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On Lacunary Statistical Convergence in Metric-Like Spaces

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Abstract

In the present study, we generalize the notions of lacunary statistical σ-convergence and lacunary statistical σ-boundedness of real valued sequences to that of vector valued sequences in an arbitrary metric-like space (X, σ). We also explore that a bounded and lacunary statistically σ-convergent sequence is lacunary strongly σ-Cesàro summable in a metric-like space. Beside this some inclusion relations are also established between the notion of lacunary statistical σ-convergence and lacunary strongly σ-Cesàro summability in (X, σ).

Keywords: Metric-like space; strongly Cesàro summability; lacunary statistical convergence; lacunary density.
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1. Introduction

Statistical convergence, which is in fact, a generalizations of usual convergence of real sequences was introduced in short by Fast [8] in 1951. Later on this concept was studied as "convergence in density" by Buck [4] in 1953. It is also a part of monograph by Zygmund [28] and referred as almost convergence. Steinhaus [25] and Schoenberg [23] introduced and studied this concept independently in connection with summability of sequences. Later on this concept of statistical convergence and its various extensions have been explored by many more mathematicians [2,5,9,11,13,15,22,27].
The concept of statistical convergence depends on the density of subsets of $\mathbb{N}$ of natural numbers. The natural density of $A \subset \mathbb{N}$ is denoted by $\delta(A)$ and defined as [20]

$$\delta(A) = \lim_{n\to\infty} \frac{1}{n} |\{m \in A : m \leq n\}|,$$

provided the limit exists. It is observed that $\delta(A) = 0$ for finite subset A of $\mathbb{N}$ and $\delta(A) + \delta(\mathbb{N} - A) = 1$ for every $A \subseteq \mathbb{N}$.

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A real valued sequence (u_m) is said to be statistically convergent to $\zeta \in \mathbb{R}$ if for each $\varepsilon > 0$,

$$\delta(\{|u_m - \zeta| \geq \varepsilon\}) = 0,$$
$$i.e., \lim_{n\to\infty} \frac{1}{n}|\{m \leq n : |u_m - \zeta| > \varepsilon\}| = 0$$

and we write $u_m \to \zeta(S)$. The set of all statistically convergent real sequences is denoted by S.

Now we recall lacunary sequence and lacunary density, before proceeding for lacunary statistical convergence as

By a lacunary sequence $\theta = (m_r)_{r=0}^{\infty}$ is an increasing sequence such that $m_r - m_{r-1} \to \infty$, where $m_0 = 0$, $m_r \geq 0$. Here we notate $I_r = (m_{r-1}, m_r]$, $h_r = m_r - m_{r-1}$ and $q_r = \dfrac{m_r}{m_{r-1}}$. Freedman *et al*. [9], defined the space of all lacunary strongly convergent sequences denoted as N_θ as follows

$$N_\theta = \left\{ (u_m) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{m\in I_r} |u_m - \zeta| = 0 \; for\ some\ \zeta \right\}.$$

There is a strong connection between N_θ and the space w of strongly Cesàro summable sequences where

$$w = \left\{ (u_m) : \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} |u_m - \zeta| = 0 \; for\ some\ \zeta \right\}.$$

A new variant of statistical convergence, named as lacunary statistical convergence was studied by Fridy and Orhan [12] in 1993 that is in the same connection with statistical convergence as N_θ with w and defined as

A real valued sequence (u_m) is said to be lacunary statistical convergent to ζ if for every $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r}|\{m \in I_r : |u_m - \zeta| \geq \varepsilon\}| = 0.$$

By S_θ we notate the class of all lacunary statistical convergent sequences of reals. As statistical convergence has its deep root in the notion of natural density, similar is the relation of lacunary statistical convergence with lacunary density (or $\theta-$density).

Definition 1.1 ([3]). *For a given lacunary sequence $\theta = (m_r)$, the lacunary density (or $\theta-$density) of $A \subseteq \mathbb{N}$ is defined as $\delta_\theta(A) = \lim_{r\to\infty} \frac{1}{h_r}|\{m \in I_r : m \in A\}|$.*

Definition 1.2. *For a given lacunary sequence $\theta = (m_r)$, if the set of indices m's for which for which (u_m) does not satisfy property P has zero lacunary density, then we say (u_m) satisfies P for "almost all m with respect to θ" abbreviated as "a.a. m w.r.t. θ"*

In view of above definition, lacunary statistical convergence can also be defined as

