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Monotonic Behavior in Subdiffusion Equation of the Parameter in Mittag-Leffler Function

Swati Mishra[1,*], Ajay Kumar Singh[1]

[1] *Department of Mathematics, Madhyanchal Professional University, Bhopal, Madhya Pradesh, India*

Abstract

In this paper, we demonstrate the strict monotonicity concerning the parameter ho of the Mittag-Leffler functions $E_ho(-t^ho)$ and $t^{ho-1}E_{ho,ho}(-t^ho)$. Furthermore, the results obtained are applicable to a broader class of subdiffusion equations than those previously examined.

Keywords: Fractional derivatives; The Caputo fractional derivatives; the Riemann-Liouville fractional derivatives; monotonicity; Mittag-Leffler function.

1. Introduction

Let Ω be an arbitrary N - dimensional domain with a sufficiently smooth boundary $\partial\Omega$. Consider the following *Initial-boundary value problem*:

$$\begin{cases} D_t^\rho u(x,t) - \Delta u(x,t) = 0, \quad x \in \Omega, \quad 0 < t \le T, \\ u(x,t)|_{\partial\Omega} = 0, \\ u(x,0) = \varphi(x), \quad x \in \Omega, \end{cases} \tag{1}$$

where $\varphi(x)$ is a continuous function, Δ is the Laplace operator, $\rho \in (0,1)$ and D_t^ρ is the fractional Caputo derivative defined as (see, for example, [14], p. 91):

$$D_t^\rho u(x,t) = \frac{1}{\Gamma(1-\rho)} \frac{d}{dt} \int_0^t \frac{u(x,\xi) - u(x,0)}{(t-\xi)^\rho} d\xi, \quad t > 0, \ x \in \Omega.$$

When modeling various processes, the order of the fractional derivative ρ is often unknown. One of the effective methods for finding the unknown order of the derivative is, by setting some additional conditions for a solution $u(x,t)$ of problem (1), to solve analytically the corresponding inverse problem. A very interesting paper [16] by G. Li, Z. Wang, X. Jia, Y. Zhang was recently published, in which the

*(Research Scholar)

order of the fractional derivative ρ in the equation

$$D_t^{\rho} u(x,t) - c\Delta u(x,t) = 0$$

where $\Omega \subset \mathbb{R}^d, d = 1,2,3$ and c is a sufficiently small positive number.The solution of the initial-boundary value problem at only one point of space-time (x_0, t_0). Note that the proof relies on the condition $d \leq 3$. However, the paper contains an interesting auxiliary result (see Theorem 1): the authors proved that the function $g(\rho) = E_\rho(-ct^\rho)$ has a negative derivative, $g'(\rho) < 0$, if t is large enough and c is small enough. In particular:

1) we have proved that the Mittag-Leffler functions $E_\rho(-t^\rho)$ and $t^{\rho-1}E_{\rho,\rho}(-t^\rho)$ increase monotonically in parameter ρ for sufficiently small t. This is the main result of this work;

2) the requirement of a sufficiently small coefficient c was removed from the considered equation;

3) the restriction on the dimension of the domain $\Omega \subset \mathbb{R}^d, d = 1,2,3$ was removed. Of course, the transition to the domain $\Omega \subset \mathbb{R}^N$, with $N \geq 1$, does not cause any particular difficulty, but we would like to draw the attention of readers to one fundamental work by V.A. Ilyin [12];

4) it is shown that the conditions of Theorem 2 of [16] is not complete and an updated version of this theorem with a proof is presented.

5) other applications of the monotonicity in parameter of Mittag-Leffler functions are given.

2. Preliminaries

Let Ω be a bounded N-dimentional domain with a sufficiently smooth boundary $\partial\Omega$ and $\{v_k(x)\}$ denote the complete system of orthonormal in $L_2(\Omega)$ eigenfunctions and $\{\lambda_k\}$ the set of positive eigenvalues of the spectral problem:

$$\begin{cases} -\Delta v(x) = \lambda v(x), \quad x \in \Omega, \\ v(x)|_{\partial\Omega} = 0. \end{cases}$$

Let us present some assertions about eigenfunctions $v_k(x)$ and eigenvalues λ_k proved by V.A.Ilyin [12].

