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Existence of Prime Labeling in Structurally Extended Wheel Graphs

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Abstract

A graph with n vertices has a prime labeling if its vertices can be assigned distinct integers from 1 to n such that adjacent vertex labels are relatively prime. The study of prime labeling for graphs obtained through structural operations has attracted considerable attention in recent years. In this paper, we introduce and investigate a novel family of graphs derived through a systematic structural extension of the standard wheel graph. The construction process initiates with the edge subdivision of the outer cycle of the wheel graph. Subsequently, we focus on the newly inserted subdivision vertices−specifically those that are non-adjacent to the central apex vertex. To enhance the structural complexity, a multi-layered expansion is applied: between every pair of consecutive subdivision vertices, we introduce k distinct intermediate vertices. This operation effectively generates k parallel paths of length two bridging these subdivision vertices. In this study, we specifically focus on graphs constructed with $k = 1, 2, 3$, and 4 layers. Furthermore, we explore the labeling properties of these extended structures and analytically prove that they admit a prime labeling for every integer $n \geq 3$.

Keywords: Prime labeling; prime graph; wheel graph; edge subdivision; path graph; Structural extended.
2020 Mathematics Subject Classification: 05C78, 05C76.

1. Introduction

Graph labeling is a specialized area of graph theory in which numerical values or labels are assigned to the vertices or edges or both of a graph according to certain predetermined mathematical rules. Entringer first proposed the fundamental idea of prime labeling, which was subsequently formalized by Tout et al. [10]. Research on prime graphs has advanced significantly, focusing primarily on classifying structures that admit a prime labeling. Prime labeling distinguishes itself from other graph labeling approaches as a crucial area of investigation, a distinction largely driven by its rigorous mathematical and number-theoretical rules.

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Definition 1.1. *For an n-vertex graph G, a prime labeling is a one-to-one mapping $f : V(G) \rightarrow \{1, 2, ..., n\}$ where adjacent vertices always receive relatively prime values (meaning for every edge $uv \in E(G), gcd(f(u), f(v)) = 1$). A graph admitting such a labeling is called a prime graph.*

A natural question emerges: can a graph retain its prime labeling after undergoing structural operations?
Extensive research has been conducted to determine the primality of various fundamental graph families; However, the impact of specific structural operations on prime labeling remains a fertile ground for discovery. For an extensive review of various graph labeling techniques, readers are referred to the dynamic survey by J. Gallian [2]. The present study is motivated by existing prime graphs such as cycle, wheel, gear, helm, and book graphs-whose proofs are discussed in [8,9], and [10]. Furthermore, this work is inspired by investigations into prime labeling through edge subdivision [6] and various structural operations [7]. In our proposed work, we enhance the complexity of a base graph by introducing additional vertices and obtain prime labeling for the extended graph. This section outlines the basic terminology and preliminary findings applied in our subsequent proofs.

Definition 1.2. *Subdividing an edge e (connecting vertices x and y in graph G) means replacing e with a new vertex u and two new edges, xu and uy. This operation adds exactly one vertex and one edge to the graph.*

Definition 1.3. *A path graph P_n is defined by the vertex set $V(P_n) = \{v_1, v_2, ..., v_n\}$ and edge set $E(P_n) = \{v_i v_{i+1} : 1 \leq i < n\}$.*

Definition 1.4. *A cycle graph C_n ($n \geq 3$) is a simple, closed loop of n vertices, with each vertex connected to exactly two others.*

Definition 1.5. *For any integer $n \geq 3$, the wheel graph W_n is defined as a graph comprising the vertex set $V(W_n) = \{v_0, v_1, ..., v_n\}$ and the edge set $E(W_n) = \{v_0 v_i : 1 \leq i \leq n\} \cup \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{v_n v_1\}$.*

Definition 1.6 ([7])**.** *A gear graph $\overline{W_n}$ is constructed from a wheel graph W_n by inserting a new vertex onto each connecting adjacent vertices in the outer n-cycle.*

Definition 1.7 ([5])**.** *The graph $\overline{W_{m,n}}$ is constructed from the gear graph $\overline{W_n}$ by introducing m new common neighbors, denoted $v_{i+1}^j (1 \leq j \leq m)$, for each vertex pair (v_i, v_{i+2}) where $i \in \{2, 4, ..., 2n-2\}$. Additionally, m new common neighbors, v_1^j, are added to connect the vertices v_{2n} and v_2.*

Essential lemmas based on the Euclidean algorithm are established below for continued application in subsequent sections.

Lemma 1.8 ([1])**.** *For positive integers a, b, and k, $gcd(a, b) = 1$ under any of these conditions:*

1. *$a = 1$;*
2. *$a = 2^k$ and b is odd;*

3. *a and b are odd integers differing by* 2^k*;*

4. *a and b are consecutive.*

Lemma 1.9 ([1]). *For any integer* $n > 1$ *and arbitrary integers a, b, c, and d, the following properties hold:*

1. *If* $a \equiv b \ (mod\ n)$ *and* $b \equiv c \ (mod\ n)$*, then* $a \equiv c \ (mod\ n)$.

2. *If* $a \equiv b \ (mod\ n)$ *and* $c \equiv d \ (mod\ n)$*, then* $a + c \equiv b + d \ (mod\ n)$ *and* $ac \equiv bd \ (mod\ n)$.