Definition 1.3. *A sequence (u_m) of reals is lacunary statistical convergent to $\zeta \in \mathbb{R}$ if for $\varepsilon > 0$,*

$$\delta_\theta(\{m \in \mathbb{N} : |u_m - \zeta| \geq \varepsilon\}) = 0,$$

$$i.e., \ |u_m - \zeta| < \varepsilon \quad a.a. \ m \ w.r.t \ \theta.$$

In recent years, various generalization of this notion, that is, $\lambda-$statistical convergence, $A-$statistical convergence, $I-$convergence, rough statistical convergence, $\mu-$statistical convergence, lacunary statistical convergence, deferred statistical convergence, convergence of order α etc. have been appeared and also these generalizations have been studied for vector valued sequences by many more mathematicians. One may refer to [6,7,10,12,14,17–19,21,24,26].

The notion of metric like space was introduced by Amini-Harandi [1] in 2012 and defined as

Definition 1.4 ([1])**.** *Let X be a non-empty set and $\mathbb{R}_{\geq 0}$ denotes the set of non-negative real numbers. A Mapping $\sigma : X \times X \longrightarrow \mathbb{R}_{\geq 0}$ is said to be metric-like on X if for any $u, v, w \in X$, the following holds*

(σ_1) $\sigma(u,v) = 0 \implies u = v$

(σ_2) $\sigma(u,v) = \sigma(v,u)$

(σ_3) $\sigma(u,w) \leq \sigma(u,v) + \sigma(v,w).$

The pair (X,σ) is called a metric-like space. It is observed that a metric-like on X satisfies all of the conditions of a metric except that $\sigma(u,u)$ may be positive for $u \in X$.

Remark 1.5 ([1])**.** *It is observed that every metric space is a partial metric space and that of every partial metric space is a metric-like space, but converse need not be true.*

Definition 1.6. *Let (u_m) be sequence in metric-like space (X,σ) and $u_0 \in X$. If for given $\varepsilon > 0$, there exists a positive integer m_0 such that $|\sigma(u_m, u_0) - \sigma(u_0, u_0)| < \varepsilon$ for all $m \geq m_0$, then we say (u_m) is σ-convergent to u_0. The class of all σ-convergent sequences is denoted by c_σ.*

Malik and Das [16] in 2024, introduced the notion of statistical convergence in metric-like space and we will denote this notion as statistical σ-convergence.

Definition 1.7. *A sequence (u_m) in metric-like space (X,σ) is said to be statistically σ-convergent to some $u_0 \in X$ if for given $\varepsilon > 0$,*

$$\delta(\{m \in \mathbb{N} : |\sigma(u_m, u_0) - \sigma(u_0, u_0)| \geq \varepsilon\}) = 0$$

and we write it as $u_m \xrightarrow{\sigma} u_0(S)$. By $S(c_\sigma)$, we notate the class of all statistically σ-convergent sequence from (X,σ).

Definition 1.8. *A sequence (u_m) in metric-like space (X,σ) is said to be σ-bounded if there exists some $u_0 \in X$ and $M > 0$ such that $|\sigma(u_m, u_0) - \sigma(u_0, u_0)| < M$ for all $m \geq 1$. We write b_σ as the class of all σ-bounded sequences.*

2. Lacunary Statistical σ-convergence and Lacunary Statistical σ-boundedness

In this section, we introduce the concepts of lacunary statistically σ-convergence and lacunary statistically σ-boundedness and investigate various properties of lacunary statistically σ-convergent sequences. It is also observed that every lacunary statistically σ-convergent sequence has a lacunary statistically dense, σ-convergent subsequence.

Definition 2.1. *A sequence (u_m) in metric like space (X, σ) is said to be lacunary statistically σ-convergent to $u_0 \in X$ if for $\varepsilon > 0$,*

$$\lim_{r\to\infty} \frac{1}{h_r} |\{m \in I_r : |\sigma(u_m, u_0) - \sigma(u_0, u_0)| \geq \varepsilon\}| = 0$$

and we write $u_m \xrightarrow{\sigma} u_0(S_\theta)$. The set of all lacunary statistically σ-convergent sequences in (X, σ) is denoted by $S_\theta(c_\sigma)$.