Lemma 2.1. *The series* $\sum\limits_{k=1}^{\infty} \lambda_k^{-([\frac{N}{2}]+1)} v_k^2(x)$ *converges uniformly in a closed domain* $\overline{\Omega}$.

Lemma 2.2. *Let the function* $g(x)$ *satisfy the conditions*

(1). $g(x) \in C^p(\overline{\Omega}), \ \frac{\partial^{p+1} g(x)}{\partial x_1^{p_1} \ldots \partial x_n^{p_n}} \in L_2(\Omega), \ p+1 = p_1 + p_2 + \ldots p_n, p \geq 1,$

(2). $g(x)|_{\partial\Omega} = \Delta g(x)|_{\partial\Omega} = \cdots = \Delta^{[\frac{p}{2}]} g(x)|_{\partial\Omega} = 0.$

Then the number series $\sum\limits_{k=1}^{\infty} g_k^2 \lambda_k^{p+1}$ *converges, where* $g_k = (g, v_k)$.

For $0 < \rho < 1$, let $E_{\rho,\mu}(z)$ denote the Mittag-Leffler function defined as:

$$E_{\rho,\mu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\rho k + \mu)}, \quad \mu, z \in \mathbb{C}.$$

If $\mu = 1$, then the Mittag-Leffler function is called the one-parameter or classical Mittag-Leffler function and is denoted by $E_\rho(z) = E_{\rho,1}(z)$. Recall the following estimate of the Mittag-Leffler functions [13].

Lemma 2.3. *For any $t \geq 0$ one has*

$$|E_{\rho,\mu}(-t)| \leq \frac{C}{1+t}, \quad \mu \in \mathbb{C},$$

where constant C does not depend on t.

3. Monotonicity of the Mittag-Leffler Function $E_\rho(-t^\rho)$

Let us first introduce some important concepts to prove the strict monotonicity of the Mittag-Leffler function $E_\rho(-t^\rho)$.

Lemma 3.1. *Let $\rho \in (0,1)$. Then the following equality holds:*

$$\lim_{n\to\infty} \frac{\Gamma(\rho n)}{\Gamma(\rho n + \rho)} = 0.$$

Proof. Using the Stirling approximation of the gamma function we have:

$$\frac{\Gamma(\rho n)}{\Gamma(\rho n + \rho)} \sim \frac{\frac{\sqrt{2\pi}}{\sqrt{\rho n}}\left(\frac{\rho n}{e}\right)^{\rho n}}{\frac{\sqrt{2\pi}}{\sqrt{\rho n+\rho}}\left(\frac{\rho n+\rho}{e}\right)^{\rho n+\rho}} \sim e^\rho \left(\frac{\rho n}{\rho n + \rho}\right)^{\rho n} \frac{1}{(\rho n+\rho)^\rho}\sqrt{\frac{\rho n+\rho}{\rho n}} \quad (n\to\infty)$$

$$\sim e^\rho \left[\left(\frac{1}{1+1/n}\right)^n\right]^\rho \frac{1}{(\rho n+\rho)^\rho}\sqrt{1+\frac{1}{n}} \quad (n\to\infty).$$

According to $\lim\limits_{n\to\infty}\left(\frac{1}{1+1/n}\right)^n = e^{-1}$, we get:

$$\lim_{n\to\infty} \frac{\Gamma(\rho n)}{\Gamma(\rho n + \rho)} = \lim_{n\to\infty} e^\rho e^{-\rho} \frac{1}{(\rho n+\rho)^\rho}\sqrt{1+\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{(\rho n+\rho)^\rho} = 0.$$

