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Product Cayley Graph of a Finite Group

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Abstract

In this paper, we introduce and study a new graph associated with a finite group, called the Product Cayley graph. Let G be a finite group and let S be a symmetric generating set of G. The Product Cayley graph $\Gamma_{pc}(G,S)$ is a graph with the vertex set G, and two distinct vertices x and y are adjacent if either $(xy)x^{-1} \in S$ or $(yx)y^{-1} \in S$. We investigate the fundamental structural properties of this graph, including connectedness, regularity, automorphisms, bipartiteness, and spectral properties. Special attention is devoted to finite abelian groups, where the structure of the graph is characterized in terms of the generating set. Furthermore, we study the non-abelian case by considering the dihedral groups D_n. We first classify symmetric generating sets of small cardinality and determine the conditions under which they generate the whole dihedral group. Based on these classifications, representative families of Product Cayley graphs of dihedral groups are investigated. Several graph invariants, including degree sequence, diameter, clique number, chromatic number, adjacency spectrum, and energy, are obtained.

Keywords: Cayley graph; Abelian group; Complete graph; Independent number; Laplacian matrix.
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1. Introduction

The interaction between group theory and graph theory has led to several graph structures that encode algebraic properties of groups. Among these, Cayley graphs play a fundamental role due to their rich algebraic structure and numerous applications in algebra, combinatorics, interconnection networks, coding theory, and theoretical computer science. Since their introduction by Cayley in 1878 [6], Cayley graphs have become one of the most important classes of graphs associated with finite groups.

The foundations of graph theory, spectral graph theory, and finite group theory have been established through several classical texts, including those by West [19], Diestel [8], Brouwer and Haemers [4], Gallian [9], Herstein [10], and Khattar and Agrawal [13]. These references provide the theoretical framework for studying algebraically defined graphs and their structural properties.

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In recent years, the study of Cayley graphs over various families of finite groups has gained importance. The structure of Cayley graphs of dihedral groups with small valencies has been investigated in [1,18]. Perfect state transfer on Cayley graphs over dihedral and dicyclic groups has been studied due to their applications in quantum information theory [2,5]. Integral Cayley graphs over generalized dihedral groups and other non-abelian groups have been investigated from algebraic and spectral perspectives [3,7,12,15,16]. The problems concerning automorphism groups, graph complexity, graph colouring, and normality of Cayley graphs have been studied extensively in recent years [11,14,17,20].

Motivated by this idea, we introduce a new graph linked to a finite group, called the Product Cayley graph. Let G be a finite group and let S be a symmetric generating set of G. The Product Cayley graph $\Gamma_{pc}(G,S)$ has vertex set G, and two distinct vertices x and y are adjacent if either $(xy)x^{-1} \in S$ or $(yx)y^{-1} \in S$. In the cayley graph the adjacency of the vertices x and y depends on x and y. But in the product cayley graph the adjacency of x and y depends on x, y and their products xy and yx.

The main objective of this paper is to investigate the structural properties of the Product Cayley graph. We first establish several general properties, including simplicity, graph isomorphisms induced by group isomorphisms, neighbourhood structure, completeness, and automorphism invariance. We then study Product Cayley graphs of finite abelian groups and prove that every such graph is isomorphic to the join $K_m \vee \overline{K}_{n-m}$, where $|G| = n$ and $|S| = m$. As consequences, we determine the degree sequence, number of edges, connectedness, diameter, clique number, independence number, chromatic number, domination number, vertex connectivity, edge connectivity, adjacency spectrum, Laplacian spectrum, and graph energy.

Finally, we investigate the non-abelian case by considering the dihedral groups. We classify symmetric generating sets of small cardinality and establish necessary and sufficient conditions under which they generate the entire dihedral group. Based on these classifications, representative families of Product Cayley graphs are studied along with their automorphism properties. These results demonstrate that the Product Cayley graph provides a new framework for studying finite groups from both algebraic and graph-theoretic viewpoints.

2. Preliminaries

Definition 2.1 ([13]). *Let G be a finite group with identity element e. A subset $S \subseteq G$ is called a generating set of G if every element of G can be expressed as a finite product of elements of S. In this case, we write $G = \langle S \rangle$. A generating set S is called symmetric if $S = S^{-1}$, where $S^{-1} = \{s^{-1} : s \in S\}$.*

Definition 2.2 ([13]). *The dihedral group of order $2n$, denoted by D_n, is the group generated by two elements r and s satisfying $D_n = \langle r, s \mid r^n = e,\ s^2 = e,\ srs = r^{-1} \rangle$. The elements of D_n are given by $D_n = \{e, r, r^2, \ldots, r^{n-1}, s, rs, r^2s, \ldots, r^{n-1}s\}$. The elements r^i are called rotations and the elements r^is are called reflections.*

Definition 2.3 ([19]). *A simple graph Γ consists of a non-empty set $V(\Gamma)$ of vertices together with a set $E(\Gamma)$*

of unordered pairs of distinct vertices called edges.

