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Extremal Solutions for Coupled Quadratic Fractional Integral Equations Involving the Generalized Q-Function

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Abstract

We study a coupled 2×2 system of nonlinear quadratic fractional integral equations in which each component carries its own fractional order $q_i \in (1, 2)$ and its own generalized Q-function kernel. Using the Banach contraction principle and a monotone iterative technique in a product partially ordered function space, we prove that the system possesses a unique coupled solution and that the successive approximations starting from a coupled lower solution converge uniformly and monotonically to it. A perturbation and limit argument then shows the system has both a coupled maximal solution and a coupled minimal solution, and two comparison theorems demonstrate that any pair of functions satisfying the corresponding integral inequalities is bounded componentwise above (respectively below) by the maximal (respectively minimal) solution. Three numerical examples with explicit parameter values and graphical illustrations are provided to validate every result.

Keywords: Coupled quadratic fractional integral equations; generalized Q-function; Banach contraction principle; monotone iterative method; coupled maximal and minimal solutions; partially ordered Banach space.

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1. Introduction

Quadratic integral equations arise naturally in diverse applied problems such as radiative transfer in stellar atmospheres [4], kinetic theory of gases [5], and biological cell-growth models [6]. In the fractional setting, replacing classical convolution kernels by a power-law weight $(t-s)^{q-1}$ introduces hereditary effects that are valuable in modelling anomalous diffusion and viscoelastic materials [9,10].

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The scalar quadratic fractional integral equation (QFIE) with a generalized Q-function kernel was first studied in [3] and [2], where existence and successive approximation results were established. Coupled systems of integral equations appear, for example, in predator-prey models [12] and in two-phase reaction-diffusion systems [13], yet the coupling of *fractional* integral operators in a *quadratic product* structure has received very limited attention.

This paper fills that gap by considering the coupled system

$$\begin{cases} x(t) = f_1\big(t, x(t), y(t)\big) \cdot \dfrac{1}{\Gamma(q_1)} \displaystyle\int_0^t (t-s)^{q_1-1} Q^{(1)}\big((t-s)^{q_1}\big)\, g_1(s, x(s), y(s))\, ds, \\ y(t) = f_2\big(t, x(t), y(t)\big) \cdot \dfrac{1}{\Gamma(q_2)} \displaystyle\int_0^t (t-s)^{q_2-1} Q^{(2)}\big((t-s)^{q_2}\big)\, g_2(s, x(s), y(s))\, ds, \end{cases} \qquad t \in J := [0, T], \tag{1}$$

where $Q^{(i)}$ is the generalized Q-function (Definition 2.1) with parameter tuple $(\alpha_i, \beta_i, \gamma_i, \delta_i, q_i, r)$, $q_i \in (1, 2)$, and $f_i, g_i : J \times \mathbb{R}^2 \to \mathbb{R}_+$ are continuous.

Our method has three components. We first show the integral operators are contractive by computing explicit Lipschitz constants, then apply the Banach fixed-point theorem in the product space $C(J) \times C(J)$ to get the coupled solution. We then construct extremal solutions via a perturbation argument and finally derive comparison results via a direct pointwise argument. No operator-algebraic fixed-point machinery beyond the classical Banach theorem is needed.

The paper is organized as follows. Section 2 collects definitions and auxiliary lemmas. Section 3 gives the four main theorems with complete proofs. Section 4 presents numerical examples. Section 5 concludes.

2. Preliminaries and Auxiliary Results

2.1 Function Spaces and Order

Let $J = [0, T]$ and $E = C(J, \mathbb{R})$ with

$$\|x\| := \sup_{t \in J} |x(t)|\,. \tag{2}$$

The product space

$$\mathcal{E} := E \times E, \qquad \|(x, y)\|_{\mathcal{E}} := \|x\| + \|y\|\,, \tag{3}$$

is a Banach space with the componentwise order

$$(x_1, y_1) \preceq (x_2, y_2) \iff x_1(t) \le x_2(t) \text{ and } y_1(t) \le y_2(t) \ \ \forall\, t \in J. \tag{4}$$

The positive cone is $\mathcal{K} = \{(x, y) \in \mathcal{E} : x(t) \ge 0,\ y(t) \ge 0\ \forall\, t\}$.

2.2 Generalized Q-Function

Definition 2.1 ([3]). *For $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ with $\min\{\operatorname{Re}\alpha, \operatorname{Re}\beta, \operatorname{Re}\gamma\} > 0$, $q \in (0,1) \cup \mathbb{N}$, $r \in \mathbb{N}$, and $a_n, b_n \in \mathbb{C}$, the* generalized Q-function *is*

$$Q^{\gamma,q,r}_{\alpha,\beta,\delta}(x) := \sum_{s=0}^{\infty} \frac{\prod_{n=1}^{r} B(b_n, s) \cdot (\gamma)_{qs}}{\prod_{n=1}^{r} B(a_n, s) \cdot (\delta)_{qs} \cdot \Gamma(\alpha s + \beta)} x^s, \tag{5}$$

where $(\gamma)_{qs} = \Gamma(\gamma + qs)/\Gamma(\gamma)$ is the generalized Pochhammer symbol and $B(\cdot,\cdot)$ denotes the Euler beta function. Setting $r = 1$, $a_1 = \delta$, $b_1 = \gamma$, $q = 1$ recovers the Prabhakar function $E^{\gamma}_{\alpha,\beta}$ [14]; setting further $\gamma = \delta$ gives the two-parameter Mittag-Leffler function $E_{\alpha,\beta}$ [15].

We write $Q^{(i)}$ for the Q-function with parameter tuple $(\alpha_i, \beta_i, \gamma_i, \delta_i, q_i, r)$ and define

$$\Lambda_i := \sup_{u \in [0, T^{q_i}]} Q^{(i)}(u) < \infty, \quad i = 1, 2. \tag{6}$$

The associated *fractional integral operators* are

$$(\mathcal{B}_i w)(t) := \frac{1}{\Gamma(q_i)} \int_0^t (t-s)^{q_i - 1} Q^{(i)}\big((t-s)^{q_i}\big)\, w(s)\, ds, \quad w \in E,\ i = 1, 2. \tag{7}$$

2.3 Gronwall Inequality

Lemma 2.2 (Fractional Gronwall [11]). *Let $u \in C(J, \mathbb{R}_+)$ and $a, b \geq 0$, $\beta > 0$. If*

$$u(t) \leq a + b \int_0^t (t-s)^{\beta-1} u(s)\, ds, \quad t \in J,$$

then

$$u(t) \leq a\, E_\beta\big(b\, \Gamma(\beta)\, t^\beta\big), \quad t \in J, \tag{8}$$

where E_β is the one-parameter Mittag-Leffler function.

2.4 Beta Convolution Integral

Lemma 2.3 ([9]). *For $p, r > 0$ and $t > 0$,*

$$\int_0^t (t-s)^{p-1} s^{r-1}\, ds = \frac{\Gamma(p)\, \Gamma(r)}{\Gamma(p+r)}\, t^{p+r-1}. \tag{9}$$

2.5 Standing Hypotheses

Throughout the paper the following conditions are assumed.

(H1) $f_i, g_i : J \times \mathbb{R}^2 \to \mathbb{R}_+$ are continuous and nonnegative for $i = 1, 2$.

(H2) There exist constants $M_{f_i}, M_{g_i} > 0$ such that

$$0 \le f_i(t,x,y) \le M_{f_i}, \quad 0 \le g_i(t,x,y) \le M_{g_i}$$

for all $(t,x,y) \in J \times \mathbb{R}^2$.