Theorem 2.2. *In a metric like space (X, σ) every σ-convergent sequence is lacunary statistically σ-convergent, i.e., $c_\sigma \subset S_\theta(c_\sigma)$ and containment is proper.*

Proof. Let $(u_m) \in S_\theta(c_\sigma)$. Then for given $\varepsilon > 0$ and $u_0 \in X$, there exists $m_0 \in \mathbb{N}$, we have $|\sigma(u_m, u_0) - \sigma(u_0, u_0)| < \varepsilon$ for all $m \geq m_0$. Now the result hold from the fact that lacunary density of a finite set is zero. For proper containment, consider the following example
Let $X = \mathbb{R}$ and σ be the metric-like defined by $\sigma(\xi, \eta) = |\xi - \eta|$; $\xi, \eta \in \mathbb{R}$. consider the sequence

$$u_i = \begin{cases} m_{r-1} + 1 & \text{for } i = m_{r-1} + 1, \\ 0 & \text{otherwise.} \end{cases} \quad \text{on } I_r,\ r = 1, 2, 3 \ldots$$

Let, if possible, (u_m) is σ-convergent to some $u_0 \in X$. Then for given $\varepsilon > 0$, there exists $m_0 \in \mathbb{N}$ we have $|\sigma(u_i, u_0) - \sigma(u_0, u_0)| < \varepsilon$ for all $i \geq m_0$, i.e., $|m_{r-1} + 1 - u_0| < \varepsilon\ \forall\ m_{r-1} + 1 \geq m_0$ for all $r = 1, 2, 3, \ldots$ a contradiction as $m_r \to \infty$. However, (u_i) is lacunary statistically σ-convergent to $0 \in X$, because for $\varepsilon > 0$, $\lim_{r\to\infty} \frac{1}{h_r} |\{i \in I_r : |\sigma(u_i, 0) - \sigma(0, 0)| \geq \varepsilon\}) = \lim_{r\to\infty} \frac{1}{h_r} = 0$. □

Theorem 2.3 ([13]). *Let f be a modulus function (unbounded). Then $S^f_{p,q} - \lim u_m = L$ iff $\exists\ M \subseteq \mathbb{N}$ such that $\delta^f_{p,q}(M) = 0$ and $\lim_{m\in\mathbb{N}-M} u_m = L$.*

For a lacunary sequence $\theta = (m_r)$, taking $f(x) = x$, $p(r) = m_{r-1}$ and $q(r) = m_r$, $r = 1, 2, 3, \ldots$ we have,

Corollary 2.4. *A sequence (u_m) of reals is lacunary statistical convergent to L iff $\exists\ M \subseteq \mathbb{N}$ such that $\delta_\theta(M) = 0$ and $\lim_{m\in\mathbb{N}-M} u_m = L$.*

Theorem 2.5. *A sequence (u_m) in metric like space (X, σ) is lacunary statistical σ-convergent to $u_0 \in X$ if and only if there exists $M \subseteq \mathbb{N}$ such that $\delta_\theta(M) = 0$ and $(u_m)_{m\in\mathbb{N}-M} \xrightarrow{\sigma} u_0$.*

Proof. Let $u_0 \in X$ be such that $u_m \xrightarrow{\sigma} u_0(S_\theta)$. Then for given $\varepsilon > 0$, we have

$$\delta_\theta(\{m \in \mathbb{N} : |v_m| \geq \epsilon\}) = 0,$$

where $v_m = \sigma(u_m, u_0) - \sigma(u_0, u_0)$. Thus (v_m) is a real valued sequence which is lacunary statistical convergent to 0, so by Corollary 2.4, there exists $M \subseteq \mathbb{N}$ such that $\delta_\theta(M) = 0$ and $\lim_{m \in \mathbb{N}-M} v_m = 0$. Hence we get $\lim_{m \in \mathbb{N}-M} \sigma(u_m, u_0) = \sigma(u_0, u_0)$

Conversely, it is given that $(v_m)_{m \in \mathbb{N}-M} \xrightarrow{\sigma} u_0$. So for $\varepsilon > 0$, there exists $m_0 \in \mathbb{N}$ such that $|\sigma(u_m, u_0) - \sigma(u_0, u_0)| < \varepsilon$ for all $m \in \mathbb{N} - M$ with $m \geq m_0$. From this we conclude

$$\{m : |\sigma(u_m, u_0) - \sigma(u_0, u_0)| \geq \varepsilon\} \subseteq M \cup \{1, 2, \ldots, m_0 - 1\}.$$

As lacunary density of finite set is zero and $\delta_\theta(M) = 0$, so the result follows. □