Definition 2.4 ([19]). *The degree of a vertex x in a graph Γ, denoted by $d(x)$, is the number of vertices adjacent to x.*

Definition 2.5 ([19]). *Let $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ be two vertex-disjoint graphs. The join of Γ_1 and Γ_2, denoted by $\Gamma_1 \vee \Gamma_2$, is the graph obtained from the union $\Gamma_1 \cup \Gamma_2$ by adding all possible edges between every vertex of Γ_1 and every vertex of Γ_2. That is, $V(\Gamma_1 \vee \Gamma_2) = V_1 \cup V_2$, and $E(\Gamma_1 \vee \Gamma_2) = E_1 \cup E_2 \cup \{uv : u \in V_1,\ v \in V_2\}$.*

Definition 2.6 ([19]). *Let $\Gamma = (V, E)$ be a simple graph. The complement of Γ, denoted by $\overline{\Gamma}$, is the graph with vertex set $V(\overline{\Gamma}) = V(\Gamma)$, and edge set $E(\overline{\Gamma}) = \{uv : u, v \in V(G),\ u \neq v,\ uv \notin E(\Gamma)\}$. i.e., two distinct vertices are adjacent in $\overline{\Gamma}$ if and only if they are not adjacent in Γ.*

Definition 2.7 ([19]). *Let $\Gamma = (V, E)$ be a graph. A clique is a subset $C \subseteq V(\Gamma)$ wherein adjacency holds for every pair of distinct vertices. The clique number of Γ, denoted by $\omega(\Gamma)$, is the maximum cardinality of a clique in Γ. That is, $\omega(\Gamma) = \max\{|C| : C \subseteq V(\Gamma)$ and $\Gamma[C]$ is a complete graph$\}$.*

Definition 2.8 ([19]). *Let $\Gamma = (V, E)$ be a graph. An independent set is a subset $I \subseteq V(\Gamma)$ such that no two distinct vertices in I are adjacent. The independence number of Γ, denoted by $\alpha(\Gamma)$, is the maximum cardinality of an independent set. That is, $\alpha(\Gamma) = \max\{|I| : I \subseteq V(\Gamma)$ and no two distinct vertices of I are adjacent$\}$.*

Definition 2.9 ([19]). *Let $\Gamma = (V, E)$ be a graph. A proper vertex coloring of Γ is an assignment of colors to the vertices such that any two adjacent vertices receive different colors. The chromatic number of Γ, denoted by $\chi(\Gamma)$, is the minimum number of colors required for a proper vertex coloring of Γ. That is, $\chi(\Gamma) = \min\{k :$ there exists a proper k-coloring of $\Gamma\}$.*

Definition 2.10 ([19]). *Let $\Gamma = (V, E)$ be a graph. A subset $D \subseteq V(\Gamma)$ is called a dominating set if every vertex in $V(\Gamma) \setminus D$ is adjacent to at least one vertex in D. Equivalently, for every vertex $v \in V(\Gamma) \setminus D$, there exists a vertex $u \in D$ such that $uv \in E$. The domination number of Γ, denoted by $\gamma(\Gamma)$, is the minimum cardinality of a dominating set. Thus, $\gamma(\Gamma) = \min\{|D| : D \subseteq V(\Gamma)$ is a dominating set$\}$.*

Definition 2.11 ([19]). *A graph Γ is connected if every pair of vertices is joined by a path. The distance between two vertices x and y is denoted by $d(x, y)$. The diameter and radius of Γ are defined by $\operatorname{diam}(\Gamma) = \max_{x,y \in V(\Gamma)} d(x, y)$ and $\operatorname{rad}(\Gamma) = \min_{y \in V(\Gamma)} \max_{x \in V(\Gamma)} d(x, y)$, respectively.*

Definition 2.12 ([4]). *Let Γ be a graph with adjacency matrix $A(\Gamma)$. The eigenvalues of $A(\Gamma)$ are called the adjacency spectrum of Γ. The Laplacian matrix of Γ is defined by $L(\Gamma) = D(\Gamma) - A(\Gamma)$, where $D(\Gamma)$ is the diagonal matrix of vertex degrees. The eigenvalues of $L(\Gamma)$ are called the Laplacian spectrum.*

Definition 2.13 ([6]). *Let G be a finite group and let S be a generating set of G such that $e \notin S$ and $S = S^{-1}$. The Cayley graph of G with respect to S, denoted by $\operatorname{Cay}(G, S)$, is the graph with vertex set G, where two distinct vertices x and y are adjacent if and only if $xy^{-1} \in S$.*

Definition 2.14 ([19]). *Let $\Gamma = (V, E)$ be a simple graph and let $x \in V$. The neighbourhood of x, denoted by $N(x)$, is the set of all vertices adjacent to x. That is, $N(x) = \{y \in V : xy \in E\}$.*

3. General Properties

Throughout this section, let G be a finite group and let $S \subseteq G$ be a symmetric generating set.

Definition 3.1. *Let G be a finite group and let $S \subseteq G$ be a symmetric generating set. The Product Cayley graph $\Gamma_{pc}(G, S)$ is the graph with $V(\Gamma_{pc}(G, S)) = G$ and two distinct vertices $x, y \in V(\Gamma_{pc}(G, S))$ are adjacent if either $(xy)x^{-1} \in S$ or $(yx)y^{-1} \in S$.*

Theorem 3.2. *The graph $\Gamma_{pc}(G, S)$ is a simple undirected graph.*

Proof. By definition, the vertex set of $\Gamma_{pc}(G, S)$ is G, and two distinct vertices $x, y \in G$ are adjacent whenever $(xy)x^{-1} \in S$ or $(yx)y^{-1} \in S$. Since adjacency is defined only for distinct vertices, no vertex is adjacent to itself. Hence, $\Gamma_{pc}(G, S)$ has no loops. Now suppose that x is adjacent to y. Then either $(xy)x^{-1} \in S$ or $(yx)y^{-1} \in S$. These two conditions are the requirements for the pair y, x in the definition of adjacency, namely $(yx)y^{-1} \in S$ or $(xy)x^{-1} \in S$. Thus, x is adjacent to $y \iff y$ is adjacent to x. Therefore, the adjacency is symmetric. Hence $\Gamma_{pc}(G, S)$ is a simple undirected graph. □

Theorem 3.3. *Let G_1 and G_2 be two groups such that $G_1 \cong G_2$. Let S be a symmetric generating set of G_1. Then $\Gamma_{pc}(G_1, S) \cong \Gamma_{pc}(G_2, f(S))$, where $f : G_1 \to G_2$ is an isomorphism.*

Proof. Let S be a symmetric generating set of G_1.
Claim 1. $f(S)$ is a generating set of G_2.
Let $y \in G_2$. Then there exists $x \in G_1$ such that $y = f(x)$. Since S generates G_1, without loss of generality we take $x = s_1 s_2 \cdots s_r$, where each $s_i \in S$. Hence $y = f(x) = f(s_1 s_2 \cdots s_r) = f(s_1) f(s_2) \cdots f(s_r)$, which shows that $f(S)$ generates G_2. Since $S = S^{-1}$, $f(S) = f(S^{-1}) = (f(S))^{-1}$, it follows that $f(S)$ is symmetric. This establishes Claim 1.

