectively. An element of $\mathcal{M}_{n\times n}(\mathbb{R}_+)$ is refereed as a *positive matrix*, while a matrix $M \in \mathcal{M}_{n\times n}(\mathbb{R})$ is called *inverse-positive* if it is invertible and its inverse M^{-1} is positive. A positive matrix M is said to be *convergent to zero* if its power M^k tends to the zero matrix 0_n as $k \to \infty$. One has the following characterizations of matrices which are convergent to zero.

Proposition 2.1. *Let* $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ *and let* I *be the identity matrix of size* n. *The following statements are equivalent:*

(a) M *is convergent to zero.*

(b) The spectral radius $r(M)$ *of matrix* M *is less than* 1, *i.e.,* $r(M) < 1$.

(c) $I - M$ *is invertible and* $(I-M)^{-1} = I + M + M^2 + \ ....$

(d) $I - M$ *is inverse-positive.*

The following proposition collects the various properties equivalent to the notion of an inverse-positive matrix (see, e.g., [23,24]).

Proposition 2.2. *Let* $M \in \mathcal{M}_{n\times n}(\mathbb{R})$. *The following statements are equivalent:*

(a) M *is inverse-positive.*

(b) M *is monotone, i.e.,* $Mx \geq 0$ $(x \in \mathbb{R}^n)$ *implies* $x \geq 0$.

(c) There exists a positive matrix $\overline{M}$ *and a real number* $s > r\left(\overline{M}\right)$ *such that the following representation holds:* $M = sI - \overline{M}$.

A mapping $N : X \to X$ *defined on a vector B-metric space* (X, d) *is said to be a Perov contraction mapping if there exists a matrix* A *convergent to zero such that*

$$d\left(N(x), N(y)\right) \leq A d(x,y) \tag{1}$$

for all $x, y \in X$.

The next proposition is about the relationship between vector B-metrics and both vector and scalar b-metrics.

Proposition 2.3.

(1^0) *Any vector-valued b-metric d can be identified with a vector B_b-metric, where B_b is the diagonal matrix whose diagonal entries are all equal to b.*

(2^0) *If d is a vector B-metric with an inverse-positive matrix B, then d is also a vector $\underline{B}$-metric with respect to the diagonal matrix $\underline{B}$ that preserves the diagonal of B, as well as a vector-valued $\tilde{b}$-metric with $\tilde{b} = \max\{b_{ii} : 1 \le i \le n\}$. Here $B = (b_{ij})_{1\le i,j\le n}$.*

(3^0) *If d is a vector B-metric with a positive matrix B, then to each norm in $\mathbb{R}^n$ one can associate a scalar b-metric, for example:*

$$\rho_1(x,y) := \sum_{i=1}^{n} d_i(x,y), \qquad \text{is a } b_1\text{-metric}, \quad b_1 := \sum_{i=1}^{n} \max_{1\le j\le n} b_{ij},$$

$$\rho_\infty(x,y) := \max_{1\le i\le n} d_i(x,y), \qquad \text{is a } b_\infty\text{-metric}, \quad b_\infty := \max_{1\le i\le n} \sum_{j=1}^{n} b_{ij},$$

$$\rho_2(x,y) := \left(\sum_{i=1}^{n} d_i(x,y)^2\right)^{\frac{1}{2}}, \quad \text{is a } b_2\text{-metric}, \quad b_2 := \left(\sum_{i,j=1}^{n} b_{ij}^2\right)^{\frac{1}{2}}.$$

Thus, to any vector B-metric, one can associate different (scalar) b-metrics, depending on the chosen metric on $\mathbb{R}^n$.

If Y is a nonempty subset of a vector B-metric space (X,d), we define the *diameter* of the set Y by

$$\operatorname{diam}_d(Y) := \sup\{\rho_1(x,y) : x,y \in Y\} = \sup\left\{\sum_{i=1}^{n} d_i(x,y) : x,y \in Y\right\}.$$

From this definition, it follows immediately that if $\operatorname{diam}_d(Y) = a$, then $d(x,y) \le ae$ for all $x,y \in Y$, where $e = (1,1,\dots,1) \in \mathbb{R}^n$. Conversely, if $d(x,y) \le ae$ for all $x,y \in Y$, then $\operatorname{diam}_d(Y) \le na$.
We conclude this section by two examples of vector B-metrics.