Definition 2.6. *A sequence (u_m) in (X, σ) is said to be lacunary statistically σ-bounded if there exists some $u_0 \in X$ and $M > 0$ such that $\lim_{r\to\infty} \frac{1}{h_r} |\{m \in I_r : |\sigma(u_m, u_0) - \sigma(u_0, u_0)| > M\}| = 0$. The set of all lacunary statistically σ-bounded sequences is denoted by $S_\theta(b_\sigma)$.*

Theorem 2.7. *Lacunary statistically σ-convergence implies lacunary statistically σ-boundedness. Converse need not hold good.*

Proof. Let (u_m) be a lacunary statistically σ-convergent to some $u_0 \in X$. Then for given $\varepsilon > 0$, $\lim_{r\to\infty} \frac{1}{h_r} |\{m \in I_r : |\sigma(u_m, u_0) - \sigma(u_0, u_0)| \geq \varepsilon\}| = 0$. Now the result hold for for sufficiently large M, and using the inequality,

$$|\{m \in I_r : |\sigma(u_m, u_0) - \sigma(u_0, u_0)| \geq M\}| \leq |\{m \in I_r : |\sigma(u_m, u_0) - \sigma(u_0, u_0)| \geq \varepsilon\}|.$$

For strict inclusion, consider the following example

Let $\theta = (2^r)$, $X = \mathbb{R}$ and σ be the metric-like defined as $\sigma(\xi, \eta) = |\xi - \eta|$; $\xi, \eta \in \mathbb{R}$. Take a sequence (u_m) as

$$u_m = \begin{cases} -1 & \text{if } m \neq 2n \\ 1 & \text{if } m = 2n \end{cases} \quad for\ n \in \mathbb{N}\,.$$

Now $\lim_{r\to\infty} \frac{1}{h_r} |\{m \in I_r : |\sigma(u_m, 1) - \sigma(1, 1)| > \varepsilon\}) = \frac{1}{2}$. Thus (u_m) is not lacunary statistically σ-convergent to 1. Similar can be proved for -1. As every bounded sequence is lacunary statistically σ-bounded, hence the result follows. □

Theorem 2.8. *In a metric like space (X, σ) every σ-bounded sequence is lacunary statistically σ-bounded, i.e., $b_\sigma \subset S_\theta(b_\sigma)$ and containment is proper.*

Proof. Let (u_m) be a σ-bounded sequence in (X, σ). Then there exists $u_0 \in X$ and $M > 0$ such that $|\sigma(u_m, u_0) - \sigma(u_0, u_0)| < M$ for all $m \geq 1$. As empty set has zero lacunary density, so (u_m) is lacunary statistically σ-bounded. For proper containment, the example cited in Theorem 2.2 gives the desired result. □

Remark 2.9. *As every σ-convergent sequence is σ-bounded. Also in view of Theorem 2.8 every σ-bounded sequence is lacunary statistically σ-bounded. So using the both result we have,every σ-convergent sequence is lacunary statistically σ-bounded, i.e., $c_\sigma \subset S_\theta(b_\sigma)$.*

Theorem 2.10. *A sequence (u_m) in (X,σ) is lacunary statistically σ-convergent to some $u_0 \in X$ iff there exists a σ-convergent sequence (v_m) such that $v_m = u_m$ a.a. m w.r.t. θ.*

Proof. Let (u_m) be a lacunary statistically σ-convergent to some $u_0 \in X$. So, for each $\varepsilon > 0$, we have $\delta_\theta(M) = 0$ where $M = \{m \in \mathbb{N} : \sigma(u_m, u_0) - \sigma(u_0, u_0) + \varepsilon\}$

$$v_m = \begin{cases} u_m & \text{for } m \in \mathbb{N} - M \\ u_0 & \text{for } m \in M. \end{cases}$$

Now $\{m \in \mathbb{N} : v_m \neq u_m\} \subseteq M$ and so $v_m = u_m$ a.a. m w.r.t. θ. Also

$$\sigma(v_m, u_0) = \begin{cases} \sigma(u_m, u_0) & \text{for } m \in \mathbb{N} - M \\ \sigma(u_0, u_0) & \text{for } m \in M \end{cases}$$

implies $|\sigma(u_m, u_0) - \sigma(u_0, u_0)| < \varepsilon$ for all $m \geq 1$. Thus (v_m) is σ-convergent sequence and $v_m = u_m$ a.a. m w.r.t. θ.