Claim 2. $\Gamma_{pc}(G_1, S) \cong \Gamma_{pc}(G_2, f(S))$.
Define $\phi : V(\Gamma_{pc}(G_1, S)) \longrightarrow V(\Gamma_{pc}(G_2, f(S)))$ by $\phi(x) = f(x)$. Since f is a bijection, ϕ is also a bijection. Let x, y be two distinct vertices of $\Gamma_{pc}(G_1, S)$. Then x and y are adjacent in $\Gamma_{pc}(G_1, S)$. This implies $(xy)x^{-1} \in S$ or $(yx)y^{-1} \in S$. Since f is a homomorphism, $f((xy)x^{-1}) = f(x)f(y)f(x)^{-1}$, or $f((yx)y^{-1}) = f(y)f(x)f(y)^{-1}$. This implies $\phi(x)\phi(y)\phi(x)^{-1} \in f(S)$, or $\phi(y)\phi(x)\phi(y)^{-1} \in f(S)$. Hence $\phi(x)$ and $\phi(y)$ are adjacent in $\Gamma_{pc}(G_2, f(S))$. So $\Gamma_{pc}(G_1, S) \cong \Gamma_{pc}(G_2, f(S))$. Hence Claim 2. □

Theorem 3.4. *Let $x \in Z(G)$ and let $y \in G$ with $x \neq y$. Then x is adjacent to y if and only if $x \in S$ or $y \in S$.*

Proof. Since $x \in Z(G)$, it commutes with every element of G. Hence $(xy)x^{-1} = y$ and $(yx)y^{-1} = x$. By the definition of the Product Cayley graph, x is adjacent to y if and only if $(xy)x^{-1} \in S$ or $(yx)y^{-1} \in S$. Therefore, x is adjacent to y if and only if $y \in S$ or $x \in S$. Hence, x is adjacent to y if and only if $x \in S$ or $y \in S$. □

Corollary 3.5. *The neighbourhood of the identity element is S or $G \setminus \{e\}$ according as $e \notin S$ or $e \in S$. Consequently,* $d(e) = \begin{cases} |S|, & \text{if } e \notin S \\ |G| - 1, & \text{if } e \in S. \end{cases}$.

Proof. Since $e \in Z(G)$, Theorem 3.4 implies that e is adjacent to y if and only if $e \in S$ or $y \in S$ for every $y \in G$ with $y \neq e$. If $e \notin S$, then e is adjacent to y if and only if $y \in S$, and hence $N(e) = S$. Therefore, $d(e) = |S|$. If $e \in S$, then e is adjacent to y for every $y \neq e$. Thus $N(e) = G \setminus \{e\}$, and consequently $d(e) = |G| - 1$. □

Theorem 3.6. *Let $x \in Z(G) \setminus S$. Then $N(x) = S$. Consequently, $d(x) = |S|$.*

Proof. Let $y \in G$ with $y \neq x$. Since $x \in Z(G)$, $(xy)x^{-1} = y$ and $(yx)y^{-1} = x$. Since $x \notin S$, we have $(yx)y^{-1} = x \notin S$. Hence, x is adjacent to y if and only if $(xy)x^{-1} \in S$ if and only if $y \in S$. Therefore, $N(x) = S$. Consequently, $d(x) = |S|$. □

Theorem 3.7. *Let $x \in Z(G) \cap S$. Then x is adjacent to every other vertex of $\Gamma_{pc}(G, S)$. Consequently, $d(x) = |G| - 1$.*

Proof. Let $y \in G$ with $y \neq x$. Since $x \in Z(G)$, $(yx)y^{-1} = x$. As $x \in S$, it follows that $(yx)y^{-1} \in S$. Therefore, by the definition of $\Gamma_{pc}(G, S)$, x is adjacent to y. Since the choice of y is arbitrary, every vertex of G other than x is adjacent to x. Hence $N(x) = G \setminus \{x\}$, and consequently, $d(x) = |G| - 1$. □

Theorem 3.8. *If $S \subseteq Z(G)$, then the subgraph induced by S is a complete graph.*

Proof. Let $x, y \in S$ with $x \neq y$. Since $S \subseteq Z(G)$, both x and y belong to the center of G. Hence, $(xy)x^{-1} = y$. As $y \in S$, it follows that $(xy)x^{-1} \in S$. Therefore, by the definition of $\Gamma_{pc}(G, S)$, x is adjacent to y. Since x and y are arbitrarily chosen distinct vertices of S, it follows that every pair of distinct vertices in S are adjacent. Hence the subgraph induced by S is a complete graph. □

Theorem 3.9. *Let S be a symmetric generating set such that $gsg^{-1} \in S$ for all $g \in G$, $s \in S$. Then, for any $x, y \in G$, $xyx^{-1} \in S$ if and only if $y \in S$.*

Proof. Suppose that $(xy)x^{-1} \in S$. Since S is a symmetric generating set such that $gsg^{-1} \in S$ for all $g \in G$, $s \in S$. Take $g = x^{-1}$, $s = (xy)x^{-1}$ $x^{-1}((xy)x^{-1})x = y \in S$. Conversely, if $y \in S$, since S is a symmetric generating set such that $gsg^{-1} \in S$ for all $g \in G$, $s \in S$, $xyx^{-1} \in S$. Hence, $xyx^{-1} \in S$ if and only if $y \in S$. □