□

Next we calculate the derivative of the Mittag-Leffler function $E_\rho(-t^\rho)$. Let $\Phi(\rho)$ be the logarithmic derivative of the gamma function $\Gamma(\rho)$. Then $\Gamma'(\rho) = \Gamma(\rho)\Phi(\rho)$ and therefore, we have:

$$\frac{d}{d\rho}E_\rho(-t^\rho) = \frac{d}{d\rho}\sum_{n=0}^{\infty}(-1)^n \frac{t^{\rho n}}{\Gamma(\rho n+1)} = \sum_{n=1}^{\infty}(-1)^n n t^{\rho n}\frac{\ln t - \Phi(\rho n+1)}{\Gamma(\rho n+1)} = \sum_{n=1}^{\infty}(-1)^n y_n, \tag{2}$$

where

$$\Phi(\rho n+1) = -\gamma - \frac{1}{\rho n+1} + \sum_{s=1}^{\infty}\left(\frac{1}{s} - \frac{1}{s+\rho n+1}\right),$$
$$y_n = nt^{\rho n}\frac{\ln t - \Phi(\rho n+1)}{\Gamma(\rho n+1)}, \tag{3}$$

and $\gamma \approx 0.57722$ is the Euler-Mascherano constant.

Lemma 3.2. *Let $\rho \in (0,1)$. Then the following two statements hold:*

1) $\lim\limits_{n\to\infty} \frac{\Phi(\rho n+\rho+1)}{\Phi(\rho n+1)} = 1.$

2) $\Phi(\rho n+1) > -\frac{1}{\rho+1}$ *for all* $n \geq 1$.

Proof. Proof of Part 1)

Consider the function $f(z) = z(\ln z - \Phi(z))$. Function $f(z)$ is strictly decreasing and strictly convex on $(0,\infty)$ (see, [1]) and the following relationships are fulfilled:

$$\lim_{z\to 0} f(z) = 1 \quad \text{and} \quad \lim_{z\to\infty} f(z) = \frac{1}{2}. \tag{4}$$

From (4) and the monotonicity of $f(z)$ we get:

$$\frac{1}{2} \leq f(z) \leq 1, \quad z \in (0,\infty). \tag{5}$$

Using estimate (5) and definition of function $f(z)$ we obtain

$$\ln z - \frac{1}{z} \leq \Phi(z) \leq \ln z - \frac{1}{2z}. \tag{6}$$

Apply estimate (6) to have:

$$\ln(\rho n+1) - \frac{1}{\rho n+1} \leq \Phi(\rho n+1) \leq \ln(\rho n+1) - \frac{1}{2(\rho n+1)}. \tag{7}$$

Therefore,

$$\frac{\ln(\rho n+\rho+1) - \frac{1}{\rho n+\rho+1}}{\ln(\rho n+1) - \frac{1}{2(\rho n+1)}} \leq \frac{\Phi(\rho n+\rho+1)}{\Phi(\rho n+1)} \leq \frac{\ln(\rho n+\rho+1) - \frac{1}{2(\rho n+\rho+1)}}{\ln(\rho n+1) - \frac{1}{\rho n+1}}.$$

Firstly we will show that

$$\lim_{n\to\infty} \frac{\ln(\rho n+\rho+1)}{\ln(\rho n+1)} = 1. \tag{8}$$

Using Lopital's theorem we get

$$\lim_{n\to\infty} \frac{\ln(\rho n+\rho+1)}{\ln(\rho n+1)} = \lim_{n\to\infty} \frac{\frac{\rho}{\rho n+\rho+1}}{\frac{\rho}{\rho n+1}} = \lim_{n\to\infty} \frac{\rho n+1}{\rho n+\rho+1} = 1$$

Apply Squeeze theorem and (8) to obtain

$$\lim_{n\to\infty} \frac{\Phi(\rho n+\rho+1)}{\Phi(\rho n+1)} = 1.$$

This complete the proof of part 1.