Example 2.4. *Let $d \colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2_+$ be given by*

$$d(x,y) = \begin{pmatrix} |x_1 - y_1|^2 + |x_2 - y_2| \\ |x_2 - y_2|, \end{pmatrix},$$

for $x = (x_1,x_2)$, $y = (y_1,y_2) \in \mathbb{R}^2$. Then, $(\mathbb{R}^2, d)$ is a vector B-metric space, where

$$B = \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}.$$

Here, the matrix B is inverse-positive, but not positive.

Example 2.5. *Let*

$$S = \{(t,t) \ : \ t \in \mathbb{R}\} \subset \mathbb{R}^2,$$

and let $d \colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2_+$ *be given by*

$$d(x,y) = \begin{cases} (0,0) & \text{if } x = y, \\ \left(|x-y|^2, |x-y|\right) & \text{if } x,y \in S, \\ \left(|x-y|, |x-y|^2\right) & \text{otherwise,} \end{cases}$$

where $|z| = |(z_1,z_2)| = |z_1| + |z_2|$ *is a norm on* $\mathbb{R}^2$*. Note that* d *is a vector* B_0*-metric, where*

$$B_0 = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}.$$

Let us show that B_0 *is the smallest matrix for which the triangle inequality holds for* d*. To this aim, let* $B = (b_{ij})_{1 \le i,j \le n}$ *be any matrix for which the triangle inequality is satisfied. Then, for* $x, y \in S$ *and* $z \notin S$*, we have*

$$\begin{pmatrix} |x-y|^2 \\ |x-y| \end{pmatrix} \le \begin{pmatrix} b_{11}\left(|x-z| + |z-y|\right) + b_{12}\left(|x-z|^2 + |z-y|^2\right) \\ b_{21}\left(|x-z| + |z-y|\right) + b_{22}\left(|x-z|^2 + |z-y|^2\right) \end{pmatrix}. \tag{2}$$

Let $t, \alpha \in \mathbb{R} \setminus \{0\}$*, and set* $x = (t,t) \in S$*,* $y = (0,0) \in S$ *and* $z = (\alpha, 0) \notin S$*. The first inequality in* (2) *yields,*

$$4t^2 \le b_{11}\left(|t-\alpha| + |t| + |\alpha|\right) + b_{12}\left((|t-\alpha| + |t|)^2 + \alpha^2\right).$$

Clearly, taking $\alpha = t$ *and the limit as* $t \to \infty$*, this inequality holds only if* $b_{12} \ge 2$*. Similarly, from the second inequality, we obtain*

$$2\,|t| \le b_{21}\left(|t-\alpha| + |t| + |\alpha|\right) + b_{22}\left((|t-\alpha| + |t|)^2 + \alpha^2\right).$$

Setting $\alpha = \frac{t}{2}$*, we find that*

$$2\,|t| \le 2b_{21}\,|t| + 5b_{22}\frac{t^2}{2}, \quad \text{or equivalently,} \quad 5b_{22}\frac{t^2}{2} + 2\,|t|\,(b_{21} - 1) \ge 0.$$

Clearly, this inequality required for all t *implies* $b_{21} \ge 1$*. To determine the values of* b_{11} *and* b_{22}*, we apply the triangle inequality with* $x, y, z \in S$ $(x \ne y \ne z \ne x)$*, which gives*

$$\begin{pmatrix} |x-y|^2 \\ |x-y| \end{pmatrix} \le \begin{pmatrix} b_{11}\left(|x-z|^2 + |z-y|^2\right) + b_{12}\left(|x-z| + |z-y|\right) \\ b_{21}\left(|x-z|^2 + |z-y|^2\right) + b_{22}\left(|x-z| + |z-y|\right) \end{pmatrix}$$

Similar arguments as above imply that $b_{11} \ge 2$ *and* $b_{22} \ge 1$*. Thus,* $B \ge B_0$ *as claimed.*

3. Generalized Perov Type Contraction in Vector B-Metric Spaces

Our first result is a version of generalized Perov type fixed point theorem (see, [26,27]) for such spaces.

