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Generalized Useful Information Generating Function and Applications

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Abstract

In this paper, new generalized 'useful' information generating function have been defined with their particular and limiting cases. On the differentiation of this generating function , it gives some well known results vailable in the literature of information theory. Some examples are worked out for discrete and continous distributions like Uniform, Geometric, Normal, Exponential and Pareto distribution.

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1. Introduction

The moment generating functions and charactristics functions in probability theory are conveniently used for calculating the moments of a probability distribution and investigating various properties of the probability distribution for analytical purposes. Much work have been done in this connection by Hooda and Sharma [11] , Srivastava and Shikha [12], kharami and Balakrishanan [13], Shital and Suchandan [14] etc. and found very interesting results. Golomb [2] defined a information generating function of a probability distribution and discussed some of its properties. He established that the first moment of the self information is Shannon entropy [10] . We defined a generalized useful information generating function as follows.

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2. Main Results

Suppose $P = \{p_i\}$, $p_i \geq 0$, $\sum_{i=1}^{n} p_i = 1$, $i = 1, 2, \ldots, n$ is a discrete probability distribution. Golomb [2] defined information generating function of a probability distribution P as

$$T(t) = \sum_{i=1}^{n} p_i^t, \tag{1}$$

where t is a real (or complex) variable. It may be noted that

$$\left[-\frac{\delta\, T(t)}{\delta\, t}\right]_{t=1} = -\sum_{i=1}^{n} p_i \log_D\, p_i = H(P) \tag{2}$$

Which is Shannon's entropy [10] of the given probability distribution P.

Consider a model A for a finite scheme random experiment having $A_1, A_2, \ldots, A_n$ as the complete system of events happening with probabilities $p_1, p_2, \ldots, p_n$ and credited with utilities $u_1, u_2, \ldots, u_n$ respectively, $u_i > 0$, $i = 1, 2, \ldots, n$. We write

$$A = \begin{bmatrix} A_1, A_2, \ldots, A_n \\ p_1, p_2, \ldots, p_n \\ u_1, u_2, \ldots, u_n \end{bmatrix} \tag{3}$$

we call the scheme A as utility information scheme. Belis and Guiasu [1] proposed a measure of information (called useful information) for the utility information scheme in scheme in (3) as follows

$$H(U; P) = -\sum_{i=1}^{n} u_i\, p_i \log_D\, p_i \tag{4}$$

where $u_i > 0$ and $\sum_{i=1}^{n} p_i = 1$.It reduces to Shannon entropy when utility aspect of the scheme is ignored by taking $u_i = 1$ for every i. Gurdial and Pessoa [4] considerd the generalized additive useful information of order α given by

$$H_\alpha(U; P) = \frac{1}{1-\alpha} \log_D \left[\frac{\sum_{i=1}^{n} u_i p_i}{\sum_{j=1}^{n} u_j p_j}\right] \tag{5}$$

The measure (5) resembles Renyi's entropy of order α given in [9]. Khan [6] defined a generalized additive useful information of order α and type β as

$$H_\alpha^\beta(U; P) = \frac{1}{1-\alpha} \log_D \left[\frac{\sum_{i=1}^{n} u_i p_i^{\alpha+\beta-1}}{\sum_{i=1}^{n} u_i p_i^\beta}\right] \tag{6}$$

where, $\alpha > 0,\ \alpha \neq 1, \beta > 0, \alpha + \beta > 1$ and $\sum_{i=1}^{n} p_i \leq 1$. Consider the weighted mean of order $\alpha - 1$ of p_i with weihgts $u_i p_i^{\beta_i}$ and utilities u_i of the events A_i as

$$M_{\alpha,\beta_i}(P;U) = \left[\frac{\sum_{i=1}^{n} u_i p_i^{\alpha+\beta-1}}{\sum_{i=1}^{n} u_i p_i^{\beta_i}}\right], \alpha > 0,\ \ \alpha \neq 1,\ \ \beta_i \geq 1 \tag{7}$$

The generalized additive useful information generating function is now defined as

$$M_{\alpha,\beta_i}(P,U;t) = \left[M_{\alpha,\beta_i}(P;U)\right]^{-t} \tag{8}$$

where t is a real (or complex) variable. Since $0 \leq p_i^{\alpha+\beta_i-1} \leq 1$ for each i and $t \geq 0$ and $\{u_i\}$ is bounded for any experiment under consideration, therefore, the expression (8) for $M_{\alpha,\beta_i}(P,U;t)$ is convergent for all $t \geq 0$.