The proof of part 2 of the lemma obviously follows from:

$$\Phi(\rho n+1) \geq \ln(\rho n+1) - \frac{1}{\rho n+1} > -\frac{1}{\rho+1}, \quad n \geq 1.$$

Lemma 3.2 is completely proved. □

Lemma 3.3. *Let $\rho_0 \in (0,1)$ and $t_1 > 0$. Then series (2) is uniformly convergent with respect to $t \in [t_1, T]$ and $\rho \in [\rho_0, 1]$.*

Proof. Since $t \in [t_1, T]$ and $\rho \in [\rho_0, 1]$ we get

$$\sum_{n=1}^{\infty} \left| (-1)^n n t^{\rho n} \frac{\ln t - \Phi(\rho n+1)}{\Gamma(\rho n+1)} \right| \leq \sum_{n=1}^{\infty} n T^n \frac{|\ln t - \Phi(\rho n+1)|}{\Gamma(\rho_0 n+1)}.$$

Using inequality (7), we have

$$\sum_{n=1}^{\infty} \frac{nT^n}{\Gamma(\rho_0 n+1)} \left(\max_{t\in\{t_1,T\}} |\ln t| + \ln(n+1) + \frac{1}{2} \right) = \sum_{n=1}^{\infty} z_n,$$

where

$$z_n = \frac{nT^n}{\Gamma(\rho_0 n+1)} \left(\max_{t\in\{t_1,T\}} |\ln t| + \ln(n+1) + \frac{1}{2} \right).$$

Let us consider the following proportion

$$\frac{z_{n+1}}{z_n} = \frac{\frac{(n+1)T^{n+1}}{\Gamma(\rho_0(n+1)+1)} \left(\max\limits_{t\in\{t_1,T\}} |\ln t| + \ln(n+2) + \frac{1}{2} \right)}{\frac{nT^n}{\Gamma(\rho_0 n+1)} \left(\max\limits_{t\in\{t_1,T\}} |\ln t| + \ln(n+1) + \frac{1}{2} \right)}$$

$$= \frac{(n+1)T^{n+1}\Gamma(\rho_0 n+1)}{nT^n\Gamma(\rho_0 n+\rho_0+1)} \frac{\max\limits_{t\in\{t_1,T\}} |\ln t| + \ln(n+2) + \frac{1}{2}}{\max\limits_{t\in\{t_1,T\}} |\ln t| + \ln(n+1) + \frac{1}{2}}.$$

By the equality $\Gamma(\rho n+1) = \rho n \Gamma(\rho n)$, we have

$$\frac{z_{n+1}}{z_n} = T \frac{\Gamma(\rho_0 n)}{\Gamma(\rho_0 n+\rho_0)} \frac{\max\limits_{t\in\{t_1,T\}} |\ln t| + \ln(n+2) + \frac{1}{2}}{\max\limits_{t\in\{t_1,T\}} |\ln t| + \ln(n+1) + \frac{1}{2}}.$$

Apply Lemma 3.1, to get

$$\lim_{n\to\infty} \frac{z_{n+1}}{z_n} = 0.$$

Thus, by D'Alembert's Ratio Test, we conclude that the series $\sum_{n=1}^{\infty} z_n$ is convergent. Therefore, by the Weierstrass M-Test, we deduce that the series (2) is uniformly convergent. This complete the proof of lemma. □

Lemma 3.4. *Let* $0 < \rho < 1$. *Then for* $t \in \left(0, \min\left(\frac{1}{2^{\frac{1}{\rho}}}, \frac{1}{e^{\frac{7}{2}}}\right)\right]$ *the inequalities* $y_{n+1} > y_n,\ n \geq 1$, *hold.*

Proof. Let us consider the following proportion

$$\frac{y_{n+1}}{y_n} = t^\rho \frac{(n+1)\Gamma(\rho n+1)}{n\Gamma(\rho n+\rho+1)} \frac{\ln(t) - \Phi(\rho n+\rho+1)}{\ln(t) - \Phi(\rho n+1)}.$$