Conversely, let there exists $m_0 \in \mathbb{N}$ such that $|\sigma(v_m, u_0) - \sigma(u_0, u_0)| < \varepsilon$ for all $m \geq m_0$. The result follows from the inclusion relation $\{m : |\sigma(u_m, u_0) - \sigma(u_0, u_0)| \geq \varepsilon\} \subseteq M \cup \{1, 2, 3, \ldots, m_0 - 1\}$. □

The proof of the following theorem run on the similar lines

Theorem 2.11. *A sequence (u_m) is lacunary statistically σ-bounded iff there exists a σ-bounded sequence (v_m) such that $v_m = u_m$ a.a. m w.r.t. θ.*

Remark 2.12. *It is observed that subsequence of a lacunary statistically σ-convergent sequence need not be lacunary statistically σ-convergent. We show this by taking the following example*
Let $X = \mathbb{R}$ with metric-like σ defined as $\sigma(\xi, \eta) = |\xi - \eta|$; $\xi, \eta \in \mathbb{R}$. Consider the sequence

$$u_i = \begin{cases} m_{r-1} + 1 & \text{for } i = m_{r-1} + 1, \\ 0 & \text{otherwise} \end{cases} \qquad \text{on } I_r,\ r = 1, 2, 3 \ldots \quad .$$

Then (u_i) is lacunary σ-statistically convergent to 0. But we have $(m_0 + 1, m_1 + 1, m_2 + 1, \ldots)$ is a subsequence of (u_i) which is not lacunary statistically σ-convergent.

Definition 2.13. *A subset M of $\mathbb{N}$ is said to be lacunary statistically dense if $\delta_\theta(M) = 1$.*

In the next theorem, we characterizes the lacunary statistically σ-convergent sequence in term of lacunary statistically dense σ-convergent subsequences.

Theorem 2.14. *A sequence (u_m) is lacunary statistically σ-convergent iff every lacunary statistically dense subsequence of (u_m) is lacunary statistically σ-convergent.*

Proof. Let (u_m) is lacunary statistically σ-convergent to u_0. So for given $\varepsilon > 0$, we have

$$\lim_{r\to\infty}\frac{1}{h_r}|\{m \in I_r : |\sigma(u_m, u_0) - \sigma(u_0, u_0) \geq \varepsilon\}| = 0, \text{ i.e., } \delta_\theta(A) = 0$$

where $A = \{m \in \mathbb{N} : |\sigma(u_m, u_0) - \sigma(u_0, u_0)| \geq \varepsilon\}$ and $(u_{m_n})_{n\in\mathbb{N}}$ is lacunary statistically dense subsequence of (u_m) which is not lacunary σ-statistically convergent, i.e.,

$$\liminf_{r\to\infty}\frac{1}{h_r}|\{m_n \in I_r : |\sigma(u_{m_n}, u_0) - \sigma(u_0, u_0)| \geq \varepsilon\}| = \lambda, \text{ where } \lambda \in (0,1).$$

Now $|\{m \in I_r : |\sigma(u_m, u_0) - \sigma(u_0, u_0)| \geq \varepsilon\}) \geq |\{m_n \in I_r : |\sigma(u_{m_n}, u_0) - \sigma(u_0, u_0)| \geq \varepsilon\})$.
So $\liminf_{r\to\infty}\frac{1}{h_r}|\{m \in I_r : |\sigma(u_m, u_0) - \sigma(u_0, u_0)| \geq \varepsilon\}| \geq \lambda \neq 0$, a contradiction to given.
Converse part follows from the fact that every sequence is a lacunary statistical dense subsequence of itself. □

Remark 2.15. *Example in Remark 2.12 asserts that a subsequence of a lacunary statistically σ-bounded sequence need not be lacunary statistically σ-bounded.*

By using the same technique as in Theorem 2.14, we give a characterization of the lacunary statistically σ-boundedness in terms of the lacunary statistically dense σ-bounded subsequences of it, in terms of following.

Theorem 2.16. *A sequence (u_m) is lacunary statistically σ-bounded iff every lacunary statistically dense subsequence of (u_m) is lacunary statistically σ-bounded.*

3. Lacunary Strongly σ-Cesàro Summable Spaces

In the present section, we introduce and study the notion of lacunary strongly σ-Cesàro summability in a metric like space. We also demonstrate that in case of σ-bounded sequence, the lacunary strongly σ-Cesàro summability and lacunary statistically σ-convergence coincides in (X, σ). Beside this, results have been established for inclusion between two arbitrary lacunary methods of σ-statistical convergence.