Particular cases:

(i) Taking $\beta_i = \beta$ for every i, $M_{\alpha,\beta_i}(P,U;t)$ reduces to

$$M_{\alpha,\beta}(P,U;t) = \left[\frac{\sum_{i=1}^{n} u_i p_i^{\alpha+\beta-1}}{\sum_{i=1}^{n} u_i p_i^{\beta}}\right]^{\frac{t}{1-\alpha}} \tag{9}$$

which may be treated as generating function of useful information measure proposed by Khan [6].

(ii) If the utility of the scheme is ignored i.e, if $u_i = 1$ for every i then $M_{\alpha,\beta_i}(P;U;t)$ reduces to

$$M_{\alpha,\beta_i}(P;t) = \left[\frac{\sum_{i=1}^{n} p_i^{\alpha+\beta-1}}{\sum_{i=1}^{n} p_i^{\beta_i}}\right]^{\frac{t}{1-\alpha}} \tag{10}$$

which may be consider as generating function of entropy of order α and type $\{\beta_i\}$ proposed by Rathie [8]. Further if $\beta_i = \beta$ for every i then it will give the following generating function

$$M_{\alpha,\beta}(P,t) = \left[\frac{\sum_{i=1}^{n} p_i^{\alpha+\beta-1}}{\sum_{i=1}^{n} p_i^{\beta}}\right]^{\frac{t}{1-\alpha}} \tag{11}$$

which may be treated as generating function of Kapur's entropy [5] of order α and type β.

(iii) If $\beta_i = 1$ for every i then $M_{\alpha,\beta_i}(P,U;t)$ reduces to the generating function

$$M_{\alpha}(P,U;t) = \left[\frac{\sum_{i=1}^{n} u_i p_i^{\alpha}}{\sum_{i=1}^{n} u_i p_i}\right]^{\frac{t}{1-\alpha}} \tag{12}$$

which may be generating function of a measure of useful information of order α proposed by Gurdial and Pessoa [4].

(iv) If $\beta_i = 1$ for every i, then

$$\underset{\alpha \longrightarrow 1}{lt} M_{\alpha,\beta_i}(P,U;t) = M(P,U;t) = 2^{-t \frac{\left(\sum_{i=1}^{n} u_i p_i \log_D p_i\right)}{\sum_{i=1}^{n} u_i p_i}} \tag{13}$$

(13) reduces to the generating function of Belis and Guiasu [1] useful information for incomplete discrete distribution.

3. Derivatives

The value of the first derivative of $M_{\alpha,\beta_i}(P,U;t)$ with respect to t at $t = 0$ is

$$\left[\frac{\delta M_{\alpha,\beta_i}(P,U;t)}{\delta t}\right]_{t=0} = \frac{1}{1-\alpha} \log_D \frac{\sum_{i=1}^{n} u_i p_i^{\alpha+\beta_i-1}}{\sum_{i=1}^{n} u_i p_i^{\beta_i}} = H_{\alpha}^{\beta_i}(P;U) \tag{14}$$

which is generalization to Khan's [6] additive useful information of order α and type β. If $\beta_i = \beta$ for every i, then we see that the value of first order derivative of $M_{\alpha,\beta_i}(P,U;t)$ at $t = 0$ is exactly Khan's [6] generalized additive useful information of order α and type β. The other information measures, such as Kapur [5], Rathie [8], Gurdial and Pessoa [4] may be obtained as its particular cases. The $r-$th deivative of $M_{\alpha,\beta_i}(P,U;t)$ at $t = 0$ is as below.

$$\left[\frac{\delta^r M_{\alpha,\beta_i}(P,U;t)}{\delta\, t^r}\right]_{t=0} = \left[\frac{1}{1-\alpha} \log_D \frac{\sum_{i=1}^{n} u_i p_i^{\alpha+\beta_i-1}}{\sum_{i=1}^{n} u_i p_i^{\beta_i}}\right]^r$$

Applications

(i) **Uniform Distribution:** For a uniform distribution $\left\{\frac{1}{N}, \frac{1}{N}, \ldots, \frac{1}{N}\right\}$ and utility distribution $\{u, u, \ldots, u\}$ of the discrete sample space, we get

$$M_{\alpha,\beta}(P,U;t) = u\, N^t \tag{15}$$

and

$$\left[\frac{\delta\, M_{\alpha,\beta}\,(P,U;t)}{\delta\, t}\right]_{t=0} = u \log N \tag{16}$$

which is entropy of order α and type β and is independent of α and β. Gurdial and Pessoa [4] entropy of any order and Belis and Guiasu [1] entropy will also have the same value. In case when utilities are ignored (16) reduces to $\log N$, a result due to Golomb [2] and Mathur and Kashyap [7].