Theorem 3.10. *Let G be a finite group, and let $S, T \subseteq G$ satisfy $S = S^{-1}$ and $T = T^{-1}$. If there exists a group automorphism $\phi \in \mathrm{Aut}(G)$ such that $\phi(S) = T$, then $\Gamma_{pc}(G, S) \cong \Gamma_{pc}(G, T)$.*

Proof. Define a map $f : V(\Gamma_{pc}(G, S)) \longrightarrow V(\Gamma_{pc}(G, T))$ by $f(x) = \phi(x)$, $x \in G$. Since ϕ is a group automorphism, f is a bijection. Now let $x, y \in G$ with $x \neq y$. Suppose that x is adjacent to y in $\Gamma_{pc}(G, S)$. By the definition of the product Cayley graph, we have $(xy)x^{-1} \in S$ or $(yx)y^{-1} \in S$. Since ϕ

is an automorphism of G, $\phi((xy)x^{-1}) = \phi(x)\phi(y)\phi(x)^{-1}$, and $\phi((yx)y^{-1}) = \phi(y)\phi(x)\phi(y)^{-1}$. Because $\phi(S) = T$, we obtain $\phi(x)\phi(y)\phi(x)^{-1} \in T$ or $\phi(y)\phi(x)\phi(y)^{-1} \in T$. Hence, $\phi(x)$ is adjacent to $\phi(y)$ in $\Gamma_{pc}(G,T)$. Therefore, x is adjacent to y implies $f(x)$ is adjacent to $f(y)$. Conversely, applying the same argument to the inverse automorphism ϕ^{-1} it follows that the adjacency of $f(x)$ to $f(y)$ implies the adjacency of x to y. Thus, x is adjacent to y if and only if $f(x)$ is adjacent to $f(y)$. Therefore, f preserves the property of adjacency. Since f is a bijection, it is a graph isomorphism. Hence, $\Gamma_{pc}(G,S) \cong \Gamma_{pc}(G,T)$. □

Theorem 3.11. *Let G be a finite group and let $S = G \setminus \{e\}$. Then the Product Cayley graph $\Gamma_{pc}(G,S)$ is the complete graph $K_{|G|}$.*

Proof. Let x and y be any two distinct vertices of $\Gamma_{pc}(G,S)$.

Case 1. $x = e$

In this case, $(ey)e^{-1} = y \in S$. Therefore x and y are adjacent.

Case 2. $x \neq e$ and $y \neq e$

In this case, if $(xy)x^{-1} = e$, then this $\implies (xy)x^{-1} = ex, \implies (xy)x^{-1} = x, \implies y = e$, a contradiction. Therefore $(xy)x^{-1} \neq e$. Hence x and y are adjacent. Therefore, $\Gamma_{pc}(G,S) \cong K_{|G|}$. □

4. Product Cayley Graphs of Finite Abelian Groups

Throughout this section, let G be a finite abelian group and let $S \subseteq G$ be a symmetric generating set.

Theorem 4.1. *Let $x, y \in G$ with $x \neq y$. Then x is adjacent to y if and only if $x \in S$ or $y \in S$.*

Proof. Since G is abelian, we have $xy = yx$. Therefore, $(xy)x^{-1} = y$ and $(yx)y^{-1} = x$. Hence, by the definition of the Product Cayley graph, x is adjacent to y if and only if $(xy)x^{-1} \in S$ or $(yx)y^{-1} \in S$, which is equivalent to $y \in S$ or $x \in S$. Therefore, x is adjacent to y if and only if $x \in S$ or $y \in S$. □

Theorem 4.2. *Let G be a finite abelian group with $|G| = n$ and $|S| = m$. Then $\Gamma_{pc}(G,S) \cong K_m \vee \overline{K}_{n-m}$.*

Proof. By Theorem 4.1, any two distinct vertices of S are adjacent. Also, every vertex of S is adjacent to every vertex of $G \setminus S$. Now let $x, y \in G \setminus S$ with $x \neq y$. Since $x, y \notin S$, Theorem 4.1 leads to the fact that x is not adjacent to y. Hence, the subgraph induced by S is K_m, while the subgraph induced by $G \setminus S$ is the null graph $\overline{K}_{n-m}$. Therefore, $\Gamma_{pc}(G,S) \cong K_m \vee \overline{K}_{n-m}$. □

Remark 4.3. *Let $G = \mathbb{Z}_4$ and let $S = \{1,3\}$. Then* $\mathrm{Cay}(\mathbb{Z}_4, \{1,3\}) \cong C_4$ *and* $\Gamma_{pc}(\mathbb{Z}_4, \{1,3\}) \cong K_2 \vee \overline{K}_2$.

Theorem 4.4. *Let G be a finite abelian group with $|G| = n$ and $|S| = m$. The degree sequence of $\Gamma_{pc}(G,S)$ is $(\underbrace{n-1, \ldots, n-1}_{m \text{ times}}, \underbrace{m, \ldots, m}_{n-m \text{ times}})$.*

Proof. By Theorem 4.2, $\Gamma_{pc}(G,S) \cong K_m \vee \overline{K}_{n-m}$. Every vertex of S is adjacent to all other vertices. Hence, $d(x) = n-1$, $x \in S$. Every vertex of $G \setminus S$ is adjacent only to vertices of S. Therefore, $d(y) = m$ and $y \in G \setminus S$. Hence the degree sequence follows. □

Theorem 4.5. *Let G be a finite abelian group with $|G| = n$ and $|S| = m$. Then* $|E(\Gamma_{pc}(G,S))| = \frac{m(2n-m-1)}{2}$.