Definition 3.1. *A sequence (u_m) in metric-like space (X, σ) is said to be lacunary strongly σ-Cesàro summable to $u_0 \in X$ if*

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{m\in I_r}|\sigma(u_m, u_0) - \sigma(u_0, u_0)| = 0$$

and we write $N_\theta(c_\sigma)$ for the set of all lacunary strongly σ-Cesàro summable sequences in (X, σ).

Theorem 3.2. *Let $\theta = (m_r)$ be a given lacunary sequence , we have the following* *(i) Every lacunary strongly σ-Cesàro summable sequence is lacunary statistically σ-convergent to same limit, i.e., $N_\theta(c_\sigma) \subset S_\theta(c_\sigma)$.*

(ii) Containment in part (i) is proper.

(iii) Every σ-bounded and lacunary statistically σ-convergent sequence is lacunary strongly σ-Cesàro summable sequence, i.e., $S_\theta(c_\sigma) \cap b_\sigma = N_\theta(c_\sigma) \cap b_\sigma$.

Proof. Throughout in the proof of theorem, let $\sum_1$ denotes the sum of $|\sigma(u_m, u_0) - \sigma(u_0, u_0)|$ for those $m \in I_r$ for which $|\sigma(u_m, u_0) - \sigma(u_0, u_0)| < \varepsilon$ while $\sum_2$ denotes the sum for those $m \in I_r$ for which $|\sigma(u_m, u_0) - \sigma(u_0, u_0)| \geq \varepsilon$.

(i) Let $\varepsilon > 0$ be given, then we have

$$\begin{aligned}\sum_{m\in I_r} |\sigma(u_m, u_0) - \sigma(u_0, u_0)| &\geq \sum_2 |\sigma(u_m, u_0) - \sigma(u_0, u_0)| \\ &\geq \varepsilon \ |\{m \in I_r : |\sigma(u_m, u_0) - \sigma(u_0, u_0)| \geq \varepsilon\}|\end{aligned}$$

and hence the result follows.

(ii) For inclusion to be proper, consider the following example:

Let $X = \mathbb{R}$ and σ be the metric-like defined by $\sigma(\xi, \eta) = |\xi - \eta|$; $\xi, \eta \in \mathbb{R}$. Consider a sequence as follows:

Take u_m to be $1, 2, 3, ..., [\sqrt{h_r}]$ at the first $[\sqrt{h_r}]$ integers on I_r and $u_m = u_0$ otherwise for all, $r = 1, 2, 3, ...$ where $u_0 \in [0, 1) - \{\frac{1}{2}\}$ and $[\cdot]$ denote the greatest integer function. Then for every $\varepsilon > 0$, $\dfrac{1}{h_r}|\{m \in I_r : |\sigma(u_m, u_0) - \sigma(u_0, u_0)| \geq \varepsilon\}| \leq \frac{1}{h_r}[\sqrt{h_r}] \to 0$ as $r \to \infty$ and so $(u_m) \in S_\theta(c_\sigma)$. On the other hand,

$$\begin{aligned}\frac{1}{h_r}\sum_{m\in I_r} |\sigma(u_m, u_0) - \sigma(u_0, u_0)| &= \frac{1}{h_r}\sum_{m\in I_r} |u_m - u_0| \\ &= \frac{1}{h_r}\left[|1 - u_0| + |2 - u_0| + ... + |[\sqrt{h_r}] - u_0|\right] \\ &= \frac{1}{h_r}\left(\frac{[\sqrt{h_r}]([\sqrt{h_r}] + 1)}{2} - [\sqrt{h_r}]u_0\right) \to \frac{1}{2} - 0 \ as \ r \to \infty\end{aligned}$$

implies $(u_m) \notin N_\theta(c_\sigma)$.

(iii) Let (u_m) be a σ-bounded and lacunary statistically σ-convergent to $u_0 \in X$. Then

$$\frac{1}{h_r}\sum_{m\in I_r}|\sigma(u_m, u_0) - \sigma(u_0, u_0)| = \frac{1}{h_r}\left(\sum_1 |\sigma(u_m, u_0) - \sigma(u_0, u_0)| + \sum_2 |\sigma(u_m, u_0) - \sigma(u_0, u_0)|\right)$$