3. *If* $a \equiv b \ (mod\ n)$*, then* $a + c \equiv b + c \ (mod\ n)$ *and* $ac \equiv bc \ (mod\ n)$.

This study establishes the prime labeling of the proposed graph structures by utilizing essential number-theoretic mechanisms, including divisibility, modular arithmetic, and coprimality. The constructive framework introduced here provides rigorous proofs that can be generalized to larger classes of graphs. Standard graph theory conventions are drawn from West [12] along with Gross and Yellen [3], and the necessary number-theoretic background is sourced from Burton [1].

2. Main Results

Theorem 2.1. *The graph* $\overline{W_{1,n}}$ *is a prime graph for every integer* $n \geq 3$.

Proof. For any integer $n \geq 3$, we construct the graph $\overline{W_{1,n}}$ by defining its vertex set as $V(\overline{W_{1,n}}) = \{v_0, v_i : 1 \leq i \leq 2n\} \cup \{u_i : 1 \leq i \leq n\}$. The corresponding edge set linking these vertices is specified by $E(\overline{W_{1,n}}) = \{v_0 v_{2i-1} : 1 \leq i \leq n\} \cup \{v_i v_{i+1} : 1 \leq i \leq 2n-1\} \cup \{v_{2n} v_1\} \cup \{u_i v_{2i} : 1 \leq i \leq n\} \cup \{v_{2i-2} u_i : 2 \leq i \leq n\} \cup \{v_{2n} u_1\}$. Moreover, $|V(\overline{W_{1,n}})| = 3n + 1$ and $|E(\overline{W_{1,n}})| = 5n$.
For $1 \leq i \leq 2n$ and $1 \leq j \leq n$, define a function $f : V(\overline{W_{1,n}}) \to \{1, 2, ..., |V(\overline{W_{1,n}})|\}$ as follows:

$$f(v_0) = 1, \quad f(v_i) = \begin{cases} 2, & i = 1; \\ \dfrac{3i+3}{2}, & i \text{ odd}, i > 1; \\ \dfrac{3i+4}{2}, & i \equiv 2 \ (mod\ 4), i \neq 2n; \\ \dfrac{3i+2}{2}, & i \equiv 0 \ (mod\ 4), i \neq 2n; \\ 3n+1, & i = 2n. \end{cases}, \text{ and } \quad f(u_j) = \begin{cases} 5 - j, & j = 1, 2; \\ 3j - 1, & j \text{ odd}, j > 1; \\ 3j - 2, & j \text{ even}, j > 2. \end{cases}.$$

Since $f(v_0) = 1$, $gcd(f(v_0), f(x)) = 1$ for all vertices x adjacent to v_0. Moreover, v_1 and u_1 are adjacent to v_2; given that $f(v_1) = 2$, $f(u_1) = 4$, and $f(v_2) = 5$, Lemma 1.8, implies $gcd(f(v_1), f(v_2))$ $= gcd(f(u_1), f(v_2)) = 1$. Furthermore, $gcd(f(u_2), f(v_4)) = gcd(3, 7) = 1$. If n is even, v_{2n} is adjacent to v_{2n-1}, u_n, v_1, and u_1. With $f(v_{2n}) = 3n + 1$ being odd, its greatest common divisor with $f(v_1) = 2$ and $f(u_1) = 4$ is 1. Furthermore, $gcd(f(v_{2n}), f(v_{2n-1})) = gcd(3n + 1, 3n) = 1$ by Lemma 1.8, and $gcd(f(v_{2n}), f(u_n)) = gcd(3n + 1, 3n - 2) = gcd(3, 3n - 2) = 1$ because of $3n - 2$ is never divisible by 3. In all other cases, the labels of adjacent vertices are either consecutive numbers or consecutive odd

numbers; hence, by Lemma 1.8, their gcd is 1. However, when n is odd, $f(v_{2n}) = 3n + 1$ is even, resulting in a $gcd > 1$ with the labels of certain vertices adjacent to v_{2n}. Therefore, when n is odd, the function f must be modified so that the modified function satisfies the labeling condition.
We define a modified labeling $g : V(\overline{W_{1,n}}) \rightarrow \{1, 2, ..., |V(\overline{W_{1,n}})|\}$ to handle cases where n is odd.
Define a function $g : V(\overline{W_{1,n}}) \rightarrow \{1, 2, ..., |V(\overline{W_{1,n}})|\}$ as follows:

$$g(v_{2n}) = f(v_{2n-1}), g(v_{2n-1}) = f(v_{2n}), \text{ and } g(x) = f(x) \quad \text{for all } x \in V(\overline{W_{1,n}}) \setminus \{v_{2n-1}, v_{2n}\}.$$