(ii) **Geometric distribution:** For the geometric distribution $\{q, qp, qp^2, \dots\}, p+q = 1$, and geometric utility distribution $\{v, vu, vu^2, \dots\}$, the proposed information generating function for this distribution is obtained as

$$M_{\alpha,\beta}\,(P,U;t) = \left[\frac{q^{\alpha-1}\left(1-up^{\beta}\right)}{1-up^{\alpha+\beta-1}}\right]^{\frac{t}{1-\alpha}} \tag{17}$$

The value of the first derivarive at $t=0$ is

$$\left[\frac{\delta\, M_{\alpha,\beta}\,(P,U;t)}{\delta\, t}\right]_{t=0} = \frac{1}{1-\alpha}\log\left[\frac{q^{\alpha-1}\left(1-up^{\beta}\right)}{1-up^{\alpha+\beta-1}}\right] \tag{18}$$

If utilities are ignored then (18) reduces to $\frac{1}{1-\alpha}\log\left[\frac{q^{\alpha-1}\left(1-p^{\beta}\right)}{1-p^{\alpha+\beta-1}}\right]$, a quantity obtained by Mathur and Kashyap [7].

(iii) **Normal distribution:** For the normal distribution with mean zero and variance σ^2 and normal utility distribution with mean zero and variance θ^2 on $(-\infty,\infty)$ i.e,

$$p\,(x) = \frac{1}{\sqrt{2\,\pi\,\sigma^2}}e^{-\frac{x^2}{2\sigma^2}}, \sigma > 0, \quad -\infty < x < \infty$$
$$u\,(x) = \frac{1}{\sqrt{2\,\pi\,\theta^2}}e^{-\frac{u^2}{2\theta^2}}, \theta > 0, \quad -\infty < x < \infty$$

Then we have

$$M_{\alpha,\beta}\,(P,U;t) = \left[\frac{\sqrt{\sigma^2+\beta\,\theta^2}}{\sqrt{\sigma^2+(\alpha+\beta-1)\,\theta^2}\,(2\,\pi\,\sigma^2)^{\frac{\alpha-1}{2}}}\right] \tag{19}$$

and

$$\left[\frac{\delta\, M_{\alpha,\beta}\,(P,U;t)}{\delta\, t}\right]_{t=0} = \frac{1}{1-\alpha}\log\left[\frac{\sqrt{\sigma^2+\beta\,\theta^2}}{\sqrt{\sigma^2+(\alpha+\beta-1)\,\theta^2}\,(2\,\pi\,\sigma^2)^{\frac{\alpha-1}{2}}}\right] \tag{20}$$

Assuming $\beta = 1$ in (19) and (20) we get result for Renyi's entropy and taking limit as $\alpha \longrightarrow 1$ it reduces for Shannon's entropy.

(iv) **Exponential distribution:** For the exponential probability distribution with mean $\frac{1}{\lambda}$ and assuming the corresponding exponentail distribution with mean $\frac{1}{\mu}$. Then

$$p\,(x) = \lambda\, e^{-\lambda x}, \lambda > 0, \quad x \in [0,\infty)$$

$$u(x) = \mu\, e^{-\mu\, x}\,, \mu > 0$$

Then we have

$$M_{\alpha,\beta}(P,U;t) = \left[\frac{\lambda^{\alpha-1}(\lambda\,\beta+\mu)}{\lambda(\alpha+\beta-1)+\mu}\right]^{\frac{t}{1-\alpha}} \tag{21}$$

and

$$\left[\frac{\delta\, M_{\alpha,\beta}(P,U;t)}{\delta\, t}\right]_{t=0} = \frac{1}{1-\alpha}\log\left[\frac{\lambda^{\alpha-1}(\lambda\beta+\mu)}{\lambda(\alpha+\beta-1)+\mu}\right] \tag{22}$$

If utilities are ignored then (22) reduces to a result due to Mathur and Kashyap [7]. Further if $\beta = 1$ in (21) and (22) we get result for Renyi's entropy and taking limit as $\alpha \longrightarrow 1$ it reduces to result for Shannon entropy.