As (u_m) is σ-bounded sequence, so there exists some $b \in X$ and $M > 0$ such that

$$|\sigma(u_m, b) - \sigma(b, b)| < M \ \forall \ m \geq 1.$$

Now $\sigma(u_m, u_0) \leq \sigma(u_m, b) + \sigma(b, u_0)$ by property (σ_2) of metric-like

$$i.e., |\sigma(u_m, u_0) - \sigma(u_0, u_0)| \leq T \text{ where } T = M + |\sigma(u_0, b) + \sigma(b, b) - \sigma(u_0, u_0)|,$$
$$i.e., \ |\sigma(u_m, u_0) - \sigma(u_0, u_0)| \leq T \text{ for all } m \geq 1$$

implies

$$\begin{aligned}
&\frac{1}{h_r} \sum_{m \in I_r} |\sigma(u_m, u_0) - \sigma(u_0, u_0)| \\
&\leq \frac{1}{h_r} \left(\sum_1 |\sigma(u_m, u_0) - \sigma(u_0, u_0)| + T\, |\{m \in I_r : |\sigma(u_m, u_0) - \sigma(u_0, u_0)| \geq \varepsilon\}| \right) \\
&\leq \frac{\varepsilon}{h_r} h_r + \frac{T}{h_r} . |\{m \in I_r : |\sigma(u_m, u_0) - \sigma(u_0, u_0)| \geq \varepsilon\}|
\end{aligned}$$

and hence the result follows.

□

Theorem 3.3. *Then* $S(c_\sigma) \subset S_\theta(c_\sigma)$ *if and only if* $\liminf_r q_r > 1$.

Proof. Let $\liminf_r q_r > 1$. Then $\exists\ \delta > 0$ such that $q_r > 1 + \delta$ for all $r \geq 1$. As $q_r = \frac{m_r}{m_{r-1}}$, so we have $\frac{h_r}{m_{r-1}} \geq \delta$, implies $\frac{m_{r-1}}{h_r} \leq \frac{1}{\delta}$. On adding 1 to both sides, we have $\frac{m_r}{h_r} \leq \frac{1+\delta}{\delta}$. Let $u_m \xrightarrow{\sigma} u_0(S)$, then for every $\varepsilon > 0$ and sufficiently large r, we get

$$\begin{aligned}
\frac{1}{m_r} |\{m < m_r : |\sigma(u_m, u_0) - \sigma(u_0, u_0)| \geq \varepsilon\}) &\geq \frac{1}{m_r} |\{m \in I_r : |\sigma(u_m, u_0) - \sigma(u_0, u_0)| \geq \varepsilon\}| \\
&\geq \frac{h_r}{m_r} \frac{1}{m_r} |\{m \in I_r : |\sigma(u_m, u_0) - \sigma(u_0, u_0)| \geq \varepsilon\}| \\
&\geq \frac{\delta}{1+\delta} \frac{1}{m_r} |\{m \in I_r : |\sigma(u_m, u_0) - \sigma(u_0, u_0)| \geq \varepsilon\}|
\end{aligned}$$

and hence the result follows.

Conversely, suppose $\liminf_r q_r = 1$. As $\theta = (m_r)$ is lacunary sequence, so following [9], we can select a subsequence $\left(m_{r_j}\right)$ of (m_r) satisfying $\frac{m_{r_j}}{m_{r_j - 1}} < 1 + \frac{1}{j}$ where $r_j > r_{j-1} + 2$. Let $X = \mathbb{R}$ be a metric-like space with metric-like defined as $\sigma(\xi, \eta) = |\xi - \eta|\ ;\ \xi, \eta \in \mathbb{R}$. Define

$$u_m = \begin{cases} 1 & \text{if } m \in I_{r_j}, \\ 0 & \text{otherwise.} \end{cases} \qquad for\ j = 1, 2, 3, ...$$

Then $\lim_{j \to \infty} \frac{1}{h_{r_j}} |\{m \in I_{r_j} : |\sigma(u_m, 0) - \sigma(0, 0)| \geq \varepsilon\}| = \lim_{j \to \infty} \frac{h_{r_j}}{h_{r_j}} = 1$ and for $r \neq r_j$; $\lim_{r \to \infty} \frac{1}{h_r} |\{m \in I_r : |\sigma(u_m, 0) - \sigma(0, 0)| \geq \varepsilon\}| = \lim_{r \to \infty} \frac{0}{h_r} = 0$, hence $(u_m) \notin S_\theta(c_\sigma)$. However for any sufficiently large integer