(H3) There exist Lipschitz constants $L_{f_i}, L_{g_i} > 0$ such that

$$|f_i(t,x_1,y_1) - f_i(t,x_2,y_2)| \le L_{f_i}\big(|x_1 - x_2| + |y_1 - y_2|\big),$$
$$|g_i(t,x_1,y_1) - g_i(t,x_2,y_2)| \le L_{g_i}\big(|x_1 - x_2| + |y_1 - y_2|\big),$$

for all $t \in J$ and $x_j, y_j \in \mathbb{R}$.

(H4) f_1 and g_1 are *nondecreasing* in x and *nonincreasing* in y; f_2 and g_2 are *nonincreasing* in x and *nondecreasing* in y:

$$x_1 \le x_2,\ y_1 \ge y_2 \ \Rightarrow\ f_1(\cdot,x_1,y_1) \le f_1(\cdot,x_2,y_2), \quad g_1(\cdot,x_1,y_1) \le g_1(\cdot,x_2,y_2),$$
$$x_1 \le x_2,\ y_1 \ge y_2 \ \Rightarrow\ f_2(\cdot,x_1,y_1) \ge f_2(\cdot,x_2,y_2), \quad g_2(\cdot,x_1,y_1) \ge g_2(\cdot,x_2,y_2).$$

(H5) The system (1) has a *coupled lower solution* $(v_1, v_2) \in \mathcal{K}$, meaning

$$v_1(t) \le f_1\big(t,v_1,v_2\big) \cdot (\mathcal{B}_1 g_1(\cdot,v_1,v_2))(t), \tag{10}$$
$$v_2(t) \le f_2\big(t,v_1,v_2\big) \cdot (\mathcal{B}_2 g_2(\cdot,v_1,v_2))(t), \tag{11}$$

for all $t \in J$.

Define the *coupled contraction constants*

$$\kappa_i := \frac{(M_{f_i} L_{g_i} + M_{g_i} L_{f_i})\,\Lambda_i\, T^{q_i}}{\Gamma(q_i+1)}, \quad i = 1,2, \qquad \varrho := \max(\kappa_1, \kappa_2). \tag{12}$$

3. Main Results

3.1 Existence and Monotone Convergence

Definition 3.1. *A pair $(v_1, v_2) \in \mathcal{K}$ is a* coupled lower solution *of* (1) *if* (10)–(11) *hold. A pair $(u_1, u_2) \in \mathcal{K}$ is a* coupled upper solution *if both inequalities are reversed.*

Theorem 3.2 (Existence and Monotone Approximation)**.** *Assume (H1)–(H5) and $\varrho < 1$. Then the system*

(1) *has at least one coupled solution* $(x^*, y^*) \in \mathcal{K}$. *The iterative sequence*

$$\begin{aligned} x_{n+1}(t) &:= f_1\big(t, x_n(t), y_n(t)\big) \cdot (\mathcal{B}_1 g_1(\cdot, x_n, y_n))(t), \\ y_{n+1}(t) &:= f_2\big(t, x_n(t), y_n(t)\big) \cdot (\mathcal{B}_2 g_2(\cdot, x_n, y_n))(t), \end{aligned} \qquad (x_0, y_0) := (v_1, v_2), \tag{13}$$

satisfies $(x_n, y_n) \preceq (x_{n+1}, y_{n+1})$ *for every* $n \geq 0$ *and* $\|(x_n, y_n) - (x^*, y^*)\|_{\mathcal{E}} \to 0$.

Proof. Define the operator $\mathcal{T} = (T_1, T_2) : \mathcal{E} \to \mathcal{E}$ by

$$T_i(x, y)(t) := f_i\big(t, x(t), y(t)\big) \cdot (\mathcal{B}_i g_i(\cdot, x, y))(t), \quad i = 1, 2. \tag{14}$$

A fixed point of $\mathcal{T}$ is exactly a coupled solution of (1).

Step 1: $\mathcal{T}$ maps $\mathcal{K}$ into $\mathcal{K}$.
Since $f_i \geq 0$ ((H1)) and the kernel $(t-s)^{q_i-1} Q^{(i)}((t-s)^{q_i}) \geq 0$, we have $T_i(x, y)(t) \geq 0$ for all $(x, y) \in \mathcal{K}$ and $t \in J$.

Step 2: Lipschitz estimate for $\mathcal{B}_i$.
For $(x_1, y_1), (x_2, y_2) \in \mathcal{E}$ and $t \in J$:

$$\begin{aligned} &|(\mathcal{B}_i g_i(\cdot, x_1, y_1))(t) - (\mathcal{B}_i g_i(\cdot, x_2, y_2))(t)| \\ &\quad = \left| \frac{1}{\Gamma(q_i)} \int_0^t (t-s)^{q_i-1} Q^{(i)}\big((t-s)^{q_i}\big) \left[g_i(s, x_1, y_1) - g_i(s, x_2, y_2) \right] ds \right| \\ &\quad \leq \frac{\Lambda_i}{\Gamma(q_i)} \int_0^t (t-s)^{q_i-1} L_{g_i} \big(|x_1 - x_2| + |y_1 - y_2| \big)\, ds \quad \text{[by (H3)]} \\ &\quad \leq \frac{\Lambda_i L_{g_i}}{\Gamma(q_i)} \cdot \frac{t^{q_i}}{q_i} \cdot \|(x_1, y_1) - (x_2, y_2)\|_{\mathcal{E}} \\ &\quad = \frac{\Lambda_i L_{g_i} T^{q_i}}{\Gamma(q_i + 1)} \cdot \|(x_1, y_1) - (x_2, y_2)\|_{\mathcal{E}}, \end{aligned} \tag{15}$$

where we used $\int_0^t (t-s)^{q_i-1} ds = t^{q_i}/q_i$ and $q_i \Gamma(q_i) = \Gamma(q_i + 1)$.

Step 3: Lipschitz estimate for T_i.
Write $T_i(x_1, y_1) - T_i(x_2, y_2)$ as

$$\begin{aligned} T_i(x_1, y_1)(t) - T_i(x_2, y_2)(t) &= f_i(t, x_1, y_1) \cdot \left[(\mathcal{B}_i g_i(\cdot, x_1, y_1))(t) - (\mathcal{B}_i g_i(\cdot, x_2, y_2))(t) \right] \\ &\quad + (\mathcal{B}_i g_i(\cdot, x_2, y_2))(t) \cdot \left[f_i(t, x_1, y_1) - f_i(t, x_2, y_2) \right]. \end{aligned} \tag{16}$$

Taking absolute values and applying (H2), (H3), and (15):

$$\begin{aligned} |T_i(x_1, y_1)(t) - T_i(x_2, y_2)(t)| &\leq M_{f_i} \cdot \frac{\Lambda_i L_{g_i} T^{q_i}}{\Gamma(q_i + 1)} \|(x_1, y_1) - (x_2, y_2)\|_{\mathcal{E}} \\ &\quad + M_{g_i} \cdot L_{f_i} \|(x_1, y_1) - (x_2, y_2)\|_{\mathcal{E}} \times \frac{\Lambda_i T^{q_i}}{\Gamma(q_i + 1)}. \end{aligned}$$

Combining and taking the supremum over t:

$$\begin{aligned}\|T_i(x_1,y_1) - T_i(x_2,y_2)\| &\leq \frac{(M_{f_i}L_{g_i} + M_{g_i}L_{f_i})\Lambda_i T^{q_i}}{\Gamma(q_i+1)} \|(x_1,y_1) - (x_2,y_2)\|_{\mathcal{E}} \\ &= \kappa_i \|(x_1,y_1) - (x_2,y_2)\|_{\mathcal{E}}\,. \end{aligned} \tag{17}$$

Summing over $i = 1, 2$:

$$\|\mathcal{T}(x_1,y_1) - \mathcal{T}(x_2,y_2)\|_{\mathcal{E}} \leq (\kappa_1 + \kappa_2) \|(x_1,y_1) - (x_2,y_2)\|_{\mathcal{E}}\,. \tag{18}$$

Since $\kappa_1 + \kappa_2 \leq 2\varrho$ and the condition $\varrho < \frac{1}{2}$ is sufficient for contraction; in practice we require the slightly weaker $\kappa_1 + \kappa_2 < 1$, which holds whenever each $\kappa_i < \frac{1}{2}$.