Observe that v_{2n} is adjacent to v_{2n-1}, u_n, v_1, and u_1. Assuming n is odd, $g(v_{2n}) = f(v_{2n-1}) = 3n$ is also odd; thus, its gcd with $f(v_1) = 2$ and $f(u_1) = 4$ is 1. Furthermore, $gcd(g(v_{2n}), g(v_{2n-1}))$ $= gcd(3n, 3n + 1) = 1$ and $gcd(g(v_{2n}), g(u_n)) = gcd(3n, 3n - 1) = 1$ by Lemma 1.8, as both are pairs of consecutive integers. Similarly, v_{2n-1} is adjacent to v_0, v_{2n-2}, and v_{2n}. $gcd(g(v_{2n-1}), g(v_0)) = 1$ since $g(v_0) = 1$. Since n is odd, let $n = 2k + 1$. Then $2n - 2 = 2(2k + 1) - 2 = 4k \equiv 0 \ (mod\ 4)$. It follows that $g(v_{2n-2}) = 3n - 2$. Therefore, $gcd(g(v_{2n-1}), g(v_{2n-2})) = gcd(3n + 1, 3n - 2) = gcd(3, 3n - 2) = 1$, because $3n - 2$ is never divisible by 3. The verification of the prime condition for the remaining adjacency is straightforward. Consequently, $\overline{W_{1,n}}$ is a prime graph for all $n \geq 3$: via f when n is even, and via g when n is odd. □

Example 2.2. *Figure 1 illustrates a prime labeling of the graph $\overline{W_{1,7}}$.*

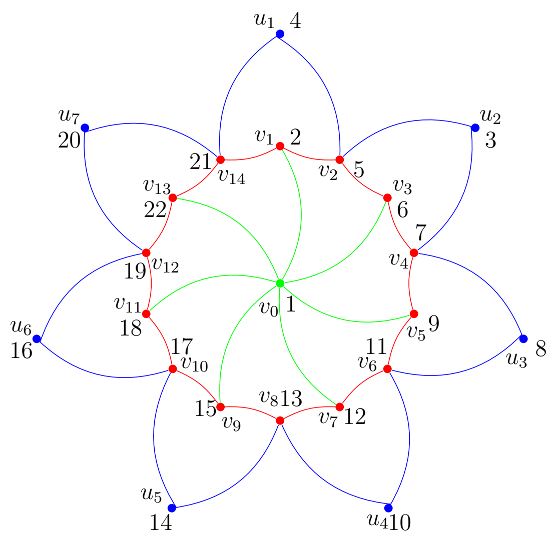

Figure 1: The graph $\overline{W_{1,7}}$ with prime labeling

Theorem 2.3. *The graph $\overline{W_{2,n}}$ is a prime graph for every integer $n \geq 3$.*

Proof. For any integer $n \geq 3$, we construct the graph $\overline{W_{2,n}}$ by defining its vertex set as $V(\overline{W_{2,n}}) = \{v_0, v_i : 1 \leq i \leq 2n\} \cup \{u_i, w_i : 1 \leq i \leq n\}$. The corresponding edge set linking these vertices is specified by $E(\overline{W_{2,n}}) = \{v_0v_{2i-1} : 1 \leq i \leq n\} \cup \{v_iv_{i+1} : 1 \leq i \leq 2n-1\} \cup \{v_{2n}v_1\} \cup \{u_iv_{2i}, w_iv_{2i} :$

$1 \leq i \leq n\} \cup \{v_{2i-2}u_i, v_{2i-2}w_i : 2 \leq i \leq n\} \cup \{v_{2n}u_1, v_{2n}w_1\}$. Moreover, $|V(\overline{W_{2,n}})| = 4n+1$ and $|E(\overline{W_{2,n}})| = 7n$. For $1 \leq i \leq 2n$ and $1 \leq j \leq n$, define a function $f : V(\overline{W_{2,n}}) \to \{1, 2, ..., |V(\overline{W_{2,n}})|\}$ as follows:

$$f(v_0) = 1, \quad f(v_i) = \begin{cases} 2i, & i \text{ odd}; \\ 2i+1, & i \text{ even}. \end{cases}, f(u_j) = 4j-1, \text{ and } f(w_j) = 4j.$$

Observe that for $1 < i \leq n$ and $i \equiv 2 \ (mod\ 3)$, the vertex v_{2i} is adjacent to v_{2i-1}, v_{2i+1}, u_i, w_i, u_{i+1}, and w_{i+1}. Because the labels of v_{2i} and any vertex in $\{v_{2i+1}, u_i, w_i, u_{i+1}\}$ are either consecutive or consecutive odd integers, their gcd is exactly 1. Since $i \equiv 2 \ (mod\ 3)$ directly implies both $4i-2 \equiv 0 \ (mod\ 3)$ and $4i+1 \equiv 0 \ (mod\ 3)$, the calculations simplify to $gcd(f(v_{2i}), f(v_{2i-1})) = gcd(4i+1, 4i-2) = gcd(3, 4i-2) = 3$ and similarly $gcd(f(v_{2i}), f(w_{i+1})) = gcd(4i+1, 4i+4) = gcd(4i+1, 3) = 3$.
Although the labeling f generally satisfies the requirements for a prime labeling, an exception occurs for $1 < i \leq n$ when $i \equiv 2 \ (mod\ 3)$. In this case, $gcd(f(v_{2i}), f(v_{2i-1})) = gcd(f(v_{2i}), f(w_{i+1})) = 3$. Therefore, f must be modified to resolve this conflict.
For $1 < j \leq n$, define a modified function $g : V(\overline{W_{2,n}}) \to \{1, 2, ..., |V(\overline{W_{2,n}})|\}$ as follows:

- $g(u_j) = f(v_{2j})$ and $g(v_{2j}) = f(u_j)$ for $j \in \{3k-1 : k \in \mathbb{N}\} \setminus \{15k-1 : k \in \mathbb{N}\}$;
- $g(u_j) = f(v_{2j-2})$ and $g(v_{2j-2}) = f(u_j)$ for $j \equiv 0 \ (mod\ 15)$;
- $g(x) = f(x)$ otherwise.