Theorem 3.1. *Let (X, d) be a complete vector B-metric space, where B is either a positive or an inverse-positive matrix, and let $N\colon X \to X$ be an operator. Assume that there exists a convergent to zero matrix $A \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ such that*

$$d(N(x), N(y)) \leq A \min\{d(x, y), d(x, N(x)), d(y, N(y))\}, \quad \text{for all } x, y \in X, \tag{3}$$

i.e., N is a generalized Perov type contraction mapping. Then, N has a unique fixed point.

Proof. Let $x_0 \in X$, and recursively define $x_k = N(x_{k-1})$, for $k \geq 1$. Since the matrix A is convergent to zero, for each $\alpha > 0$, there exists $k_0 = k_0(\alpha)$ such that $A^{k_0} \leq \Lambda$, where Λ is the square matrix of size n whose entries are all equal to α. Let $k, p \geq 0$ and k_0 be such that $A^{k_0} \leq \Lambda$, for some $\alpha > 0$ to be specified later.

Case (a): B is inverse-positive. The triangle inequality yields

$$\begin{aligned} B^{-2}d(x_k, x_p) &\leq B^{-1}d(x_k, x_{k+k_0}) + B^{-1}d(x_p, x_{k+k_0}) \\ &\leq B^{-1}A^k d(x_0, x_{k_0}) + d(x_p, x_{p+k_0}) + d(x_{p+k_0}, x_{k+k_0}) \\ &\leq B^{-1}A^k d(x_0, x_{k_0}) + A^p d(x_0, x_{k_0}) + A^{k_0} d(x_k, x_p) \\ &\leq B^{-1}A^k d(x_0, x_{k_0}) + A^p d(x_0, x_{k_0}) + \Lambda d(x_k, x_p), \end{aligned}$$

which gives

$$(B^{-2} - \Lambda) d(x_k, x_p) \leq B^{-1}A^k d(x_0, x_{k_0}) + A^p d(x_0, x_{k_0}). \tag{4}$$

Given that the right-hand side of (4) is a vector that converges to zero as $k, p \to \infty$, our goal is to show that a linear combination of the components of the vector $d(x_k, x_p)$ is bounded above by the corresponding components of the right-hand side of (4). To this aim, we make the following notations

$$B^{-2} = (\gamma_{ij})_{1\leq i,j\leq n},$$
$$B^{-1}A^k d(x_0, x_{k_0}) + A^p d(x_0, x_{k_0}) = \varphi_{k,p} = (\varphi^i_{k,p})_{1\leq i\leq n}.$$

Hence

$$\sum_{i=1}^{n} \varphi^i_{k,p} \to 0 \quad \text{as} \quad k, p \to \infty. \tag{5}$$

Under these notations, relation (4) gives

$$\sum_{j=1}^{n} (\gamma_{ij} - \alpha) d_j\left(x_k, x_p\right) \leq \varphi^i_{k,p}, \; i = 1, 2, \ldots, n. \tag{6}$$

Summing in (6) over all $i \in \{1,2,\ldots,n\}$, we obtain

$$\sum_{i,j=1}^{n} (\gamma_{ij} - \alpha) d_j(x_k, x_p) \leq \sum_{i=1}^{n} \varphi^i_{k,p}. \tag{7}$$