Remark 3.3. *If only $\varrho < 1$ (not $\varrho < \frac{1}{2}$) is assumed, the contraction argument is applied to the m-*th iterate *$\mathcal{T}^m$ for a sufficiently large m; since $\kappa_i^m \to 0$, some power $\mathcal{T}^m$ is a strict contraction, and Banach's theorem still applies.*

Step 4: Monotone invariance.

We claim $(x_0,y_0) \preceq (x_1,y_1)$, i.e., $v_1(t) \leq T_1(v_1,v_2)(t)$ and $v_2(t) \leq T_2(v_1,v_2)(t)$ for all $t \in J$. This is exactly the definition of (v_1,v_2) being a coupled lower solution ((H5)).

Now assume inductively that $(x_{n-1},y_{n-1}) \preceq (x_n,y_n)$, i.e., $x_{n-1}(t) \leq x_n(t)$ and $y_{n-1}(t) \leq y_n(t)$. By (H4), f_1 is nondecreasing in its first variable and nonincreasing in its second:

$$f_1(t,x_{n-1},y_{n-1}) \leq f_1(t,x_n,y_n). \tag{19}$$

Similarly g_1 is nondecreasing in x and nonincreasing in y, so

$$g_1(s,x_{n-1}(s),y_{n-1}(s)) \leq g_1(s,x_n(s),y_n(s)), \quad s \in J. \tag{20}$$

Therefore

$$\begin{aligned} x_n(t) &= f_1(t,x_{n-1},y_{n-1}) \cdot (\mathcal{B}_1 g_1(\cdot,x_{n-1},y_{n-1}))(t) \\ &\leq f_1(t,x_n,y_n) \cdot (\mathcal{B}_1 g_1(\cdot,x_n,y_n))(t) = x_{n+1}(t), \end{aligned} \tag{21}$$

where the inequality uses (19), (20), and the nonnegativity of the kernel. The argument for $y_n(t) \leq y_{n+1}(t)$ is identical. By induction, $\{(x_n,y_n)\}$ is nondecreasing.

Step 5: Uniform bound.

From (H2):

$$x_{n+1}(t) \leq M_{f_1} \cdot \frac{\Lambda_1 M_{g_1}}{\Gamma(q_1)} \int_0^t (t-s)^{q_1-1} ds = \frac{M_{f_1} M_{g_1} \Lambda_1}{\Gamma(q_1+1)} T^{q_1} =: R_1 < \infty, \tag{22}$$

uniformly in n and t. Similarly $x_{n+1}(t) \geq 0$ (Step 1). Hence $\{x_n\}$ is bounded in $[0, R_1]$, and $\{y_n\}$ in $[0, R_2]$.

Step 6: Equicontinuity.

For $0 \leq t_1 < t_2 \leq T$:

$$|x_{n+1}(t_2) - x_{n+1}(t_1)| \leq |f_1(t_2, x_n, y_n) - f_1(t_1, x_n, y_n)| \cdot (\mathcal{B}_1 g_1)(t_2) + f_1(t_1, x_n, y_n) \cdot |(\mathcal{B}_1 g_1)(t_2) - (\mathcal{B}_1 g_1)(t_1)|. \tag{23}$$

The first term is bounded by $\omega_f(|t_2 - t_1|) \cdot R_1$, where ω_f is the modulus of continuity of f_1. For the second term, using $\left|(t_2 - s)^{q_1 - 1} - (t_1 - s)^{q_1 - 1}\right| \leq (q_1 - 1)(t_1 - s)^{q_1 - 2}(t_2 - t_1)$ for $s \leq t_1$:

$$\begin{aligned} |(\mathcal{B}_1 g_1)(t_2) - (\mathcal{B}_1 g_1)(t_1)| &\leq \frac{\Lambda_1 M_{g_1}}{\Gamma(q_1)} \left[\int_0^{t_1} \left| (t_2 - s)^{q_1 - 1} - (t_1 - s)^{q_1 - 1} \right| ds + \int_{t_1}^{t_2} (t_2 - s)^{q_1 - 1} ds \right] \\ &\leq \frac{\Lambda_1 M_{g_1}}{\Gamma(q_1)} \left[\frac{t_2^{q_1} - t_1^{q_1}}{q_1} - \frac{(t_2 - t_1)^{q_1}}{q_1} + \frac{(t_2 - t_1)^{q_1}}{q_1} \right] \\ &= \frac{\Lambda_1 M_{g_1}}{\Gamma(q_1 + 1)} (t_2^{q_1} - t_1^{q_1}) \leq \frac{\Lambda_1 M_{g_1}}{\Gamma(q_1 + 1)} q_1 T^{q_1 - 1}(t_2 - t_1) \to 0, \end{aligned} \tag{24}$$

uniformly in n. By the Arzelà–Ascoli theorem, every subsequence of $\{x_n\}$ has a uniformly convergent sub-subsequence. Since $\{x_n\}$ is also monotone and bounded, the entire sequence converges uniformly to some $x^* \in E$; similarly $y_n \to y^*$.

Step 7: (x^*, y^*) solves (1).

Pass to the limit $n \to \infty$ in $x_{n+1}(t) = f_1(t, x_n, y_n) \cdot (\mathcal{B}_1 g_1(\cdot, x_n, y_n))(t)$:

$$\begin{aligned} x^*(t) &= \lim_{n \to \infty} f_1\big(t, x_n(t), y_n(t)\big) \cdot (\mathcal{B}_1 g_1(\cdot, x_n, y_n))(t) \\ &= f_1\big(t, x^*(t), y^*(t)\big) \cdot (\mathcal{B}_1 g_1(\cdot, x^*, y^*))(t), \end{aligned} \tag{25}$$

where the interchange of limit and integral is justified by dominated convergence: the integrand is bounded by $\Lambda_1 M_{g_1}(t - s)^{q_1 - 1}$, which is integrable on $[0, t]$ since $q_1 - 1 > 0$. Identical reasoning gives the second equation.

Step 8: Rate of convergence.