Using Lemma 1.9, we proceed as follows.
For $1 < j \leq n$ and $j \in \{3k-1 : k \in \mathbb{N}\} \setminus \{15k-1 : k \in \mathbb{N}\}$, the vertex v_{2j} (with the modified label $g(v_{2j}) = f(u_j) = 4j-1$) is adjacent to v_{2j-1}, v_{2j+1}, u_j, w_j, u_{j+1}, and w_{j+1}, which have the respective modified labels of $4j-2$, $4j+2$, $4j+1$, $4j$, $4j+3$, and $4j+4$. Because the labels of v_{2j} and the label of any vertex in $\{v_{2j-1}, u_j, w_j\}$ are either consecutive or consecutive odd integers, their gcd is exactly 1. Since $j \in \{3k-1 : k \in \mathbb{N}\} \setminus \{15k-1 : k \in \mathbb{N}\}$ directly implies $4j-1 \equiv 1 \ (mod\ 3)$, we have $gcd(g(v_{2j}), g(v_{2j+1})) = gcd(4j-1, 4j+2) = gcd(4j-1, 3) = 1$ and $gcd(g(v_{2j}), g(u_{j+1})) = gcd(4j-1, 4j+3) = gcd(4j-1, 4) = 1$ (since $4j-1$ is odd). Furthermore, given the established constraints on j, $gcd(g(v_{2j}), g(w_{j+1})) = gcd(4j-1, 4j+4) = gcd(4j-1, 5) = 1$. This is because $4j-1 \equiv 0 \ (mod\ 5)$ requires $j \equiv -1 \ (mod\ 5)$. Given $j = 3k-1$, this only occurs if $j \equiv -1 \ (mod\ 15)$. Since $j \notin \{15k-1 : k \in \mathbb{N}\}$, $4j-1$ is never a multiple of 5. Moreover, u_j is adjacent to v_{2j-2}. $gcd(g(u_j), g(v_{2j-2})) = gcd(4j+1, 4j-3) = gcd(4, 4j-3) = 1$ (since $4j-3$ is odd).
For $1 < j \leq n$ and $j \equiv 0 \ (mod\ 15)$, the new label is $g(v_{2j-2}) = f(u_j) = 4j-1$. The adjacent vertices v_{2j-1}, u_j, and w_j take labels $4j-2$, $4j-3$, and $4j$; these form consecutive or consecutive odd integers with $4j-1$, immediately yielding a gcd of 1. For the remaining neighbors u_{j-1}, w_{j-1}, and v_{2j-3} (labeled $4j-5$, $4j-4$, and $4j-6$), the gcd calculations reduce to $gcd(4j-1, 4j-5) = gcd(4j-1, 4) = 1$ (since $4j-1$ is odd), $gcd(4j-1, 4j-4) = gcd(4j-1, 3) = 1$, and $gcd(4j-1, 4j-6) = gcd(4j-1, 5) = 1$. The last two hold because $j \equiv 0 \ (mod\ 15)$ implies $4j-1 \equiv -1 \ (mod\ p)$ for $p \in \{3, 5\}$, meaning neither 3 nor

5 divides $4j-1$. Thus, the gcd is strictly 1 in all cases. Moreover, u_j is adjacent to v_{2j}. $gcd(g(u_j), g(v_{2j}))$ $= gcd(4j-3, 4j+1) = gcd(4j-3, 4) = 1$ (since $4j-3$ is odd).

The remaining cases are straightforward; hence g is a prime labeling of $\overline{W_{2,n}}$. $\square$

Example 2.4. *Figure 2 illustrates a prime labeling of the graph* $\overline{W_{2,6}}$.

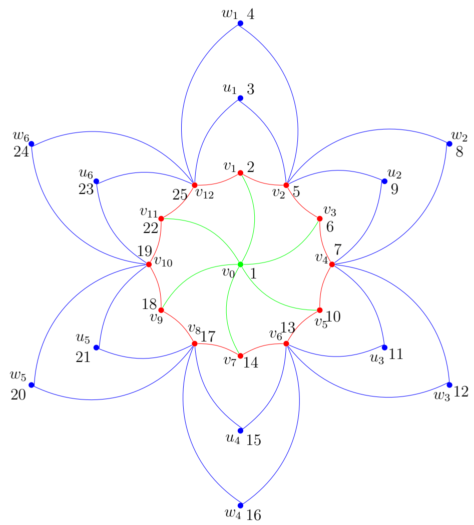

Figure 2: The graph $\overline{W_{2,6}}$ with prime labeling

Theorem 2.5. *The graph* $\overline{W_{3,n}}$ *is a prime graph for every integer* $n \geq 3$.