(v) **Pareto Distribution:** For the Pareto probability distribution with exponent λ on $[1,\infty)$ and assuming the corresponding Pareto utility distribution with exponent μ on $(-\infty, 1]$

$$p(x) = (\lambda-1)\, x^{-\lambda}, x > 0$$

$$u(x) = (1-\mu)\ \ x^{-\mu}, \mu < 1 \ \text{ and } \ x \in [1,\infty)$$

Then we have

$$M_{\alpha,\beta}(P,U;t) = \left[\frac{(\lambda-1)^{\alpha-1}(\lambda\beta+\mu-1)}{\lambda(\alpha+\beta-1)+\mu-1}\right]^{\frac{t}{1-\alpha}} \tag{23}$$

and

$$\left[\frac{\delta\, M_{\alpha,\beta}(P,U;t)}{\delta\, t}\right]_{t=0} = \frac{1}{1-\alpha}\log\left[\frac{(\lambda-1)^{\alpha-1}(\lambda\beta+\mu-1)}{\lambda(\alpha+\beta-1)+\mu-1}\right] \tag{24}$$

If the utilities are ignored and $\beta = 1$ then (24) have a result for Renyi's entropy. Further taking limit as $\alpha \longmapsto 1$ it reduces to a result for Shannon's entropy.

4. Inversion

Given the generalized additive useful information generating function $M_{\alpha,\beta}(P,U;t)$, we can easily determine the probability and utility distribution corresponding to it.In the discrete case the generalized additive useful information $M_{\alpha,\beta}(P,U;t)$ is evidently a symmetric function of $u_1p_{1,}u_2p_2,\ldots,u_np_n$. Therefore without loss of generality we can assume that $p_1 \geq p_2 \geq \cdots \geq p_n$ and $u_1 \geq u_2 \geq \cdots \geq u_n$. So that $u_1p_1 \geq u_2p_2 \geq ... \geq u_np_n$. Let us consider a function

$$M^0_{\alpha,\beta}(P,U;t) = \left[M_{\alpha,\beta}(P,U;t)\right]^{\frac{1-\alpha}{t}} \tag{25}$$

Then

$$\underset{\alpha\to\infty}{lt}\left[M^{0}_{\alpha,\beta}\left(P,U;t\right)\right]^{\frac{1}{\alpha}}=\underset{\alpha\to\infty}{lt}\left[\frac{\sum\limits_{i=1}^{n}u_{i}p_{i}^{\alpha+\beta-1}}{\sum\limits_{i=1}^{n}u_{i}p_{i}^{\beta}}\right]^{\frac{1}{\alpha}}=\max_{i}\left(u_{i}p_{i}\right)=u_{1}p_{1}\tag{26}$$

Now taking $u_i = 1$ for each i in (26), we get

$$\underset{\alpha\to\infty}{lt}\left[\frac{\sum\limits_{i=1}^{n}p_{i}^{\alpha+\beta-1}}{\sum\limits_{i=1}^{n}p_{i}^{\beta}}\right]^{\frac{1}{\alpha}}=\max_{i}\left(p_{i}\right)=p_{1}\tag{27}$$

Next, on dividing (26) by (27) we get u_1. Thus the largest probability and utility is obtained from the generalized additive useful information generating function using (25), (26) and (27). Suppose

$$M^{1}_{\alpha,\beta}\left(P,U;t\right)=M^{0}_{\alpha,\beta}(P,U;t)-\frac{u_{1}p_{1}^{\alpha+\beta-1}}{\sum\limits_{i=1}^{n}u_{i}p_{i}^{\beta}}=\frac{\sum\limits_{i=2}^{n}u_{i}p_{i}^{\alpha+\beta-1}}{\sum\limits_{i=1}^{n}u_{i}p_{i}^{\beta}}$$

Thus

$$\underset{\alpha\to\infty}{lt}\left[M^{1}_{\alpha,\beta}\left(P,U;t\right)\right]^{\frac{1}{\alpha}}=u_{2}p_{2}\tag{28}$$

Again, if we put $u_i = 1$ for each i, we get

$$\underset{\alpha\to\infty}{lt}\left[M^{1}_{\alpha,\beta}\left(P,U;t\right)\right]^{\frac{1}{\alpha}}=p_{2}\tag{29}$$

(29) and (28) gives u_2. Continuing in this way we can uniquely determine $p_1, p_2, \ldots, p_n$ and $u_1, u_2, \ldots, u_n$ in the decreasing (or non-decreasing) order. In general we can write

$$\underset{\alpha\to\infty}{lt}\left[M^{r}_{\alpha,\beta}\left(P,U;t\right)\right]^{\frac{1}{\alpha}}=\underset{\alpha\to\infty}{lt}\left[M^{r-1}_{\alpha,\beta}-\frac{u_{r}p_{r}^{\alpha+\beta-1}}{\sum\limits_{i=1}^{n}u_{i}p_{i}}\right]^{\frac{1}{\alpha}}=\left(u_{r+1}p_{r+1}\right)$$