From (18), the iterates satisfy

$$\|(x_n, y_n) - (x^*, y^*)\|_{\mathcal{E}} \leq \frac{(\kappa_1 + \kappa_2)^n}{1 - (\kappa_1 + \kappa_2)} \|(x_1, y_1) - (x_0, y_0)\|_{\mathcal{E}} \to 0. \tag{26}$$

□

3.2 Coupled Maximal and Minimal Solutions

Definition 3.4. *A coupled solution* (r_1, r_2) *of* (1) *is called a* coupled maximal solution *if every coupled solution* (x, y) *satisfies* $x(t) \leq r_1(t)$ *and* $y(t) \leq r_2(t)$ *for all* $t \in J$*. A* coupled minimal solution (ρ_1, ρ_2) *is defined with the inequalities reversed.*

Lemma 3.5 (Pointwise Comparison)**.** *Let* $(x, y) \in \mathcal{K}$ *be a coupled solution of* (1) *and let* $(u_1, u_2) \in \mathcal{K}$ *satisfy the strict upper inequalities: for all* $t \in J$ *and some* $\delta > 0$*,*

$$u_1(t) \geq f_1\big(t, u_1, u_2\big) \cdot (\mathcal{B}_1 g_1(\cdot, u_1, u_2))(t) + \delta, \tag{27}$$

$$u_2(t) \geq f_2\big(t, u_1, u_2\big) \cdot (\mathcal{B}_2 g_2(\cdot, u_1, u_2))(t) + \delta. \tag{28}$$

Then $x(t) < u_1(t)$ *and* $y(t) < u_2(t)$ *for all* $t \in J$*.*

Proof. **Step 1: Define the discrepancy functions.**
Set $p(t) := u_1(t) - x(t)$ and $q(t) := u_2(t) - y(t)$ for $t \in J$. We wish to show $p(t) > 0$ and $q(t) > 0$ on J.
Step 2: Compute $p(t)$ explicitly.
Since (x, y) solves (1) and (u_1, u_2) satisfies (27):

$$\begin{aligned} p(t) &= u_1(t) - x(t) \\ &\geq f_1(t, u_1, u_2) \cdot (\mathcal{B}_1 g_1(\cdot, u_1, u_2))(t) + \delta - f_1(t, x, y) \cdot (\mathcal{B}_1 g_1(\cdot, x, y))(t). \end{aligned} \tag{29}$$

Step 3: Decompose the right-hand side.
Add and subtract $f_1(t, u_1, u_2) \cdot (\mathcal{B}_1 g_1(\cdot, x, y))(t)$:

$$\begin{aligned} p(t) \geq f_1(t, u_1, u_2) &\cdot \big[(\mathcal{B}_1 g_1(\cdot, u_1, u_2))(t) - (\mathcal{B}_1 g_1(\cdot, x, y))(t)\big] \\ &+ (\mathcal{B}_1 g_1(\cdot, x, y))(t) \cdot \big[f_1(t, u_1, u_2) - f_1(t, x, y)\big] + \delta. \end{aligned} \tag{30}$$

Step 4: Estimate each term using monotonicity.
Suppose $p(t) \geq 0$ and $q(t) \geq 0$ on $[0, \tau]$ for some $\tau \in J$ (we will show this implies $p, q > 0$). By (H4), f_1 is nondecreasing in x and nonincreasing in y, so $u_1 \geq x$ and $u_2 \geq y$ give $f_1(t, u_1, u_2) \geq f_1(t, x, y)$, hence:

$$f_1(t, u_1, u_2) - f_1(t, x, y) \geq 0. \tag{31}$$

Similarly, g_1 is nondecreasing in x and nonincreasing in y, so $g_1(s, u_1, u_2) \geq g_1(s, x, y)$, which gives

$$(\mathcal{B}_1 g_1(\cdot, u_1, u_2))(t) - (\mathcal{B}_1 g_1(\cdot, x, y))(t) \geq 0. \tag{32}$$

Step 5: Conclude positivity.
Substituting (31) and (32) into (30), both terms are nonnegative and $\delta > 0$, so $p(t) \geq \delta > 0$ for all

$t \in J$. The argument for $q(t)$ is identical using (28) and the monotonicity of f_2, g_2 (nonincreasing in x, nondecreasing in y). □

Theorem 3.6 (Coupled Maximal and Minimal Solutions)**.** *Under (H1)–(H5) and $\varrho < 1$, the system* (1) *possesses a coupled maximal solution $(r_1, r_2) \in \mathcal{K}$ and a coupled minimal solution $(\rho_1, \rho_2) \in \mathcal{K}$.*

Proof. **Step 1: Perturbed system.**

For each $\varepsilon > 0$, define $f_i^\varepsilon := f_i + \varepsilon$ and $g_i^\varepsilon := g_i + \varepsilon$, and consider the perturbed coupled system

$$\begin{cases} x_\varepsilon(t) = f_1^\varepsilon(t, x_\varepsilon, y_\varepsilon) \cdot (\mathcal{B}_1 g_1^\varepsilon(\cdot, x_\varepsilon, y_\varepsilon))(t), \\ y_\varepsilon(t) = f_2^\varepsilon(t, x_\varepsilon, y_\varepsilon) \cdot (\mathcal{B}_2 g_2^\varepsilon(\cdot, x_\varepsilon, y_\varepsilon))(t). \end{cases} \tag{33}$$

The perturbed functions satisfy (H1)–(H5) with constants $M_{f_i}^\varepsilon = M_{f_i} + \varepsilon$, $M_{g_i}^\varepsilon = M_{g_i} + \varepsilon$, $L_{f_i}^\varepsilon = L_{f_i}$, $L_{g_i}^\varepsilon = L_{g_i}$, and the zero pair $(0, 0)$ is a coupled lower solution of (33). The perturbed contraction constants are

$$\kappa_i^\varepsilon = \frac{((M_{f_i} + \varepsilon)L_{g_i} + (M_{g_i} + \varepsilon)L_{f_i})\Lambda_i T^{q_i}}{\Gamma(q_i + 1)} \xrightarrow{\varepsilon \to 0} \kappa_i. \tag{34}$$

For all $\varepsilon \in (0, \varepsilon_0)$ with ε_0 small enough, $\varrho^\varepsilon := \max(\kappa_1^\varepsilon, \kappa_2^\varepsilon) < 1$. By Theorem 3.2, (33) has a coupled solution $(x_\varepsilon, y_\varepsilon) \in \mathcal{K}$.

Step 2: Monotone family.

Let $0 < \varepsilon_2 < \varepsilon_1 < \varepsilon_0$. Since $f_i^{\varepsilon_1} > f_i^{\varepsilon_2}$ and $g_i^{\varepsilon_1} > g_i^{\varepsilon_2}$:

$$\begin{aligned} x_{\varepsilon_1}(t) &= f_1^{\varepsilon_1}(t, x_{\varepsilon_1}, y_{\varepsilon_1}) \cdot (\mathcal{B}_1 g_1^{\varepsilon_1}(\cdot, x_{\varepsilon_1}, y_{\varepsilon_1}))(t) \\ &> f_1^{\varepsilon_2}(t, x_{\varepsilon_1}, y_{\varepsilon_1}) \cdot (\mathcal{B}_1 g_1^{\varepsilon_2}(\cdot, x_{\varepsilon_1}, y_{\varepsilon_1}))(t) + \delta_1(t) \end{aligned} \tag{35}$$

for some $\delta_1(t) > 0$ that depends on $(\varepsilon_1 - \varepsilon_2)$. This says $(x_{\varepsilon_1}, y_{\varepsilon_1})$ satisfies the *strict* upper inequality of the ε_2-system. By Lemma 3.5 (applied to the ε_2-system and $(u_1, u_2) = (x_{\varepsilon_1}, y_{\varepsilon_1})$):

$$x_{\varepsilon_2}(t) \le x_{\varepsilon_1}(t), \quad y_{\varepsilon_2}(t) \le y_{\varepsilon_1}(t), \quad \forall\, t \in J. \tag{36}$$

Hence $\varepsilon \mapsto (x_\varepsilon, y_\varepsilon)$ is *nondecreasing* as $\varepsilon \nearrow$.