Proof. For any integer $n \geq 3$, we construct the graph $\overline{W_{3,n}}$ by defining its vertex set as $V(\overline{W_{3,n}}) = \{v_0, v_i : 1 \leq i \leq 2n\} \cup \{u_i, w_i, y_i : 1 \leq i \leq n\}$. The corresponding edge set linking these vertices is specified by $E(\overline{W_{3,n}}) = \{v_0v_{2i-1} : 1 \leq i \leq n\} \cup \{v_iv_{i+1} : 1 \leq i \leq 2n-1\}$ $\cup\{v_{2n}v_1\} \cup \{u_iv_{2i}, w_iv_{2i}, y_iv_{2i} : 1 \leq i \leq n\} \cup \{v_{2i-2}u_i, v_{2i-2}w_i, v_{2i-2}y_i : 2 \leq i \leq n\} \cup \{v_{2n}u_1, v_{2n}w_1, v_{2n}y_1\}$. Moreover, $|V(\overline{W_{3,n}})| = 5n+1$ and $|E(\overline{W_{3,n}})| = 9n$.

For $1 \leq i \leq 2n-1$ and $1 \leq j \leq n$, define a function $f : V(\overline{W_{3,n}}) \to \{1, 2, ..., |V(\overline{W_{3,n}})|\}$ as follows:

$$f(v_0) = 5, \quad f(v_i) = \begin{cases} \dfrac{5i-1}{2}, & i \equiv 1 \ (mod\ 4); \\ \dfrac{5i+4}{2}, & i \equiv 2 \ (mod\ 4); \\ \dfrac{5i-3}{2}, & i \equiv 3 \ (mod\ 4); \\ \dfrac{5i+2}{2}, & i \equiv 4 \ (mod\ 4). \end{cases}, \ f(v_{2n}) = 5n+1, \quad f(u_j) = \begin{cases} 13-5j, & 1 \leq j \leq 2; \\ 5j-2, & 3 \leq j \leq n. \end{cases},$$

$$f(w_j) = 5j-1, \text{ and } \quad f(y_j) = \begin{cases} 1, & j = 1; \\ 5j, & 1 < j \leq n. \end{cases}.$$

For $4 \leq j \leq n$, the labeling function f holds true for most edges ($gcd = 1$), but the prime labeling condition fails ($gcd > 1$) in the following three exceptional cases:

1. When n is an odd integer, $f(v_{2n}) = 5n+1$ becomes even, resulting in a $gcd \geq 2$ for the edges $\{v_{2n}, v_1\}$, $\{v_{2n}, u_1\}$, and $\{v_{2n}, w_1\}$.

2. When $j \equiv 4\ (mod\ 6)$, the pairs $\{v_{2j}, u_j\}$ and $\{v_{2j}, w_{j+1}\}$ yield a $gcd > 1$.

3. When $j \equiv 5\ (mod\ 6)$, the pairs $\{v_{2j}, w_j\}$ and $\{v_{2j}, y_{j+1}\}$ yield a $gcd > 1$.

Therefore, a suitable modification to the labeling function for these specific vertices is necessary to achieve a valid prime labeling for the entire graph.
Define a modified function $g : V(\overline{W_{3,n}}) \to \{1, 2, ..., |V(\overline{W_{3,n}})|\}$ as follows:

- $g(v_{2j}) = f(w_j)$ and $g(w_j) = f(v_{2j})$ for $4 \le j \le n$ and $j \equiv 4\ (mod\ 6)$;
- $g(v_{2j}) = f(u_j)$ and $g(u_j) = f(v_{2j})$ for $4 \le j \le n$, $j \equiv 5\ (mod\ 6)$ and $j \ne n$;
- $g(v_{2n}) = f(y_n)$ and $g(y_n) = f(v_{2n})$ if n is odd and $n \not\equiv 3\ (mod\ 6)$;
- $g(v_{2n}) = f(u_n)$ and $g(u_n) = f(v_{2n})$ if n is odd and $n \equiv 3\ (mod\ 6)$;
- $g(x) = f(x)$ otherwise.