According to equality $\Gamma(\rho n+1) = \rho n\Gamma(\rho n)$ we have

$$\begin{aligned} \frac{y_{n+1}}{y_n} &= t^\rho \frac{\Gamma(\rho n)}{\Gamma(\rho n+\rho)} \frac{\ln(t) - \Phi(\rho n+\rho+1)}{\ln(t) - \Phi(\rho n+1)} \\ &= t^\rho \frac{\Gamma(\rho n)}{\Gamma(\rho n+\rho)} \left(1 + \frac{\Phi(\rho n+1) - \Phi(\rho n+\rho+1)}{\ln(t) - \Phi(\rho n+1)}\right) \end{aligned} \tag{9}$$

Now we will show that

$$\left|\frac{\Phi(\rho n+1) - \Phi(\rho n+\rho+1)}{\ln(t) - \Phi(\rho n+1)}\right| < 1.$$

Based on estimate (7), we examine four cases to prove the estimate above.

Case 1: We use the following estimates

$$\Phi(\rho n+1) \leq \ln(\rho n+1) - \frac{1}{2(\rho n+1)}, \quad \text{and} \quad \Phi(\rho n+\rho+1) \geq \ln(\rho n+\rho+1) - \frac{1}{\rho n+\rho+1}.$$

From these estimates we get:

$$\begin{aligned} \left|\frac{\Phi(\rho n+1) - \Phi(\rho n+\rho+1)}{\ln t - \Phi(\rho n+1)}\right| &\leq \left|\frac{\ln(\rho n+1) - \frac{1}{2(\rho n+1)} - \ln(\rho n+\rho+1) + \frac{1}{\rho n+\rho+1}}{\ln t - \ln(\rho n+1) + \frac{1}{\rho n+1}}\right| \\ &\leq \left|\frac{\ln\left(\frac{\rho n+1}{\rho n+\rho+1}\right) + \frac{\rho n+1-\rho}{2(\rho n+1)(\rho n+\rho+1)}}{\ln t - \ln(\rho n+1) + 1}\right| \\ &\leq \left|\frac{\ln\left(\frac{\rho n+1}{\rho n+\rho+1}\right)}{\ln t - \ln(\rho n+1) + 1}\right| + \left|\frac{\frac{\rho n+1-\rho}{2(\rho n+1)(\rho n+\rho+1)}}{\ln t - \ln(\rho n+1) + 1}\right| \\ &\leq \frac{1}{|\ln t - \ln(\rho n+1)| - 1} + \frac{\left|\frac{\rho n+1}{2(\rho n+1)(\rho n+\rho+1)}\right|}{|\ln t - \ln(\rho n+1)| - 1} \\ &\leq \frac{1}{|\ln t| - 1} + \frac{\frac{1}{2}}{|\ln t| - 1} \\ &= \frac{\frac{3}{2}}{|\ln t| - 1}. \end{aligned}$$

If $\ln t < -\frac{5}{2}$ then we have

$$\left|\frac{\Phi(\rho n+1) - \Phi(\rho n+\rho+1)}{\ln t - \Phi(\rho n+1)}\right| < 1.$$

Case 2: We use the following estimates

$$\Phi(\rho n+1) \geq \ln(\rho n+1) - \frac{1}{\rho n+1}, \quad \text{and} \quad \Phi(\rho n+\rho+1) \leq \ln(\rho n+\rho+1) - \frac{1}{2(\rho n+\rho+1)}.$$

From these estimates we get:

$$\begin{aligned}
\left|\frac{\Phi(\rho n+1)-\Phi(\rho n+\rho+1)}{\ln t-\Phi(\rho n+1)}\right| &\leq \left|\frac{\ln(\rho n+1)-\frac{1}{\rho n+1}-\ln(\rho n+\rho+1)+\frac{1}{2(\rho n+\rho+1)}}{\ln t-\ln(\rho n+1)+\frac{1}{\rho n+1}}\right| \\
&\leq \left|\frac{\ln\left(\frac{\rho n+1}{\rho n+\rho+1}\right)-\frac{\rho n+2\rho+1}{2(\rho n+1)(\rho n+\rho+1)}}{\ln t-\ln(\rho n+1)+1}\right| \\
&\leq \left|\frac{\ln\left(\frac{\rho n+1}{\rho n+\rho+1}\right)}{\ln t-\ln(\rho n+1)+1}\right| + \left|\frac{\frac{\rho n+2\rho+1}{2(\rho n+1)(\rho n+\rho+1)}}{\ln t-\ln(\rho n+1)+1}\right| \\
&\leq \frac{1}{|\ln t-\ln(\rho n+1)|-1} + \frac{\left|\frac{\rho n+3}{2(\rho n+1)(\rho n+\rho+1)}\right|}{|\ln t-\ln(\rho n+1)|-1} \\
&\leq \frac{1}{|\ln t|-1} + \frac{\frac{3}{2}}{|\ln t|-1} \\
&= \frac{\frac{5}{2}}{|\ln t|-1}.
\end{aligned}$$

If $\ln t < -\frac{7}{2}$ then we have

$$\left|\frac{\Phi(\rho n+1)-\Phi(\rho n+\rho+1)}{\ln t-\Phi(\rho n+1)}\right| < 1.$$

Case 3: We use the following estimates

$$\Phi(\rho n+1) \leq \ln(\rho n+1) - \frac{1}{2(\rho n+1)}, \quad \text{and} \quad \Phi(\rho n+\rho+1) \leq \ln(\rho n+\rho+1) - \frac{1}{2(\rho n+\rho+1)}.$$

From these estimates we get:

$$\begin{aligned}
\left|\frac{\Phi(\rho n+1)-\Phi(\rho n+\rho+1)}{\ln t-\Phi(\rho n+1)}\right| &\leq \left|\frac{\ln(\rho n+1)-\frac{1}{2(\rho n+1)}-\ln(\rho n+\rho+1)+\frac{1}{2(\rho n+\rho+1)}}{\ln t-\ln(\rho n+1)+\frac{1}{\rho n+1}}\right| \\
&\leq \left|\frac{\ln\left(\frac{\rho n+1}{\rho n+\rho+1}\right)-\frac{\rho}{2(\rho n+1)(\rho n+\rho+1)}}{\ln t-\ln(\rho n+1)+1}\right| \\
&\leq \left|\frac{\ln\left(\frac{\rho n+1}{\rho n+\rho+1}\right)}{\ln t-\ln(\rho n+1)+1}\right| + \left|\frac{\frac{\rho}{2(\rho n+1)(\rho n+\rho+1)}}{\ln t-\ln(\rho n+1)+1}\right| \\
&\leq \frac{1}{|\ln t-\ln(\rho n+1)|-1} + \frac{\left|\frac{1}{2(\rho n+1)(\rho n+\rho+1)}\right|}{|\ln t-\ln(\rho n+1)|-1} \\
&\leq \frac{1}{|\ln t|-1} + \frac{\frac{1}{2}}{|\ln t|-1} \\
&= \frac{\frac{3}{2}}{|\ln t|-1}.
\end{aligned}$$

If $\ln t < -\frac{5}{2}$ then we have

$$\left|\frac{\Phi(\rho n+1)-\Phi(\rho n+\rho+1)}{\ln t-\Phi(\rho n+1)}\right|<1.$$

Case 4: We use the following estimates

$$\Phi(\rho n+1)\geq \ln(\rho n+1)-\frac{1}{\rho n+1}, \quad \text{and} \quad \Phi(\rho n+\rho+1)\geq \ln(\rho n+\rho+1)-\frac{1}{\rho n+\rho+1}.$$