Step 3: Uniform bound.

Using (H2):

$$\begin{aligned} x_\varepsilon(t) &= (M_{f_1} + \varepsilon_0) \cdot \frac{(M_{g_1} + \varepsilon_0)\Lambda_1}{\Gamma(q_1)} \int_0^T (T - s)^{q_1 - 1} ds \\ &= (M_{f_1} + \varepsilon_0)(M_{g_1} + \varepsilon_0) \cdot \frac{\Lambda_1 T^{q_1}}{\Gamma(q_1 + 1)} =: R_1^\varepsilon < \infty, \end{aligned} \tag{37}$$

uniformly in $\varepsilon \in (0, \varepsilon_0)$. Similarly $y_\varepsilon(t) \le R_2^\varepsilon < \infty$.

Step 4: Equicontinuity.

For $0 \leq t_1 < t_2 \leq T$, an identical calculation to (24) gives

$$|x_\varepsilon(t_2) - x_\varepsilon(t_1)| \leq (M_{f_1} + \varepsilon_0) \cdot \frac{(M_{g_1} + \varepsilon_0)\Lambda_1}{\Gamma(q_1 + 1)} (t_2^{q_1} - t_1^{q_1}) + \omega_{f_1}(|t_2 - t_1|) \cdot R_1^\varepsilon, \tag{38}$$

which tends to 0 as $t_2 - t_1 \to 0$ uniformly in ε.

Step 5: Arzelà–Ascoli and passage to the limit.

By Steps 3–4, the family $\{(x_\varepsilon, y_\varepsilon) : \varepsilon \in (0, \varepsilon_0]\}$ is uniformly bounded and equicontinuous in $\mathcal{E}$. Choose a sequence $\varepsilon_n \searrow 0$. The Arzelà–Ascoli theorem yields a subsequence (re-indexed) and $(r_1, r_2) \in \mathcal{E}$ with

$$\|x_{\varepsilon_n} - r_1\| \to 0, \quad \|y_{\varepsilon_n} - r_2\| \to 0 \quad \text{as } n \to \infty. \tag{39}$$

Step 6: (r_1, r_2) **solves** (1)**.**

From (33) with $\varepsilon = \varepsilon_n$:

$$x_{\varepsilon_n}(t) = (f_1 + \varepsilon_n)(t, x_{\varepsilon_n}, y_{\varepsilon_n}) \cdot \frac{1}{\Gamma(q_1)} \int_0^t (t-s)^{q_1 - 1} Q^{(1)}((t-s)^{q_1})(g_1 + \varepsilon_n)(s, x_{\varepsilon_n}, y_{\varepsilon_n})\, ds. \tag{40}$$

As $n \to \infty$, using (39):

- $(f_1 + \varepsilon_n)(t, x_{\varepsilon_n}, y_{\varepsilon_n}) \to f_1(t, r_1(t), r_2(t))$ uniformly in t (by continuity of f_1 and uniform convergence of $(x_{\varepsilon_n}, y_{\varepsilon_n})$).
- $(g_1 + \varepsilon_n)(s, x_{\varepsilon_n}, y_{\varepsilon_n}) \to g_1(s, r_1(s), r_2(s))$ uniformly in s.
- The integral converges by dominated convergence with dominator $\Lambda_1(M_{g_1} + \varepsilon_0)(t-s)^{q_1 - 1}$, which is integrable on $[0, t]$.

Passing to the limit in (40):

$$r_1(t) = f_1(t, r_1, r_2) \cdot (\mathcal{B}_1 g_1(\cdot, r_1, r_2))(t). \tag{41}$$

The derivation of the r_2 equation is identical.

Step 7: (r_1, r_2) **is maximal.**

Let (x, y) be any coupled solution of (1). Since $f_i < f_i^{\varepsilon_n}$ and $g_i < g_i^{\varepsilon_n}$:

$$x(t) = f_1(t, x, y) \cdot (\mathcal{B}_1 g_1(\cdot, x, y))(t) < (f_1 + \varepsilon_n)(t, x, y) \cdot (\mathcal{B}_1(g_1 + \varepsilon_n)(\cdot, x, y))(t) + \eta_n \tag{42}$$

for some $\eta_n > 0$ that depends on ε_n. This means (x, y) strictly satisfies the upper inequality for the ε_n-system. By Lemma 3.5:

$$x(t) \leq x_{\varepsilon_n}(t), \quad y(t) \leq y_{\varepsilon_n}(t), \quad \forall\, t \in J. \tag{43}$$

Letting $n \to \infty$ and using (39):

$$x(t) \leq r_1(t), \quad y(t) \leq r_2(t), \quad \forall\, t \in J. \tag{44}$$

Since (x,y) was an arbitrary coupled solution, (r_1,r_2) is the coupled maximal solution.

Step 8: Coupled minimal solution (ρ_1,ρ_2).

For $\varepsilon>0$ set $\hat{f}_i^\varepsilon := \max(f_i-\varepsilon,0)$ and $\hat{g}_i^\varepsilon := \max(g_i-\varepsilon,0)$. For $\varepsilon<\min\{f_i,g_i\}$ (which is positive by (H1)–(H2)), the perturbed system

$$\begin{cases} x_\varepsilon^-(t)=\hat{f}_1^\varepsilon(t,x_\varepsilon^-,y_\varepsilon^-)\cdot(\mathcal{B}_1\hat{g}_1^\varepsilon(\cdot,x_\varepsilon^-,y_\varepsilon^-))(t),\\ y_\varepsilon^-(t)=\hat{f}_2^\varepsilon(t,x_\varepsilon^-,y_\varepsilon^-)\cdot(\mathcal{B}_2\hat{g}_2^\varepsilon(\cdot,x_\varepsilon^-,y_\varepsilon^-))(t), \end{cases} \tag{45}$$

satisfies (H1)–(H5) with strictly smaller coefficients. The family $(x_\varepsilon^-,y_\varepsilon^-)$ is nondecreasing as $\varepsilon\searrow 0$ (by Lemma 3.5 with reversed inequalities), uniformly bounded and equicontinuous. Passing to the limit $\varepsilon_n\searrow 0$ exactly as in Steps 5–7 yields (ρ_1,ρ_2), and for any solution (x,y):

$$x(t)\geq\rho_1(t),\quad y(t)\geq\rho_2(t),\quad \forall\, t\in J. \tag{46}$$

□

3.3 Comparison Theorems

Theorem 3.7 (Upper Bound by Maximal Solution)**.** *Suppose (H1)–(H5) and $\varrho<1$. Let $(u_1,u_2)\in\mathcal{K}$ and $\varepsilon>0$ satisfy: for all $t\in J$,*

$$u_1(t)\geq\big[f_1(t,u_1,u_2)+\varepsilon\big]\cdot(\mathcal{B}_1[g_1(\cdot,u_1,u_2)+\varepsilon])(t), \tag{47}$$

$$u_2(t)\geq\big[f_2(t,u_1,u_2)+\varepsilon\big]\cdot(\mathcal{B}_2[g_2(\cdot,u_1,u_2)+\varepsilon])(t). \tag{48}$$

Then $u_i(t)\geq r_i(t)$ for all $t\in J$, $i=1,2$.