For $4 \le j \le n$ with $j \equiv 4\ (mod\ 6)$, the new label is $g(v_{2j}) = f(w_j) = 5j - 1$. The adjacent vertices u_j, w_j, and y_j are labeled $5j-2$, $5j+1$, and $5j$, which form consecutive or consecutive odd integers with $5j-1$, immediately yielding a gcd of 1. For the remaining neighbors v_{2j-1}, v_{2j+1}, u_{j+1}, w_{j+1}, and y_{j+1} (labeled $5j-4, 5j+2, 5j+7, 5j+4$, and $5j+5$), the gcd calculations reduce to $gcd(5j-1, 5j-4) = gcd(5j-1, 3)$ and $gcd(5j-1, 5j+2) = gcd(5j-1, 3)$. Since $j \equiv 4\ (mod\ 6)$, we can write $j = 6k+4$, which implies $5j - 1 = 30k + 19 \equiv 1\ (mod\ 3))$; hence, the gcd is 1. Similarly, $gcd(5j-1, 5j+7) = gcd(5j-1, 8) = 1$ (since $j \equiv 4\ (mod\ 6)$ then $5j-1$ is odd), $gcd(5j-1, 5j+4) = gcd(5j-1, 5) = 1$ ($5j - 1 \equiv 4\ (mod\ 5)$ implies $5j-1$ is never divisible by 5), and $gcd(5j-1, 5j+5) = gcd(5j-1, 6) = 1$ because $5j-1$ is odd, it shares no factor of 2 with 6, and it is not divisible by 3; therefore, the gcd is 1. Moreover, w_j is adjacent to v_{2j-2}. $gcd(5j+1, 5j-3) = gcd(5j+1, 4) = 1$ since $5j+1$ is odd.
For $4 \le j \le n$ with $j \equiv 5\ (mod\ 6)$ and $j \ne n$, the new label is $g(v_{2j}) = f(u_j) = 5j-2$. The adjacent vertices v_{2j-1}, w_j, and y_j are labeled $5j-3$, $5j-1$, and $5j$, which form consecutive or consecutive odd integers with $5j-2$, immediately yielding a gcd of 1. For the remaining neighbors v_{2j+1}, u_j, u_{j+1}, w_{j+1}, and y_{j+1} (labeled $5j+1$, $5j+2$, $5j+3$, $5j+4$, and $5j+5$), the gcd calculations reduce to $gcd(5j-2, 5j+1) = gcd(5j-2, 3) = 1$. Since $j \equiv 5\ (mod\ 6)$, we can write $j = 6k+5$, which implies $5j-2 = 30k+23 \equiv 2\ (mod\ 3)$. Additionally, $gcd(5j-2, 5j+2) = gcd(5j-2, 4) = 1$ (since $j \equiv 5\ (mod\ 6)$, then $5j-2$ is odd), and $gcd(5j-2, 5j+3) = gcd(5j-2, 5) = 1$ ($5j-2 \equiv 3\ (mod\ 5)$ implies $5j-2$ is never divisible by 5), and $gcd(5j-2, 5j+4) = gcd(5j-2, 6) = 1$ because $5j-2$ is odd, shares no factor of 2 with 6, and is not divisible by 3; therefore, the gcd is 1. Finally, $gcd(5j-2, 5j+5) = gcd(5j-2, 7) = 1$ because $5j-2 = 30k+23 \equiv 2\ (mod\ 7)$.
When n is odd and $n \not\equiv 3\ (mod\ 6)$, the new label is $g(v_{2n}) = f(y_n) = 5n$. The adjacent vertices u_n, w_n and y_n are labeled $5n-2$, $5n-1$, and $5n+1$,respectively. These form pair of consecutive or consecutive odd integers with $5n$, immediately yielding a gcd of 1. For the remaining neighbors v_{2n-1}, v_1, u_1, w_1,

and y_1 (labeled $5n-3$, 2, 8, 4, and 1), the calculations reduce to $gcd(5n, 5n-3) = gcd(5n, 3) = 1$. This holds true because n being odd and $n \not\equiv 3 \ (mod\ 6)$ guarntees n is not a multiple of 3. Thus, $5n \not\equiv 0 \ (mod\ 3)$, meaning 3 does not divide $5n$, yielding a gcd of 1. Since n is odd, $5n$ must also be odd; therefore, $gcd(5n, p) = 1$ for all $p \in \{1, 2, 4, 8\}$. Moreover, y_n is adjacent to v_{2n-2}. Here, $gcd(5n+1, 5n-4) = gcd(5n+1, 5) = 1$, since $5n+1$ is never divisible by 5.

When n is odd and $n \equiv 3 \ (mod\ 6)$, the new label is $g(v_{2n}) = f(u_n) = 5n-2$. The adjacent vertices u_{2n-1}, w_n and y_n are labeled $5n-3$, $5n-1$, and $5n$,respectively. These form pair of consecutive or consecutive odd integers with $5n-2$, immediately yielding a gcd of 1. For the remaining neighbors u_n, v_1, u_1, w_1, and y_1 (labeled $5n+1$, 2, 8, 4, and 1), the calculations reduce to $gcd(5n-2, 5n+1) = gcd(3, 5n+1) = 1$. Since $n \equiv 3 \ (mod\ 6)$, we can write $n = 6k+3$, which implies $5j - 2 = 30k + 13 \equiv 1 \ (mod\ 3)$. Since n is odd, $5n-2$ must also be odd; therefore, $gcd(5n-2, p) = 1$ for all $p \in \{1, 2, 4, 8\}$. Moreover, u_n is adjacent to v_{2n-2}. Here, $gcd(5n+1, 5n-4) = gcd(5n+1, 5) = 1$, since $5n+1$ is never divisible by 5.

The remaining cases are straightforward; hence g is a prime labeling of $\overline{W_{3,n}}$. □

Example 2.6. *Figure 3 illustrates a prime labeling for the graph $\overline{W_{3,5}}$.*

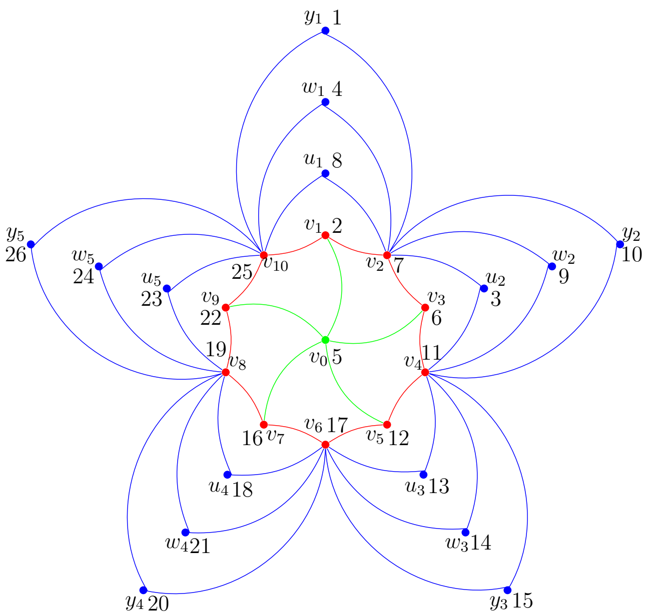

Figure 3: The graph $\overline{W_{3,5}}$ with prime labeling

Theorem 2.7. *The graph $\overline{W_{4,n}}$ is a prime graph for every integer $n \geq 3$.*