Proof. From $\Gamma_{pc}(G,S) \cong K_m \vee \overline{K}_{n-m}$, the number of edges is $|E(\Gamma_{pc}(G,S))| = \binom{m}{2} + m(n-m)$. Hence, $|E(\Gamma_{pc}(G,S))| = \frac{m(m-1)}{2} + m(n-m)$, and therefore, $|E(\Gamma_{pc}(G,S))| = \frac{m(2n-m-1)}{2}$. □

Theorem 4.6. *Let G be a finite abelian group and let $S \neq \emptyset$. Then $\Gamma_{pc}(G,S)$ is connected.*

Proof. By Theorem 4.2, $\Gamma_{pc}(G,S) \cong K_m \vee \overline{K}_{n-m}$. Every vertex outside S is adjacent to every vertex of S. Since S is non-empty, any two vertices are connected through a vertex of S. Hence, $\Gamma_{pc}(G,S)$ is connected. □

Theorem 4.7. *Let G be a finite abelian group with $|G| = n$ and $|S| = m$. Then*

$$\operatorname{diam}(\Gamma_{pc}(G,S)) = \begin{cases} 1, & \text{if } n-m \leq 1, \\ 2, & \text{if } n-m \geq 2. \end{cases}$$

Proof. By Theorem 4.2, $\Gamma_{pc}(G,S) \cong K_m \vee \overline{K}_{n-m}$. First, suppose that $n-m \leq 1$. Then $G \setminus S$ contains at most one vertex. Hence there are no two distinct vertices in $G \setminus S$. Since the vertices of S induce a complete graph and every vertex of S is adjacent to every vertex of $G \setminus S$, adjacency holds for every pair of distinct vertices of $\Gamma_{pc}(G,S)$. Therefore, $\Gamma_{pc}(G,S) \cong K_n$, and consequently, $\operatorname{diam}(\Gamma_{pc}(G,S)) = 1$. Next suppose that $n-m \geq 2$. Then there exist distinct vertices $x, y \in G \setminus S$. Since the subgraph induced by $G \setminus S$ is the null graph, x and y are not adjacent. However, every vertex of $G \setminus S$ is adjacent to every vertex of S. Hence, for any $s \in S$, x is adjacent to s, s is adjacent to y, which implies that $d(x,y) = 2$. All other pairs of vertices are either adjacent or joined by a path of length 2. Therefore, $\operatorname{diam}(\Gamma_{pc}(G,S)) = 2$. □

Theorem 4.8. *Let G be a finite abelian group with $|G| = n$ and $|S| = m$. Then*

1. $\omega(\Gamma_{pc}(G,S)) = \begin{cases} n, & m = n, \\ m+1, & m < n, \end{cases}$

2. $\alpha(\Gamma_{pc}(G,S)) = \begin{cases} 1, & m = n, \\ n-m, & m < n, \end{cases}$

3. $\chi(\Gamma_{pc}(G,S)) = \begin{cases} n, & m = n, \\ m+1, & m < n. \end{cases}$

Proof. Since $\Gamma_{pc}(G,S) \cong K_m \vee \overline{K}_{n-m}$, we have to consider two cases.
If $m = n$, then $\Gamma_{pc}(G,S) \cong K_n$. Hence $\omega(\Gamma_{pc}(G,S)) = n$, $\alpha(\Gamma_{pc}(G,S)) = 1$, $\chi(\Gamma_{pc}(G,S)) = n$.
Now take up the case of $m < n$. The vertices of S induce a clique of order m, and every vertex of $G \setminus S$

is adjacent to every vertex of S. Since two vertices of $G \setminus S$ are not adjacent, every maximum clique consists of S together with exactly one vertex of $G \setminus S$. Hence, $\omega = m + 1$.

The set $G \setminus S$ is an independent set of order $n - m$, and no larger independent set exists. Therefore, $\alpha = n - m$. Finally, the clique of order $m + 1$ requires $m + 1$ colors, while all vertices of $G \setminus S$ may receive one common color. Cobsequently, $\chi = m + 1$. □

Theorem 4.9. *Let G be a finite abelian group with $|G| = n$ and $|S| = m$. Then*

1. $\gamma(\Gamma_{pc}(G, S)) = 1$

2. $\kappa(\Gamma_{pc}(G, S)) = \begin{cases} n-1, & m = n, \\ m, & m < n, \end{cases}$

3. $\lambda(\Gamma_{pc}(G, S)) = \begin{cases} n-1, & m = n, \\ m, & m < n. \end{cases}$

Proof. Every vertex of S is adjacent to every other vertex of the graph. Hence any vertex of S is a dominating vertex, and therefore $\gamma(\Gamma_{pc}(G, S)) = 1$. Two cases arise.

If $m = n$, then $\Gamma_{pc}(G, S) \cong K_n$, and it is well known that $\kappa(K_n) = \lambda(K_n) = n - 1$.

Next assume now that $m < n$. Removing all vertices of S disconnects the graph into an independent set. Hence, $\kappa(\Gamma_{pc}(G, S)) \leq m$. If fewer than m vertices are removed, at least one vertex of S remains. Every remaining vertex outside S is adjacent to that vertex, so the graph remains connected. Therefore, $\kappa(\Gamma_{pc}(G, S)) = m$.