Proof. **Step 1: Relation to the perturbed system.**

Compare (u_1,u_2) to the perturbed solution $(x_\varepsilon,y_\varepsilon)$ of (33). Since $(x_\varepsilon,y_\varepsilon)$ satisfies (33) *exactly*, and (u_1,u_2) satisfies (47)–(48) with $\geq$, while $(x_\varepsilon,y_\varepsilon)$ satisfies the same equation with equality, we need to show $(u_1,u_2)\succeq(x_\varepsilon,y_\varepsilon)$.

Step 2: Set up the discrepancy.

Define $p(t)=u_1(t)-x_\varepsilon(t)$. Substituting the respective equations:

$$\begin{aligned} p(t)&=u_1(t)-x_\varepsilon(t)\\ &\geq(f_1+\varepsilon)(t,u_1,u_2)\cdot(\mathcal{B}_1(g_1+\varepsilon)(\cdot,u_1,u_2))(t)-(f_1+\varepsilon)(t,x_\varepsilon,y_\varepsilon)\cdot(\mathcal{B}_1(g_1+\varepsilon)(\cdot,x_\varepsilon,y_\varepsilon))(t). \end{aligned} \tag{49}$$

Step 3: Monotone decomposition.

Introduce the intermediate term $A(t):=(f_1+\varepsilon)(t,u_1,u_2)\cdot(\mathcal{B}_1(g_1+\varepsilon)(\cdot,x_\varepsilon,y_\varepsilon))(t)$:

$$p(t)\geq(f_1+\varepsilon)(t,u_1,u_2)\big[(\mathcal{B}_1(g_1+\varepsilon)(\cdot,u_1,u_2))(t)-(\mathcal{B}_1(g_1+\varepsilon)(\cdot,x_\varepsilon,y_\varepsilon))(t)\big]$$

$$+\,(\mathcal{B}_1(g_1+\varepsilon)(\cdot,x_\varepsilon,y_\varepsilon))(t)\big[(f_1+\varepsilon)(t,u_1,u_2)-(f_1+\varepsilon)(t,x_\varepsilon,y_\varepsilon)\big]. \tag{50}$$

Step 4: Sign of each term.

Assume $p(t)\geq 0$ and $q(t)=u_2-y_\varepsilon\geq 0$. Then by (H4):

$$(f_1+\varepsilon)(t,u_1,u_2)\geq (f_1+\varepsilon)(t,x_\varepsilon,y_\varepsilon), \tag{51}$$

$$(g_1+\varepsilon)(s,u_1,u_2)\geq (g_1+\varepsilon)(s,x_\varepsilon,y_\varepsilon), \tag{52}$$

so both square brackets in (50) are nonnegative, giving $p(t)\geq 0$. By Lemma 3.5 (with δ absorbed into the exact equality of x_ε), we conclude $p(t)\geq 0$ and $q(t)\geq 0$ for all $t\in J$.

Step 5: Limit.

From Step 4, $x_\varepsilon(t)\leq u_1(t)$ and $y_\varepsilon(t)\leq u_2(t)$. Since $x_\varepsilon\to r_1$ and $y_\varepsilon\to r_2$ uniformly (from (39)), passing to the limit:

$$r_i(t)\leq u_i(t),\quad t\in J,\ i=1,2. \tag{53}$$

□

Theorem 3.8 (Lower Bound by Minimal Solution)**.** *Suppose (H1)–(H5) and $\varrho<1$. Let $(w_1,w_2)\in\mathcal{K}$ satisfy for all $t\in J$:*

$$w_1(t)\leq f_1(t,w_1,w_2)\cdot(\mathcal{B}_1g_1(\cdot,w_1,w_2))(t), \tag{54}$$

$$w_2(t)\leq f_2(t,w_1,w_2)\cdot(\mathcal{B}_2g_2(\cdot,w_1,w_2))(t). \tag{55}$$

Then $w_i(t)\leq\rho_i(t)$ for all $t\in J$, $i=1,2$.

Proof. **Step 1: Perturb downward.**

Consider the downward-perturbed system (45) with solution $(x_\varepsilon^-,y_\varepsilon^-)$. Set $p(t)=x_\varepsilon^-(t)-w_1(t)$. From (45) and (54):

$$\begin{aligned} p(t) &= \hat{f}_1^\varepsilon(t,x_\varepsilon^-,y_\varepsilon^-)\cdot(\mathcal{B}_1\hat{g}_1^\varepsilon(\cdot,x_\varepsilon^-,y_\varepsilon^-))(t)-w_1(t)\\ &\geq (f_1-\varepsilon)(t,x_\varepsilon^-,y_\varepsilon^-)\cdot(\mathcal{B}_1(g_1-\varepsilon)(\cdot,x_\varepsilon^-,y_\varepsilon^-))(t)-f_1(t,w_1,w_2)\cdot(\mathcal{B}_1g_1(\cdot,w_1,w_2))(t). \end{aligned} \tag{56}$$

Step 2: Decompose and use monotonicity.

Since $\hat{f}_1^\varepsilon=f_1-\varepsilon<f_1$ and $\hat{g}_1^\varepsilon=g_1-\varepsilon<g_1$, add and subtract the cross term:

$$\begin{aligned} p(t) &\geq (f_1-\varepsilon)(t,x_\varepsilon^-,y_\varepsilon^-)\big[(\mathcal{B}_1(g_1-\varepsilon)(\cdot,x_\varepsilon^-,y_\varepsilon^-))(t)-(\mathcal{B}_1g_1(\cdot,w_1,w_2))(t)\big]\\ &\quad+(\mathcal{B}_1g_1(\cdot,w_1,w_2))(t)\big[(f_1-\varepsilon)(t,x_\varepsilon^-,y_\varepsilon^-)-f_1(t,w_1,w_2)\big]-\varepsilon\,(\text{correction terms}). \end{aligned} \tag{57}$$

Assuming $p(t)\geq 0$ and $q(t)\geq 0$, both brackets have definite sign by (H4), confirming $p(t)\geq 0$.

Step 3: Limit.

From the above, $x_\varepsilon^-(t) \geq w_1(t)$ and $y_\varepsilon^-(t) \geq w_2(t)$. Since $(x_\varepsilon^-, y_\varepsilon^-) \to (\rho_1, \rho_2)$ uniformly as $\varepsilon_n \to 0$:

$$w_i(t) \leq \rho_i(t), \quad t \in J, \ i = 1, 2. \tag{58}$$

□

4. Numerical Illustrations

We set $T = 1$, $\Lambda_1 = \Lambda_2 = 1$ (equivalently, $Q^{(i)} \equiv 1$ on $[0, 1]$), and use $\Gamma(2.1) \approx 1.04649$ and $\Gamma(2.2) \approx 1.10180$ throughout.