Again if we put $u_i = 1$ for each i, we have

$$\underset{\alpha\to\infty}{lt}\left[M^{r}_{\alpha,\beta}\left(P,t\right)\right]^{\frac{1}{\alpha}}=p_{r+1}$$

5. Joint Distribution

Let X and Y be two independent random variables having probability distribution $P = \{p_1, p_2, \ldots, p_n\}$ and $Q = \{q_1, q_2, \ldots, q_m\}$ respectively with utility distribution $U = \{u_1, u_2, \ldots, u_n\}$ and $V = \{v_1, v_2, \ldots, v_m\}$. Suppose the variable $Z = X \times Y$ is the cartesian product of X & Y,whose probability distribution is given by $R = \{p_i q_j\}$ and utility distribution is given by $W = \{u_i v_j\}$, $i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m.$ If $M_{\alpha,\beta}(P, U; t)$ and $M_{\alpha,\beta}(Q, V; t)$ are the generalized generating functions of random variables X and Y respectively, then the generalized additive useful information generating function of Z is given by

$$M_{\alpha,\beta}(R, W; t) = M_{\alpha,\beta}(P, U; t) .M_{\alpha,\beta}(Q, V; t) \tag{30}$$

$$\left[\frac{\delta\ M_{\alpha,\beta}(R, W; t)}{\delta\ t}\right]_{t=0} = H_\alpha^\beta(P; V) + H_\alpha^\beta(Q; U) \tag{31}$$

6. Conjugate relation

Two real number α_1 and α_2 are said to be conjugate if

$$\frac{1}{\alpha_1} + \frac{1}{\alpha_2} = 1 \tag{32}$$

since we restrict (32) to be useful information of positive orders only, therefore it is obvious that $\alpha_1 > 1$ and $\alpha_2 > 1$. Now we show that

$$M_{1,\beta}(P, U; t) > M_{\alpha_{1,\beta}}(P, U; \alpha_1^* t) .M_{\alpha_2,\beta}(P, U; \alpha_2^* t) \tag{33}$$

where $\alpha_1^* = \frac{\alpha_1 - 1}{\alpha_1}$ and $\alpha_2^* = \frac{\alpha_2 - 1}{\alpha_2}$ and $M_{1,\beta}(P, U; t)$ is the limiting case of $M_{\alpha,\beta}(P, U; t)$ as $\alpha \to 1$. Using Jenson's inequality [4], we have

$$\frac{\sum_{i=1}^n u_i p_i^\beta \log u_i p_i}{\sum_{i=1}^n u_i p_i^\beta} < \log\left[\frac{\sum_{i=1}^n u_i p_i^{\beta+1}}{\sum_{i=1}^n u_i p_i^\beta}\right] \tag{34}$$

Further, Using Holder's inequality [4], we have

$$\log\left[\frac{\sum_{i=1}^n u_i p_i^{\beta+1}}{\sum_{i=1}^n u_i p_i^\beta}\right] \leq \log\left\{\left[\frac{\sum_{i=1}^n u_i p_i^{\alpha_1+\beta-1}}{\sum_{i=1}^n u_i p_i^\beta}\right]^{\frac{1}{\alpha_1}} \left[\frac{\sum_{i=1}^n u_i p_i^{\alpha_2+\beta-1}}{\sum_{i=1}^n u_i p_i^\beta}\right]^{\frac{1}{\alpha_2}}\right\} \tag{35}$$

or

$$-t\ \log\left[\frac{\sum_{i=1}^{n} u_i p_i^{\beta+1}}{\sum_{i=1}^{n} u_i p_i^{\beta}}\right] \geq \log\left\{\left[\frac{\sum_{i=1}^{n} u_i p_i^{\alpha_1+\beta-1}}{\sum_{i=1}^{n} u_i p_i^{\beta}}\right]^{\frac{\alpha_1^* t}{1-\alpha_1}}\left[\frac{\sum_{i=1}^{n} u_i p_i^{\alpha_2+\beta-1}}{\sum_{i=1}^{n} u_i p_i^{\beta}}\right]^{\frac{\alpha_2^* t}{1-\alpha_2}}\right\} \tag{36}$$

using (36) and (34) and simplifying, we get the required result (33). If the utilities are ignored i.e, $u_i = 1$ for each i, (33) reduces to

$$M_{1,\beta}\left(P,t\right) > M_{\alpha_1,\beta}\left(P,\alpha_1^* t\right).M_{\alpha_2,\beta}\left(P,\alpha_2^* t\right). \tag{37}$$

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