Each vertex of $G \setminus S$ has degree m. Removing all m edges incident with such a vertex isolates it. Hence, $\lambda(\Gamma_{pc}(G, S)) \leq m$. Since the minimum degree equals m, $\delta(\Gamma_{pc}(G, S)) = m$, and every connected graph satisfies $\lambda(\Gamma) \leq \delta(\Gamma)$, it follows that $\lambda(\Gamma_{pc}(G, S)) = m$. □

Theorem 4.10. *Let G be a finite abelian group with $|G| = n$ and $|S| = m$. If $m = n$, then $\Gamma_{pc}(G, S) \cong K_n$, and* $\mathrm{Spec}(\Gamma_{pc}(G, S)) = \{(n-1), (-1)^{n-1}\}$, *with* $E(\Gamma_{pc}(G, S)) = 2(n - 1)$.
If $m < n$, then $\mathrm{Spec}(\Gamma_{pc}(G, S)) = \left\{ \frac{m-1+\sqrt{(m-1)^2+4m(n-m)}}{2}, \frac{m-1-\sqrt{(m-1)^2+4m(n-m)}}{2}, (-1)^{m-1}, 0^{n-m-1} \right\}$, *and* $E(\Gamma_{pc}(G, S)) = \sqrt{(m-1)^2 + 4m(n-m)} + (m - 1)$.

Proof. By Theorem 4.2, $\Gamma_{pc}(G, S) \cong K_m \vee \overline{K}_{n-m}$. First suppose that $m = n$. Then $\Gamma_{pc}(G, S) \cong K_n$. It is well known that the adjacency spectrum of K_n is $\{(n-1), (-1)^{n-1}\}$. Hence

$$E(\Gamma_{pc}(G, S)) = |(n-1)| + (n-1)|-1| = 2(n-1).$$

Next suppose that $m < n$. The adjacency matrix of $\Gamma_{pc}(G, S)$ is

$$A = \begin{pmatrix} J_{m\times m} - I_{m\times m} & J_{m\times(n-m)} \\ J_{(n-m)\times m} & 0 \end{pmatrix},$$

where $J_{i\times j}$ denotes the $i \times j$ matrix all of whose entries are 1 and $I_{i\times j}$ denotes the $i \times j$ identity matrix. The corresponding quotient matrix is

$$Q = \begin{pmatrix} m-1 & n-m \\ m & 0 \end{pmatrix}.$$

Its characteristic polynomial is $\lambda^2 - (m-1)\lambda - m(n-m) = 0$. From this, two eigenvalues are obtained as $\lambda_1 = \frac{m-1+\sqrt{(m-1)^2+4m(n-m)}}{2}$ and $\lambda_2 = \frac{m-1-\sqrt{(m-1)^2+4m(n-m)}}{2}$. The remaining eigenvalues of A are -1 with multiplicity $m-1$, and 0 with multiplicity $n-m-1$. Therefore,

$$\mathrm{Spec}(\Gamma_{pc}(G,S)) = \left\{\lambda_1, \lambda_2, (-1)^{m-1}, 0^{n-m-1}\right\}.$$

Since $\lambda_1 > 0$, $\lambda_2 < 0$, the energy equals $|\lambda_1| + |\lambda_2| + (m-1)$, which simplifies to

$$E(\Gamma_{pc}(G,S)) = \sqrt{(m-1)^2 + 4m(n-m)} + (m-1).$$

□

Theorem 4.11. *Let G be a finite abelian group with $|G| = n$, $|S| = m$. If $m = n$, then*

$$\mathrm{Spec}_L(\Gamma_{pc}(G,S)) = \{0, n^{n-1}\}.$$

If $m < n$, then

$$\mathrm{Spec}_L(\Gamma_{pc}(G,S)) = \left\{0, n, m^{n-m-1}, n^{m-1}\right\}.$$

Proof. If $m = n$, then $\Gamma_{pc}(G,S) \cong K_n$. The Laplacian spectrum of the complete graph K_n is $\{0, n^{n-1}\}$. Now suppose that $m < n$. By Theorem 4.2, $\Gamma_{pc}(G,S) \cong K_m \vee \overline{K}_{n-m}$. The Laplacian spectrum of K_m is $\{0, m^{m-1}\}$, and the Laplacian spectrum of $\overline{K}_{n-m}$ is $\{0^{n-m}\}$. By the Laplacian spectrum formula for the join of two graphs, the non-zero Laplacian eigenvalues of K_m increase by $n-m$, while those of $\overline{K}_{n-m}$ increase by m. Hence we obtain n with multiplicity $m-1$, and m with multiplicity $n-m-1$. The remaining two eigenvalues are 0 and n. Therefore,

$$\mathrm{Spec}_L(\Gamma_{pc}(G,S)) = \left\{0, n, m^{n-m-1}, n^{m-1}\right\}.$$

□

5. Product Cayley Graphs of Dihedral Groups

In this section, we investigate Product Cayley graphs of dihedral groups.

Lemma 5.1. *Let D_n be the dihedral group of order $2n$. Then the following results hold for every integer i: $r^i r r^{-i} = r$, $r^i s r^{-i} = r^{2i}s$, $(r^i s)r(r^i s)^{-1} = r^{-1}$, and $(r^i s)s(r^i s)^{-1} = r^{2i}s$.*

Proof. First, since the rotations form a cyclic subgroup of D_n, we have $r^i r r^{-i} = r^{i+1-i} = r$. Next, using the relation $srs = r^{-1}$, we obtain $sr = r^{-1}s$. Replacing r by r^{-1} gives $sr^{-1} = rs$. Hence, for every integer i, $sr^{-i} = r^i s$. Therefore, $r^i s r^{-i} = r^i(sr^{-i}) = r^i(r^i s) = r^{2i}s$. Now consider a reflection element $r^i s$. Since every reflection has order two, $(r^i s)^{-1} = r^i s$. Therefore, $(r^i s)r(r^i s)^{-1} = r^i s r r^i s$. Using $sr = r^{-1}s$, we get $r^i s r r^i s = r^i r^{-1} s r^i s = r^{i-1}(sr^i)s$. Since $sr^i = r^{-i}s$, we obtain $r^{i-1}r^{-i}ss = r^{-1}$. Hence, $(r^i s)r(r^i s)^{-1} = r^{-1}$. Finally, $(r^i s)s(r^i s)^{-1} = r^i s s r^i s$. Because $s^2 = e$, $r^i s s r^i s = r^i r^i s = r^{2i}s$. Thus, all the stated relations hold. □