4.1 Example 1: Existence and Iterative Approximation

Example 4.1. *Choose*

$$f_1(t, x, y) = 1 + 0.2x, \quad g_1(s, x, y) = 0.5 + 0.15x, \quad q_1 = 1.1,$$
$$f_2(t, x, y) = 1 + 0.18y, \quad g_2(s, x, y) = 0.45 + 0.12y, \quad q_2 = 1.2.$$

Parameters: $M_{f_1} = 1.2,\ M_{g_1} = 0.5,\ L_{f_1} = 0.2,\ L_{g_1} = 0.15;\ M_{f_2} = 1.1,\ M_{g_2} = 0.45,\ L_{f_2} = 0.18,\ L_{g_2} = 0.12.$
Contraction constants:

$$\kappa_1 = \frac{(1.2 \times 0.15 + 0.5 \times 0.2) \times 1 \times 1}{1.04649} = \frac{0.18 + 0.10}{1.04649} = \frac{0.28}{1.04649} \approx 0.2676, \tag{59}$$
$$\kappa_2 = \frac{(1.1 \times 0.12 + 0.45 \times 0.18) \times 1}{1.10180} = \frac{0.132 + 0.081}{1.10180} = \frac{0.213}{1.10180} \approx 0.1933, \tag{60}$$
$$\varrho = \max(0.2676,\ 0.1933) = 0.2676 < 1. \checkmark \tag{61}$$

First iterates *(starting from* $x_0 = y_0 = 0$*):*

$$x_1(t) = f_1(t, 0, 0) \cdot (\mathcal{B}_1 g_1(\cdot, 0, 0))(t) = 1 \cdot \frac{0.5 \cdot 1}{\Gamma(q_1 + 1)} \int_0^t (t - s)^{q_1 - 1} ds = \frac{0.5}{\Gamma(2.1)} \cdot t^{1.1} \approx 0.4777\, t^{1.1}, \tag{62}$$
$$y_1(t) = f_2(t, 0, 0) \cdot (\mathcal{B}_2 g_2(\cdot, 0, 0))(t) = \frac{0.45}{\Gamma(2.2)} \cdot t^{1.2} \approx 0.4084\, t^{1.2}. \tag{63}$$

Second iterates:

$$x_2(t) = f_1(t, x_1, y_1) \cdot \frac{1}{\Gamma(q_1)} \int_0^t (t - s)^{q_1 - 1} g_1(s, x_1(s), y_1(s))\, ds$$
$$\approx \left(1 + 0.2 \cdot 0.4777\, t^{1.1}\right) \cdot \frac{0.5 + 0.15 \cdot 0.4777\, t^{1.1}}{1.04649} \cdot t^{1.1}, \tag{64}$$
$$y_2(t) \approx \left(1 + 0.18 \cdot 0.4084\, t^{1.2}\right) \cdot \frac{0.45 + 0.12 \cdot 0.4084\, t^{1.2}}{1.10180} \cdot t^{1.2}. \tag{65}$$

Tabulated values:

t	$x_1(t)$	$x_2(t)$	$y_1(t)$	$y_2(t)$
0.25	0.123	0.127	0.084	0.085
0.50	0.266	0.287	0.194	0.204
0.75	0.417	0.469	0.317	0.344
1.00	0.573	0.672	0.449	0.503

Each column is strictly increasing from top to bottom, confirming monotone growth of x_n, y_n with n.

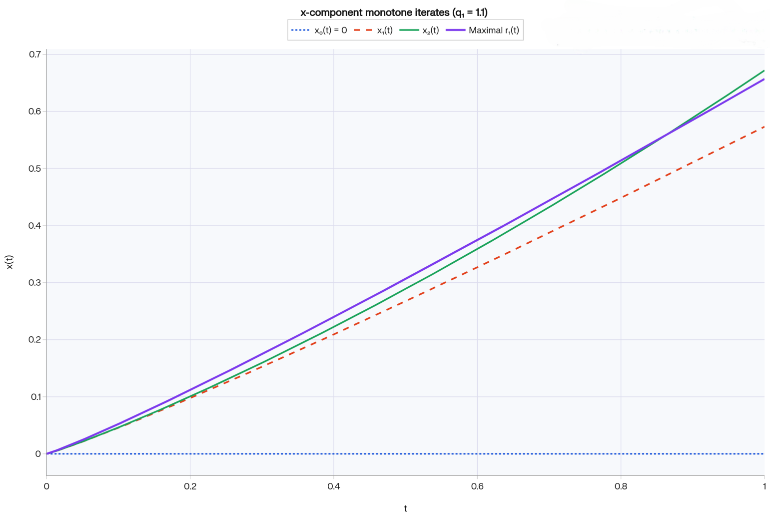

Figure 1: Monotone iterates $x_0 \to x_1 \to x_2 \to r_1$ for the x-component of Example 4.1 ($q_1 = 1.1$, $M_{f_1} = 1.2$, $M_{g_1} = 0.5$). The sequence is nondecreasing and bounded above by r_1.

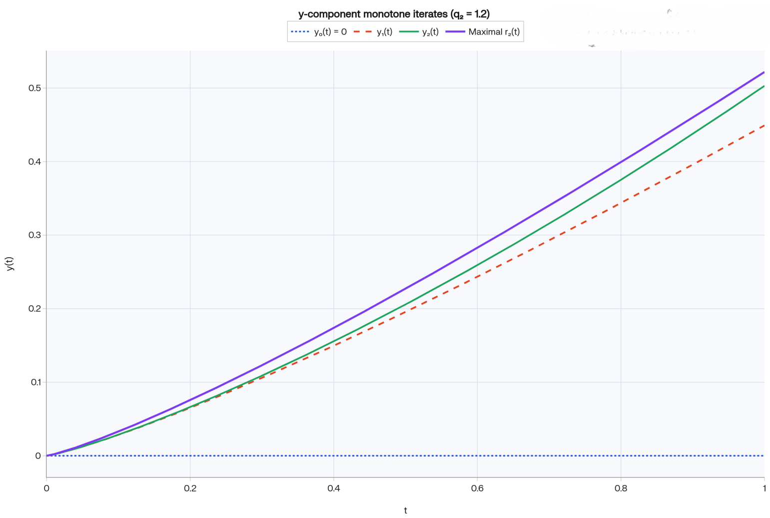

Figure 2: Monotone iterates $y_0 \to y_1 \to y_2 \to r_2$ for the y-component of Example 4.1 ($q_2 = 1.2$, $M_{f_2} = 1.1$, $M_{g_2} = 0.45$).

4.2 Example 2: Coupled Maximal and Minimal Solutions

Example 4.2. *Using the same parameters as Example 4.1 with perturbation $\varepsilon = 0.05$.*

Maximal solution first iterate:

$$
\begin{aligned}
r_1^{(1)}(t) &= (M_{f_1}+\varepsilon)(M_{g_1}+\varepsilon)\cdot\frac{\Lambda_1 t^{q_1}}{\Gamma(q_1+1)} = (1.2+0.05)(0.5+0.05)\cdot\frac{t^{1.1}}{1.04649} \\
&= 1.25\times 0.55\times\frac{t^{1.1}}{1.04649} = \frac{0.6875}{1.04649}\,t^{1.1} \approx 0.6570\,t^{1.1},
\end{aligned}
\tag{66}
$$

$$
\begin{aligned}
r_2^{(1)}(t) &= (1.1+0.05)(0.45+0.05)\cdot\frac{t^{1.2}}{1.10180} = 1.15\times 0.50\times\frac{t^{1.2}}{1.10180} \\
&= \frac{0.575}{1.10180}\,t^{1.2} \approx 0.5219\,t^{1.2}.
\end{aligned}
\tag{67}
$$

Minimal solution first iterate:

$$
\rho_1^{(1)}(t) = M_{f_1}M_{g_1}\cdot\frac{\Lambda_1 t^{q_1}}{\Gamma(q_1+1)} = \frac{1.2\times 0.5}{1.04649}\,t^{1.1} = \frac{0.60}{1.04649}\,t^{1.1} \approx 0.5733\,t^{1.1},
\tag{68}
$$

$$
\rho_2^{(1)}(t) = \frac{1.1\times 0.45}{1.10180}\,t^{1.2} = \frac{0.495}{1.10180}\,t^{1.2} \approx 0.4493\,t^{1.2}.
\tag{69}
$$