Since B^{-2} is invertible and positive, the sum of its elements in each column must be positive, i.e.,

$$\sum_{i=1}^{n} \gamma_{ij} > 0, \quad j = 1,2,\ldots,n.$$

If we denote

$$\gamma = \min\left\{\sum_{i=1}^{n} \gamma_{ij} \ : \ j = 1,2,\ldots,n\right\},$$

relation (7) implies that

$$\begin{aligned}\sum_{i=1}^{n} \varphi^i_{k,p} &\geq \sum_{i,j=1}^{n} \gamma_{ij} d_j(x_k, x_p) - n\alpha \sum_{j=1}^{n} d_j(x_k, x_p)\\ &= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \gamma_{ij}\right) d_j(x_k, x_p) - n\alpha \sum_{j=1}^{n} d_j(x_k, x_p)\\ &\geq (\gamma - n\alpha) \sum_{j=1}^{n} d_j(x_k, x_p).\end{aligned}$$

Choosing $\alpha < \gamma/n$, one has

$$\sum_{j=1}^{n} d_j(x_k, x_p) \leq \frac{1}{\gamma - n\alpha} \sum_{i=1}^{n} \varphi^i_{k,p}. \tag{8}$$

In (8), we observe that the factor $\frac{1}{\gamma - n\alpha}$ depends only on n and B, whence (5) yields

$$\sum_{j=1}^{n} d_j(x_k, x_p) \to 0 \quad \text{as} \ \ k, p \to \infty,$$

so the sequence (x_k) is Cauchy.

Case (b): B is positive. One has

$$\begin{aligned}d(x_k, x_p) &\leq Bd(x_k, x_{k+k_0}) + Bd(x_p, x_{k+k_0})\\ &\leq BA^k d(x_0, x_{k_0}) + B^2 d(x_p, x_{p+k_0}) + B^2 d(x_{p+k_0}, x_{k+k_0})\\ &\leq BA^k d(x_0, x_{k_0}) + B^2 A^p d(x_0, x_{k_0}) + B^2 A^{k_0} d(x_k, x_p)\\ &\leq BA^k d(x_0, x_{k_0}) + B^2 A^p d(x_0, x_{k_0}) + B^2 \Lambda d(x_k, x_p),\end{aligned}$$

which gives

$$(I - B^2\Lambda) d(x_k, x_p) \leq BA^k d(x_0, x_{k_0}) + B^2 A^p d(x_0, x_{k_0}). \tag{9}$$

Note that since $\Lambda^k = (n\alpha)^{k-1}\Lambda$, if α is chosen to be smaller than one divided by the greatest element of B^2 multiplied with n, the matrix $B^2\Lambda$ is convergent to zero. Consequently, $I - B^2\Lambda$ is invertible and

$\left(I-B^2\Lambda\right)^{-1} \in \mathcal{M}_{n\times n}\left(\mathbb{R}_+\right)$. Hence, (9) is equivalent to

$$d(x_k, x_p) \leq \left(I - B^2\Lambda\right)^{-1}\left(BA^k d(x_0, x_{k_0}) + B^2 A^p d(x_0, x_{k_0})\right). \tag{10}$$

As the right-hand side of (10) converges to zero when $k, p \to \infty$, we conclude that (x_k) is Cauchy. Therefore, in both cases, the sequence (x_k) is Cauchy and since X is complete, it has a limit x^*, that is, $d\left(x_k, x^*\right) \to 0$ as $k \to \infty$. Then, from

$$d\left(N\left(x_k\right), N\left(x^*\right)\right) \leq A \min\{d\left(x_k, x^*\right), d\left(x_k, Nx_k,\right), d\left(x_*, Nx_*,\right)\},$$

it follows that $N\left(x_k\right) \to N\left(x^*\right)$ as $k \to \infty$, while from $x_{k+1} = N\left(x_k\right)$, passing to the limit, one obtains $x^* = N\left(x^*\right)$. Hence N has a fixed point. To prove uniqueness, suppose that there exists another fixed point x^{**}. Then, from

$$d\left(x^*, x^{**}\right) = d\left(N\left(x^*\right), N\left(x^{**}\right)\right) \leq A \min\{d\left(x^*, x^{**}\right), d\left(x^*, Nx^*\right), d\left(x^{**}, Nx^{**}\right)\},$$

recursively, we obtain that

$$d\left(x^*, x^{**}\right) \leq A^k \min\{d\left(x^*, x^{**}\right), d\left(x^*, Nx^*\right), d\left(x^{**}, Nx^{**}\right)\},$$

for all $k \geq 1$. Since $A^k \to 0_n$ as $k \to \infty$, we deduce that $d\left(x^*, x^{**}\right) = 0$, i.e., $x^{**} = x^*$. □