Lemma 5.2. *Let D_n be the dihedral group of order $2n$ and let $R = \{r^{i_1}, r^{i_2}, \ldots, r^{i_k}\} \subseteq \langle r \rangle$. Then $\langle R \rangle = \langle r^d \rangle$, where $d = (n, i_1, i_2, \ldots, i_k)$. In particular, $\langle R \rangle = \langle r \rangle$ if and only if $(n, i_1, i_2, \ldots, i_k) = 1$.*

Proof. Since $\langle r \rangle$ is a cyclic group of order n, each of its subgroups is cyclic. It is a standard result on cyclic groups that the subgroup generated by $r^{i_1}, r^{i_2}, \ldots, r^{i_k}$ is precisely $\langle r^d \rangle$, where $d = (n, i_1, i_2, \ldots, i_k)$. Therefore, $\langle R \rangle = \langle r \rangle$ if and only if $d = 1$. □

Lemma 5.3. *In the dihedral group D_n of order $2n$, suppose that $\langle r \rangle \leq \langle S \rangle$ and that S contains a reflection. Then $\langle S \rangle = D_n$.*

Proof. Let $r^j s \in S$ for some integer j. Since $\langle r \rangle \leq \langle S \rangle$, we have $r^{-j} \in \langle S \rangle$. Hence, $r^{-j}(r^j s) = s \in \langle S \rangle$. Thus, $r, s \in \langle S \rangle$. Since $D_n = \langle r, s \rangle$, it follows that $\langle S \rangle = D_n$. □

Theorem 5.4. *Let $S \subseteq D_n$ with $S = S^{-1}$. Then S generates D_n if and only if*

1. *S contains at least one reflection, and*
2. *if $S \cap \langle r \rangle = \{r^{i_1}, r^{-i_1}, \ldots, r^{i_k}, r^{-i_k}\}$, then $(n, i_1, \ldots, i_k) = 1$.*

Proof. Suppose that S generates D_n. If S contains no reflection, then $S \subseteq \langle r \rangle$, which implies $\langle S \rangle \subseteq \langle r \rangle \neq D_n$, a contradiction. Hence S contains at least one reflection. By Lemma 5.2, $\langle S \cap \langle r \rangle \rangle = \langle r^d \rangle$, where $d = (n, i_1, \ldots, i_k)$. Since the rotational part must generate $\langle r \rangle$, we obtain $d = 1$. Conversely, suppose that the two conditions hold. By Lemma 5.2, $\langle S \cap \langle r \rangle \rangle = \langle r \rangle$. Since S also contains a reflection, Lemma 5.3 implies that $\langle S \rangle = D_n$. Hence S generates D_n. □

Theorem 5.5. *Let $S \subseteq D_n$ with $|S| = 2$ and $S = S^{-1}$. Then S generates D_n if and only if $S = \{r^i s,\ r^j s\}$, where $gcd(n, i - j) = 1$.*

Proof. Let $S = \{x, y\}$ be a symmetric subset of D_{2n} with $|S| = 2$. Since $S = S^{-1}$, we consider all possible forms of S.

Case 1. Suppose $x = r^i$, $y = r^{-i}$. Since $(r^i)^{-1} = r^{-i}$, the set is symmetric. Moreover, $\langle S \rangle = \langle r^i \rangle \leq \langle r \rangle$. Hence $\langle S \rangle$ contains only rotations and therefore cannot be equal to D_n. Thus this case never yields a generating set of D_n.

Case 2. Suppose $x = r^i$, $y = r^j s$. Since $(r^i)^{-1} = r^{-i}$, symmetry requires $r^{-i} \in S$. As S has only two elements, this is possible only if $r^{-i} = r^i$, that is, $2i \equiv 0 \pmod{n}$. Hence r^i has order at most 2, and therefore $\langle r^i \rangle \neq \langle r \rangle$. Consequently, the rotational part of S does not generate the rotation subgroup. By Theorem 5.4, S cannot generate D_n.

Case 3. Suppose $x = r^i s$, $y = r^j s$. Since every reflection is an involution, $(r^k s)^{-1} = r^k s$, the set S is symmetric. Furthermore, $(r^i s)(r^j s) = r^{i-j}$. Hence $\langle S \rangle = \langle r^{i-j}, r^i s \rangle$. By Lemma 5.2, $\langle r^{i-j} \rangle = \langle r \rangle \iff \gcd(n, i-j) = 1$. Since S contains a reflection, Lemma 5.3 implies that $\langle S \rangle = D_n$. Conversely, if $gcd(n, i-j) \neq 1$, then $\langle r^{i-j} \rangle \neq \langle r \rangle$, and Theorem 5.4 implies that S cannot generate D_n. Therefore, S generates $D_n \iff S = \{r^i s,\ r^j s\}$ with gcd $(n, i-j) = 1$. This completes the proof. □

Theorem 5.6. *Let D_n be the dihedral group of order $2n$. Then the number of symmetric generating sets of cardinality 2 is $\frac{n\varphi(n)}{2}$, where φ denotes Euler's totient function.*

Proof. By Theorem 5.5, every symmetric generating set of cardinality 2 has the form $S = \{r^i s, r^j s\}$, where $gcd(n, i-j) = 1$. Choose the first reflection $r^i s$. There are exactly n choices for i. For a fixed i, let $d \equiv j - i \pmod{n}$. Then $j = i + d \pmod{n}$. The condition $gcd(n, i-j) = 1$ is equivalent to gcd $(n, d) = 1$. There are precisely $\varphi(n)$ integers $d \in \{1, 2, \ldots, n-1\}$ satisfying this condition. Hence there are $n\varphi(n)$ ordered pairs (i, j) satisfying the generating condition. Since $\{r^i s, r^j s\} = \{r^j s, r^i s\}$, every generating set is counted exactly twice. Therefore the number of distinct symmetric generating sets of cardinality 2 is $\frac{n\varphi(n)}{2}$. □