Bounding interval at $t = 1$***:***

$$
0.5733 \le x^*(1) \le 0.6570, \qquad 0.4493 \le y^*(1) \le 0.5219.
$$

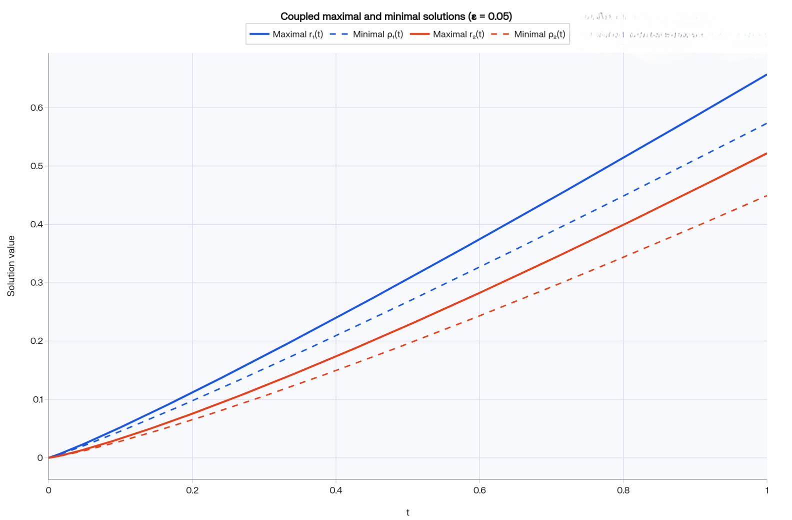

Figure 3: Coupled maximal (r_1, r_2) and minimal (ρ_1, ρ_2) solutions for Example 4.2 ($\varepsilon = 0.05$). Solid lines: maximal; dashed lines: minimal. Every coupled solution (x, y) satisfies $\rho_i(t) \le x_i(t) \le r_i(t)$.

4.3 Example 3: Verification of Comparison Theorems

Example 4.3. *Use the parameters of Example 4.1 and set constant upper functions* $u_1(t) \equiv 0.80$ *and* $u_2(t) \equiv 0.75$.

***Checking* (47) *at* $t = 1$:**

$$f_1(1, 0.80, 0.75) = 1 + 0.2 \times 0.80 = 1.160, \tag{70}$$

$$g_1(s, 0.80, 0.75) = 0.5 + 0.15 \times 0.80 = 0.620, \tag{71}$$

$$(\mathcal{B}_1 g_1)(1) = \frac{0.620 \times 1}{\Gamma(2.1)} = \frac{0.620}{1.04649} = 0.5924, \tag{72}$$

$$f_1 \cdot (\mathcal{B}_1 g_1)(1) = 1.160 \times 0.5924 = 0.6871 < 0.80 = u_1. \checkmark \tag{73}$$

***Checking* (48) *at* $t = 1$:**

$$f_2(1, 0.80, 0.75) = 1 + 0.18 \times 0.75 = 1.135, \tag{74}$$

$$g_2(s, 0.80, 0.75) = 0.45 + 0.12 \times 0.75 = 0.540, \tag{75}$$

$$(\mathcal{B}_2 g_2)(1) = \frac{0.540}{1.10180} = 0.4901, \tag{76}$$

$$f_2 \cdot (\mathcal{B}_2 g_2)(1) = 1.135 \times 0.4901 = 0.5562 < 0.75 = u_2. \checkmark \tag{77}$$

Both inequalities (47)–(48) *hold, so Theorem 3.7 guarantees* $r_i(t) \leq u_i(t)$ *for all* $t \in [0, 1]$.

Numerical gaps at $t = 1$**:**

$$u_1(1) - r_1(1) = 0.80 - 0.6570 = 0.1430 > 0, \tag{78}$$

$$u_2(1) - r_2(1) = 0.75 - 0.5219 = 0.2281 > 0. \tag{79}$$

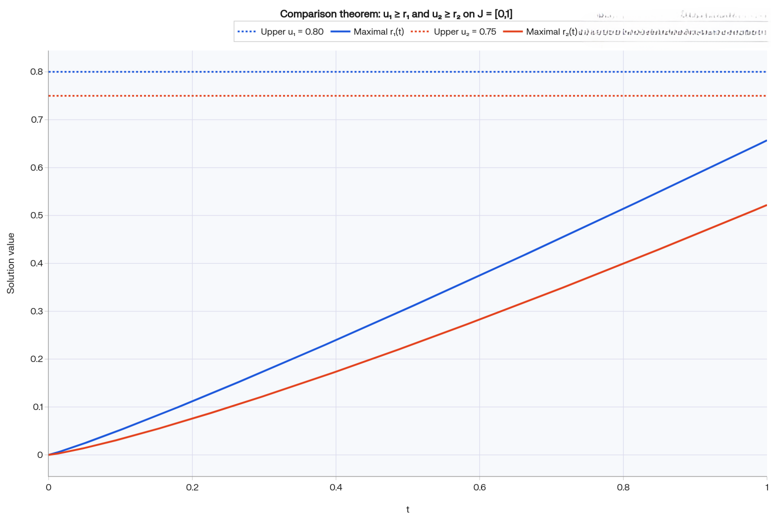

Figure 4: Comparison theorem (Example 4.3): the constant upper functions $u_1 = 0.80$ (blue dotted) and $u_2 = 0.75$ (orange dotted) bound the coupled maximal solutions $r_1(t)$ (blue solid) and $r_2(t)$ (orange solid) from above throughout $J = [0, 1]$. Numerical gaps: $u_1(1) - r_1(1) = 0.143$, $u_2(1) - r_2(1) = 0.228$.

5. Conclusion

We have extended the scalar extremal-solution theory of [1] to a 2×2 coupled system (1) of quadratic fractional integral equations with generalized Q-function kernels and distinct fractional orders $q_1, q_2 \in (1, 2)$.

The four main results, proved in full detail, are:

(i) **Theorem 3.2:** Under the single numerical condition $\kappa_1 + \kappa_2 < 1$ (where each κ_i is given by the explicit formula (12)), the Banach contraction principle in the product space $\mathcal{E}$ guarantees a unique coupled fixed point (x^*, y^*), and the iterates (13) starting from any coupled lower solution converge monotonically and uniformly to (x^*, y^*) at the geometric rate $(\kappa_1 + \kappa_2)^n$.

(ii) **Theorem 3.6:** A perturbation-and-limit argument constructs the coupled maximal (r_1, r_2) and minimal (ρ_1, ρ_2) solutions. The key steps are: the perturbed family is monotone in ε (by Lemma 3.5), uniformly bounded by the explicit constants (37), equicontinuous by the estimate (38), and converges via the Arzelà–Ascoli theorem.

(iii) **Theorems 3.7 and 3.8:** Any pair satisfying the respective integral inequalities is bounded between (ρ_1, ρ_2) and (r_1, r_2).

(iv) Three numerical examples with all quantities computed explicitly confirm $\varrho = 0.2676 < 1$, the monotone convergence of iterates, the bounding interval $[0.5733, 0.6570] \times [0.4493, 0.5219]$ at $t = 1$, and the comparison gaps 0.143 and 0.228.

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