Proof. For any integer $n \geq 3$, we construct the graph $\overline{W_{4,n}}$ by defining its vertex set as $V(\overline{W_{4,n}}) = \{v_0, v_i : 1 \leq i \leq 2n\} \cup \{u_i, w_i, y_i, z_i : 1 \leq i \leq n\}$. The corresponding edge set linking these vertices is specified by $E(\overline{W_{4,n}}) = \{v_0 v_{2i-1} : 1 \leq i \leq n\} \cup \{v_i v_{i+1} : 1 \leq i \leq 2n-1\}$ $\cup \{v_{2n} v_1\} \cup \{u_i v_{2i}, w_i v_{2i}, y_i v_{2i}, z_i v_{2i} : 1 \leq i \leq n\} \cup \{v_{2i-2} u_i, v_{2i-2} w_i, v_{2i-2} y_i, , v_{2i-2} z_i : 2 \leq i \leq n\} \cup \{v_{2n} u_1, v_{2n} w_1, v_{2n} y_1, v_{2n} z_1\}$. Moreover, $|V(\overline{W_{4,n}})| = 6n+1$ and $|E(\overline{W_{4,n}})| = 11n$.

For $1 \le i \le 2n$ and $1 \le j \le n$, define a function $f : V(\overline{W_{4,n}}) \to \{1, 2, ..., |V(\overline{W_{4,n}})|\}$ as follows:

$$f(v_0) = 1, \quad f(v_i) = \begin{cases} 3i-1, & i \; odd; \\ 3i+1, & i \; even. \end{cases}, f(u_j) = 6j-3, f(w_j) = 6j-2, f(y_j) = 6j-1, \text{ and } f(z_j) = 6j.$$

The defined labeling satisfies the prime condition ($gcd = 1$) for most edges. However, it fails for edges $v_{2j-1}v_{2j} and v_{2j}w_{j+1}$ whenever $j \equiv 4 \; (mod \; 5)$, because their respective labels become multiples of 5. Thus, a modification is required for these specific vertices.

For $1 \le j \le n$ and $j \equiv 4 \; (mod \; 5)$, define a modified function $g : V(\overline{W_{4,n}}) \to \{1, 2, ..., |V(\overline{W_{4,n}})|\}$ as follows:

$g(v_{2j}) = f(y_j)$ and $g(y_j) = f(v_{2j})$ for $j \notin \{35k + 34 : k \in \mathbb{Z}\} \setminus \{n\}$;

$g(v_{2j}) = f(y_{j+1})$ and $g(y_{j+1}) = f(v_{2j})$ for $j \in \{35k + 34 : k \in \mathbb{Z}\} \setminus \{n\}$;

$g(x) = f(x)$ otherwise.

For $1 \le j \le n$ with $j \equiv 4 \; (mod \; 5)$ and $j \notin \{35k + 34 : k \in \mathbb{Z}\} \setminus \{n\}$, the new label is $g(v_{2j}) = f(y_j) = 6j - 1$. The adjacent vertices u_j, w_j, y_j, and z_j are labeled $6j - 3$, $6j - 2$, $6j + 1$, and $6j$, respectively. These form consecutive or consecutive odd integers with $6j - 1$, immediately yielding a gcd of 1. For the remaining neighbors v_{2j-1}, v_{2j+1}, u_{j+1}, w_{j+1}, y_{j+1}, and z_{j+1} (labeled $6j - 4$, $6j + 2$, $6j + 3$, $6j + 4$, $6j + 5$, and $6j + 6$), the gcd calculations reduce to $gcd(6j - 1, 6j - 4) = gcd(6j - 1, 3) = 1$ and $gcd(6j - 1, 6j + 2) = gcd(6j - 1, 3) = 1$ (because $6j - 1 \equiv 2 \; (mod \; 3)$). Next, $gcd(6j - 1, 6j + 3) = gcd(6j - 1, 4) = 1$ (since $6j - 1$ is odd), and $gcd(6j - 1, 6j + 4) = gcd(6j - 1, 5) = 1$. This latter reduction holds because $j \equiv 4 \; (mod \; 5)$ implies $j = 5k + 4$, meaning $6j - 1 \equiv 30k + 23 \equiv 3 \; (mod \; 5)$. Additionally, $gcd(6j - 1, 6j + 5) = gcd(6j - 1, 6) = 1$ (since $6j - 1 \equiv 5 \; (mod \; 6)$). Because $j \equiv 4 \; (mod \; 5)$ and $j \notin \{35k + 34 : k \in \mathbb{Z}\} \setminus \{n\}$, we find $gcd(6j - 1, 6j + 6) = gcd(6j - 1, 7) = 1$. This is valid because the restriction $j \notin \{35k + 34 : k \in \mathbb{Z}\}$ ensure that $j \not\equiv 6 \; (mod \; 7)$. As a result, $6j - 1$ is not divisible by the prime 7, making the terms strictly coprime. Furthermore, y_j is adjacent to v_{2j-2}, and $gcd(6j + 1, 6j - 5) = gcd(6j + 1, 6) = 1$ (since $6j + 1 \equiv 1 \; (mod \; 6)$).