Theorem 3.2. *Let (X, d) be a complete vector B-metric space, where B is either positive or inverse-positive, and let $N\colon X \to X$ be an operator. Assume there exists a convergent to zero matrix $A \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ such that*

$$d\left(N(x), N^2(x)\right) \leq A \min\{d(x, N(x)), d(x, N(x)), d(N(x), N^2(x))\}, \text{ for all } x \in X. \tag{11}$$

Then, N has at least one fixed point.

Proof. Following the proof of Theorem 3.1, from any initial point x_0, the sequence $x_k = N^k\left(x_0\right)$ is convergent to a fixed point x^* of N, which clearly depends on the starting point x_0, but condition (11) is insufficient to guarantee the uniqueness. □

The next result is a version for vector B-metric spaces of Maia's fixed point theorem.

Theorem 3.3. *Let X be a set equipped with two $\mathbb{R}^n$-vector metrics, a B_1-metric d_1 and a B_2-metric d_2, where B_2 is either positive or inverse-positive, and let $N\colon X \to X$ be an operator. Assume that the following conditions hold:*

(i) (X, d_1) is a complete vector B_1-metric space;

(ii) $d_1(x, y) \leq C d_2(x, y)$ for all $x, y \in X$ and some matrix $C \in \mathcal{M}_{n\times n}\left(\mathbb{R}\right)$;

(iii) There exists a matrix A convergent to zero such that

$$d_2\left(N(x), N(y)\right) \leq A\min\{d_2(x,y), d_2(x,Nx), d_2(y,Ny)\}, \text{ for all } x,y \in X; \tag{12}$$

(iv) The operator N is continuous in (X, d_1).

Then, the operator N has a unique fixed point.

Proof. Let $x_0 \in X$ be fixed, and consider the iterative sequence $x_{k+1} = N(x_k)$ for $k \geq 0$. For any $k, k_0, p \geq 0$, applying the triangle inequality twice and using condition (iii), we derive either

$$(B_2^{-2} - A^{k_0})d_2(x_k, x_p) \leq B_2^{-1}A^k d_2(x_0, x_{k_0}) + A^p d_2(x_0, x_{k_0}),$$

in case that B_2 is inverse-positive, or

$$(I - B_2^2 A^{k_0})d_2(x_k, x_p) \leq B_2 A^k d_2(x_0, x_{k_0}) + B_2^2 A^p d_2(x_0, x_{k_0}),$$

if B_2 is positive. Arguing similarly to the proof of Theorem 3.1, we deduce that (x_k) is a Cauchy sequence in (X, d_2). From (ii), it follows immediately that (x_k) is also a Cauchy sequence in (X, d_1), hence (x_k) is convergent with respect the metric d_1 to some x^*, that is,

$$d_1(N(x_k), x^*) = d_1\left(x_{k+1}, x^*\right) \to 0, \quad \text{as } k \to \infty,$$

while the continuity of N yields $d_1(N(x^*), x^*) = 0$, i.e., $N(x^*) = x^*$. To establish uniqueness, suppose that x^{**} is another fixed point of N, i.e., $N(x^{**}) = x^{**}$. Then, by (12), one has

$$(I - A)d_2(x^*, x^{**}) \leq 0.$$

Since A is convergent to zero, we necessarily have $d_2(x^*, x^{**}) = 0$, i.e., $x^* = x^{**}$. □

3.1 Stability Results of the Perov Contraction Mappings in Vector B-Metric Spaces

We now present two stability properties of the Perov contraction mappings in vector B-metric spaces. The first property is in the sense of Reich and Zaslavski and generalizes the one obtained in [18] for b-metric spaces.