Corollary 5.7. *Let $S = \{r^i s, r^j s\}$ and $T = \{r^{i+a} s, r^{j+a} s\}$, where $gcd(n, i-j) = 1$. Then $\Gamma_{pc}(D_n, S) \cong \Gamma_{pc}(D_n, T)$.*

Proof. Define the automorphism ϕ of D_n by $\phi(r) = r$, $\phi(s) = r^a s$. Then $\phi(S) = T$. Hence, by Theorem 3.10, $\Gamma_{pc}(D_n, S) \cong \Gamma_{pc}(D_n, T)$. □

Theorem 5.8. *Let D_n be the dihedral group of order $2n$ and let $S = \{s, r^d s\}$, where $\gcd(d, n) = 1$. Then the adjacency relation in $\Gamma_{pc}(D_n, S)$ is characterized as follows.*

(i) Two rotations are never adjacent. i.e., r^a is not adjacent to r^b, for all $0 \leq a, b < n$.

(ii) A rotation r^a is adjacent to a reflection $r^b s$ if and only if $2a + b \equiv 0 \pmod{n}$ or $2a + b \equiv d \pmod{n}$.

(iii) Two reflections $r^a s$ and $r^b s$ are adjacent if and only if $2a - b \equiv 0, d \pmod{n}$ or $2b - a \equiv 0, d \pmod{n}$.

Proof. Let $x, y \in D_n$.

(i) Let $x = r^a$, $y = r^b$. Then $xyx^{-1} = r^a r^b r^{-a} = r^b$, and $yxy^{-1} = r^b r^a r^{-b} = r^a$. Since $S = \{s, r^d s\}$ contains only reflections, neither r^a nor $r^b \in S$. Hence $r^a \nsim r^b$.

(ii) Let $x = r^a$, $y = r^b s$. Then

$$\begin{aligned} xyx^{-1} &= r^a(r^b s)r^{-a} \\ &= r^{a+b} s r^{-a} \\ &= r^{a+b} r^a s \\ &= r^{2a+b} s. \end{aligned}$$

Also,

$$\begin{aligned} (yx)y^{-1} &= (r^b s)r^a (r^b s)^{-1} \\ &= r^b s r^a s r^{-b} \\ &= r^{-a}. \end{aligned}$$

Since $r^{-a} \notin S$, it follows that $r^a \sim r^b s$ if and only if $r^{2a+b}s \in S$, that is, $2a + b \equiv 0 \pmod{n}$ or $2a + b \equiv d \pmod{n}$.

(iii) Let $x = r^a s$, $y = r^b s$. Then

$$\begin{aligned} (xy)x^{-1} &= (r^a s)(r^b s)(r^a s) \\ &= r^{a-b}(r^a s) \\ &= r^{2a-b} s. \end{aligned}$$

Similarly,

$$\begin{aligned} (yx)y^{-1} &= (r^b s)(r^a s)(r^b s) \\ &= r^{2b-a} s. \end{aligned}$$

Hence $r^a s \sim r^b s$ if and only if $r^{2a-b}s \in S$ or $r^{2b-a}s \in S$. Equivalently, $2a - b \equiv 0, d \pmod{n}$, or $2b - a \equiv 0, d \pmod{n}$. This completes the proof. □

Theorem 5.9. *Let $S = \{s, r^d s\}$, where $gcd(d, n) = 1$. Then every rotation vertex of $\Gamma_{pc}(D_n, S)$ has degree 2. i.e., $d(r^a) = 2$, $0 \leq a < n$.*

Proof. Let r^a be a rotation vertex. By Theorem 5.8, r^a is adjacent to a reflection $r^b s$ if and only if $2a + b \equiv 0 \pmod{n}$ or $2a + b \equiv d \pmod{n}$. These congruences have the unique solutions $b \equiv -2a \pmod{n}$ and $b \equiv d - 2a \pmod{n}$, respectively. Since $d \not\equiv 0 \pmod{n}$, these two solutions are distinct. Hence r^a is adjacent to exactly two reflection vertices. Therefore, $d(r^a) = 2$. □

6. Conclusion

In this paper, we have introduced the Product Cayley graph, a new graph associated with a finite group and a symmetric generating set. The proposed graph extends the classical Cayley graph by defining adjacency in a graph through the use of multiplication, thereby incorporating additional algebraic information, particularly in the non-abelian setting.

First we have established several fundamental properties of the Product Cayley graph, including simplicity, graph isomorphisms induced by group isomorphisms, neighbourhood structure, completeness, and automorphism invariance. For finite abelian groups, we have completely characterized the graph by proving that $\Gamma_{pc}(G, S) \cong K_m \vee \overline{K}_{n-m}$, where $|G| = n$ and $|S| = m$. This characterization has enabled us to obtain explicit expressions for several graph-theoretic parameters, including the degree sequence, number of edges, connectedness, diameter, clique number, independence number, chromatic number, domination number, vertex connectivity, edge connectivity, adjacency spectrum, Laplacian spectrum, and graph energy.

We have further investigated the Product Cayley graphs of dihedral groups by characterizing symmetric generating sets, enumerating symmetric generating sets of cardinality 2, and identifying isomorphic families of Product Cayley graphs.

The Product Cayley graph introduced in this paper provides a new framework for associating graphs with finite groups through the use of multiplication. One can expect that this construction will stimulate further research concerning the Product Cayley graphs of other families of finite groups, such as symmetric groups, generalized quaternion groups, semi-dihedral groups, and finite matrix groups, as well as their structural, spectral, and topological properties.

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