For $1 \le j \le n$ with $j \equiv 4 \; (mod \; 5)$ and $j \in \{35k + 34 : k \in \mathbb{Z}\} \setminus \{n\}$, the new label is $g(v_{2j}) = f(y_{j+1}) = 6j + 5$. The adjacent vertices u_{j+1}, w_{j+1}, and z_{j+1} are labeled $6j + 3$, $6j + 4$, and $6j + 6$, respectively. These form consecutive or consecutive odd integers with $6j + 5$, immediately yielding a gcd of 1. For the remaining neighbors v_{2j-1}, v_{2j+1}, u_j, w_j, y_j, z_j, and y_{j+1} (labeled $6j - 4$, $6j + 2$, $6j - 3$, $6j - 2$, $6j - 1$, $6j$ and $6j + 1$), the gcd calculations reduce to $gcd(6j + 5, 6j - 4) = gcd(6j + 5, 9) = 1$ (since $6j + 5 \equiv 2 \; (mod \; 3)$, thus $6j + 5$ is never divisible by 3 or 9) and $gcd(6j + 5, 6j + 2) = gcd(6j + 5, 3) = 1$. Next, $gcd(6j + 5, 6j - 3) = gcd(6j + 5, 8) = 1$ (since $6j + 5$ is odd), and $gcd(6j + 5, 6j - 2) = gcd(6j + 5, 7) = 1$. This latter reduction holds because $j \in \{35k + 34 : k \in \mathbb{Z}\} \setminus \{n\}$, meaning $6j + 5 \equiv 210k + 209 \equiv 6 \; (mod \; 7)$. Additionally, $gcd(6j + 5, 6j - 1) = gcd(6j + 5, 6) = 1$ (since $6j + 5 \equiv 5 \; (mod \; 6)$) and $gcd(6j + 5, 6j) = gcd(6j + 5, 5) = 1$ (since $6j + 5 = 210k + 209 \equiv 4 \; (mod \; 5)$). Moreover, $gcd(6j + 5, 6j + 1) = gcd(6j + 5, 4) = 1$ (since $6j + 5$ is odd). Furthermore, y_{j+1} is adjacent to v_{2j+2}, and $gcd(6j + 1, 6j + 7) =$

$gcd(6j+1,6) = 1$ (since $6j+1 \equiv 1 \ (mod\ 6)$).

The remaining cases are straightforward; hence g is a prime labeling of $\overline{W_{4,n}}$. □

Example 2.8. *Figure 4 illustrates a prime labeling for the graph $\overline{W_{4,5}}$.*

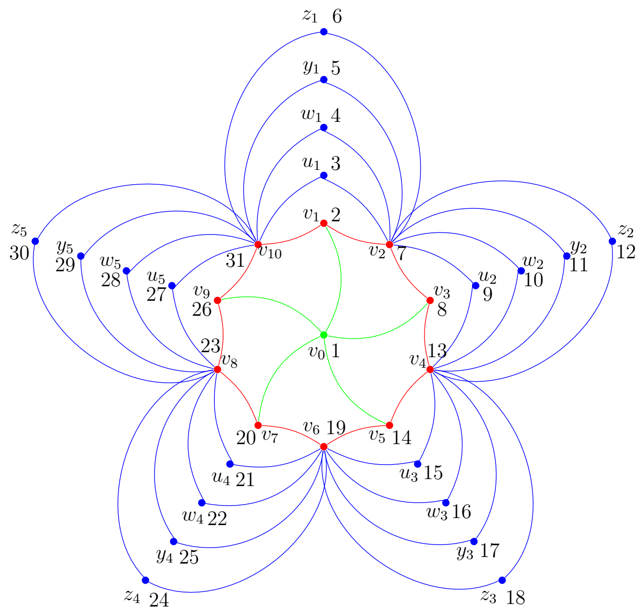

Figure 4: The graph $\overline{W_{4,5}}$ with prime labeling

3. Conclusion

This paper investigates the existence of prime labelings within generalized topological augmentations of the wheel graph W_n. By introducing edges incident to the subdivision vertices of the rim cycle C_n and internal vertices, we construct complex network topologies characterized by $k-$layered parallel paths, where $k \in \{1,2,3,4\}$. Despite the increased edge density and higher average vertex degrees in these configurations, we establish rigorously that these extended wheel graphs admit a valid prime labeling for all integers $n \geq 3$. Furthermore, this study underscores the inherent fragility of prime labeling in dense networks; modifying the graph by adding or omitting just one edge is sufficient to disrupt the labeling entirely. Moving forward, applying analogous extension operations to other graph families presents a promising avenue for discovering new labeling paradigms in network design.

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