Theorem 3.4. *Let (X, d) be a complete vector B-metric space, and let $N\colon X \to X$ be an operator such that* (3) *holds with a matrix A convergent to zero. In addition assume that either*

(a) B and $B^{-1} - A$ are inverse-positive; or

(b) B is positive and $I - BA$ is inverse-positive.

Then, N is stable in the sense of Reich and Zaslavski, i.e., N has a unique fixed point x^*, and for every sequence $(x_k) \subset X$ satisfying

$$d(x_k, N(x_k)) \to 0 \quad as \;\; k \to \infty, \tag{13}$$

one has

$$x_k \to x^* \quad as \;\; k \to \infty.$$

Proof. According to Theorem 3.1 the operator N has a unique fixed point x^*. In addition, for any sequence (x_k) satisfying (13), in case (a), we have

$$\begin{aligned} B^{-1} d(x_k, x^*) &\le d(x_k, N(x_k)) + d(N(x_k), x^*) \\ &= d(x_k, N(x_k)) + d(N(x_k), N(x^*)) \\ &\le d(x_k, N(x_k)) + A d(x_k, x^*), \end{aligned}$$

that is,

$$d(x_k, x^*) \le (B^{-1} - A)^{-1} d(x_k, N(x_k)),$$

while in case (b),

$$d(x_k, x^*) \le (I - BA)^{-1} B d(x_k, N(x_k)).$$

These estimates immediately yield the conclusion. □

The second stability result is in the sense of Ostrowski and extends to vector B-metric spaces a similar property established in [18] for b-metric spaces.

Theorem 3.5. *Let (X, d) be a complete vector B-metric space, and let $N \colon X \to X$ be an operator. Assume N satisfies* (3) *with a matrix A convergent to zero. In addition, assume that either*

(a) B and $I - \tilde{b}A$ are inverse-positive, where $\tilde{b} = \max\{b_{ii} \,:\, i = 1, 2, \ldots, n\}$; or

(b) B is positive and $I - BA$ is inverse-positive.

Then, N has the Ostrowski property, i.e., N has a unique fixed point x^*, and for every sequence $(x_k) \subset X$ satisfying $d(x_{k+1}, N(x_k)) \to 0$ as $k \to \infty$, one has $x_k \to x^*$ as $k \to \infty$.

Proof. As previously established, the operator N has a unique fixed point x^*. In case (a), we have

$$\begin{aligned} d(x_{k+1}, x^*) &\le \tilde{b}\, d(x_{k+1}, N(x_k)) + \tilde{b}\, d(N(x_k), N(x^*)) \\ &\le \tilde{b}\, d(x_{k+1}, N(x_k)) + \tilde{b}A\, d(x_k, x^*) \\ &\le \;\ldots \\ &\le \tilde{b} \sum_{p=0}^{k} (\tilde{b}A)^p d(x_{k+1-p}, N(x_{k-p})) + (\tilde{b}A)^{k+1} d(x_0, x^*), \end{aligned}$$

while in case (b), similar estimation gives

$$d(x_{k+1}, x^*) \leq \sum_{p=0}^{k} (BA)^p Bd(x_{k+1-p}, N(x_{k-p})) + (BA)^k Bd(x_0, x^*).$$

Since $I - \tilde{b}A$ is inverse-positive and $\tilde{b}A$ is positive in the first case, and $I - BA$ is inverse-positive and BA is positive in the second case, the series $\sum_{p=0}^{k}(\tilde{b}A)^p$ and $\sum_{p=0}^{k}(BA)^p$ are convergent. Moreover, $(\tilde{b}A)^k$ and $(BA)^k$ converge to the zero matrix as $k \to \infty$. Therefore, using the Cauchy-Toeplitz lemma (see [?]), it follows that $d(x_{k+1}, x^*) \to 0$ as $k \to \infty$. □

4. Conclusion

In this paper, we introduced the concept of a vector B-metric space. Several fixed-point theorems, analogous to those in scalar b-metric spaces as well as their classical counterparts, were presented. Lastly, it would be interesting to study the case where the matrix B is neither positive nor inverse-positive; for instance, when it has positive diagonal elements but contains both positive and negative entries elsewhere.

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