From these estimates we get:

$$\begin{aligned}\left|\frac{\Phi(\rho n+1)-\Phi(\rho n+\rho+1)}{\ln t-\Phi(\rho n+1)}\right| &\leq \left|\frac{\ln(\rho n+1)-\frac{1}{\rho n+1}-\ln(\rho n+\rho+1)+\frac{1}{\rho n+\rho+1}}{\ln t-\ln(\rho n+1)+\frac{1}{\rho n+1}}\right|\\ &\leq \left|\frac{\ln\left(\frac{\rho n+1}{\rho n+\rho+1}\right)-\frac{\rho}{(\rho n+1)(\rho n+\rho+1)}}{\ln t-\ln(\rho n+1)+1}\right|\\ &\leq \left|\frac{\ln\left(\frac{\rho n+1}{\rho n+\rho+1}\right)}{\ln t-\ln(\rho n+1)+1}\right|+\left|\frac{\frac{\rho}{(\rho n+1)(\rho n+\rho+1)}}{\ln t-\ln(\rho n+1)+1}\right|\\ &\leq \frac{1}{|\ln t-\ln(\rho n+1)|-1}+\frac{\left|\frac{1}{(\rho n+1)(\rho n+\rho+1)}\right|}{|\ln t-\ln(\rho n+1)|-1}\\ &\leq \frac{1}{|\ln t|-1}+\frac{1}{|\ln t|-1}\\ &= \frac{2}{|\ln t|-1}.\end{aligned}$$

If $\ln t < -3$ then we have

$$\left|\frac{\Phi(\rho n+1)-\Phi(\rho n+\rho+1)}{\ln t-\Phi(\rho n+1)}\right|<1.$$

Since $\Gamma(\rho n) \leq \Gamma(\rho n+\rho)$ and the equality (9) we have the following inequality

$$\frac{y_{n+1}}{y_n}\leq t^{\rho}\left(1+\frac{\Phi(\rho n+1)-\Phi(\rho n+\rho+1)}{\ln(t)-\Phi(\rho n+1)}\right).$$

Based on the four cases considered, we select t from the interval $\left(0,\min\left(\frac{1}{2^{\frac{1}{\rho}}},\frac{1}{e^{\frac{7}{2}}}\right)\right]$. For such t, according to assertion 2 of Lemma 3.2, the sequence y_n, defined by the equality (3), is negative for all $n \geq 1$. Therefore, we get the inequality $y_{n+1} > y_n$.

This complete the proof of Lemma. □

Theorem 3.5. *Let $\rho_0 \in (0,1)$. Then, for any $t \in \left(0,\min\left(\frac{1}{2^{\frac{1}{\rho_0}}},\frac{1}{e^{\frac{7}{2}}}\right)\right]$, the Mittag-Leffler function $E_\rho(-t^\rho)$ is monotonically increasing in $\rho \in [\rho_0, 1]$.*

Proof. From Lemma 3.3 the series (2) converges absolutely. Let us divide this series into groups as follows

$$\frac{d}{d\rho}E_\rho(-t^\rho)=\sum_{n=1}^{\infty}(-1)^n y_n=-(y_1-y_2)-(y_3-y_4)-\ldots$$

Then by Lemma 3.4 it follows that $\frac{d}{d\rho}E_\rho(-t^\rho) > 0$ for all t and ρ from the conditions of the lemma.

Theorem 3.5 is proved. □

Theorem 3.6. *Let* $\rho_0 \in (0,1)$. *Then, for any* $t \in \left(0, \min\left(\frac{1}{2^{\frac{1}{\rho_0}}}, \frac{1}{e^{\frac{13}{6}}}\right)\right]$, *the function* $t^{\rho-1}E_{\rho,\rho}(-t^{\rho})$ *is monotonically decreasing in* $\rho \in [\rho_0, 1]$.

The theorem is proved in the same way as Theorem 3.5.

Using Theorems 3.5 and 3.6, we can update the results presented in several previous works. In particular, in [2,5], the inverse problem of determining the order of the fractional derivative is solved under the over-determination condition

$$||u(x,t_0)||^2_{L_2(\Omega)} = d_0, \tag{10}$$

where t_0 is a sufficiently large number. Our results show that under the over-determination condition (10), the time instant t_0 can also be chosen sufficiently small, which is quite useful for real-life applications.

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