t, we can find the unique j such that $m_{r_j - 1} < t \le m_{r_{j+1} - 1}$ and

$$\begin{aligned}\frac{1}{t}|\{m < t : |\sigma(u_m, 0) - \sigma(0,0)| \ge \varepsilon\}| &= \frac{1}{t}|\{m < t : |u_m| \ge \varepsilon\}) \\ &\le \frac{m_{r_{j-1}} + h_{r_j}}{m_{r_j - 1}} \\ &\le \frac{m_{r_{j-1}}}{m_{r_j-1}} + \frac{h_{r_j}}{m_{r_j-1}} \\ &< \frac{1}{j} + \frac{1}{j} \to 0 \ as \ t \to \infty\end{aligned}$$

hence implies that $(u_m) \in S(c_\sigma)$. □

Theorem 3.4. *For any lacunary sequence $\theta = (m_r)$, $S_\theta(c_\sigma) \subset S(c_\sigma)$ iff* $\limsup_r q_r < \infty$.

Proof. Let $\limsup_r q_r < \infty$. Then there exists $M > 0$ such that $q_r < M$ for all $r \ge 1$. Suppose that $(u_m) \in S_\theta(c_\sigma)$. Then there exist $u_0 \in X$ and $\varepsilon > 0$ such that

$$\lim_{r\to\infty} \frac{1}{h_r}|\{m \in I_r : |\sigma(u_m, u_0) - \sigma(u_0, u_0)| \ge \varepsilon\}| = 0, \ i.e., \lim_{r\to\infty} \frac{M_r}{h_r} = 0$$

where $M_r = |\{m \in I_r : |\sigma(u_m, u_0) - \sigma(u_0, u_0)| \ge \varepsilon\}|$. So for given $\varepsilon > 0$, $\exists\, r_0 \in \mathbb{N}$ such that $\frac{M_r}{h_r} < \varepsilon$ for all $r > r_0$. Let $K = \sup\{M_r : 1 \le r \le r_0\}$ and n be any integer $m_{r-1} < n \le m_r$. Then we may have,

$$\begin{aligned}&\frac{1}{n}\,|\{m \le n : |\sigma(u_m, u_0) - \sigma(u_0, u_0)| \ge \varepsilon\}| \\ &< \frac{1}{m_{r-1}}\,|\{m \le m_r : |\sigma(u_m, u_0) - \sigma(u_0, u_0)| \ge \varepsilon\}| \\ &= \frac{1}{m_{r-1}}\,\{M_1 + M_2 + ... + M_{r_0} + M_{r_0+1} + ... + M_r\} \\ &\le \frac{r_0 K}{m_{r-1}} + \frac{1}{m_{r-1}}\{M_{r_0+1} + M_{r_0+2} + ... + M_r\} \\ &= \frac{r_0 K}{m_{r-1}} + \frac{1}{m_{r-1}}\left\{h_{r_0+1}\frac{M_{r_0+1}}{h_{r_0+1}} + h_{r_0+2}\frac{M_{r_0+2}}{h_{r_0+2}} + ... + h_r\frac{M_r}{h_r}\right\} \\ &\le \frac{r_0 K}{m_{r-1}} + \frac{1}{m_{r-1}}\left(\sup_{r>r_0}\frac{M_r}{h_r}\right)\{h_{r_0+1} + h_{r_0+2} + ... + h_r\} \\ &\le \frac{r_0 K}{m_{r-1}} + \frac{1}{m_{r-1}}\varepsilon\,(m_r - m_{r_0}) \\ &= \frac{r_0 K}{m_{r-1}} + \varepsilon\left(\frac{m_r}{m_{r-1}} - \frac{m_{r_0}}{m_{r-1}}\right) \\ &\le \frac{r_0 K}{m_{r-1}} + \varepsilon.M\end{aligned}$$

and hence the result follows.

Conversely, suppose that $\limsup_r q_r = \infty$. We can select a subsequence $\left(m_{r_j}\right)$ of the lacunary sequence $\theta = (m_r)$ such that $q_{r_j} > j$. Consider the metric-like space (X, σ) where $X = \mathbb{R}$ and $\sigma(\xi, \eta) =$

$|\xi-\eta|$; $\xi,\eta\in\mathbb{R}$. Define

$$u_i = \begin{cases} 1 & \text{if } m_{r_j-1} < i \leq 2m_{r_j-1}, \\ 0 & \text{otherwise.} \end{cases} \qquad for\ j=1,2,3,...$$

Then $\tau_{r_j} = \frac{1}{h_{r_j}}\sum_{i\in I_{r_j}} |\sigma(u_i,0)-\sigma(0,0)| = \frac{m_{r_j-1}}{h_{r_j}} = \frac{m_{r_j-1}}{m_{r_j}-m_{r_j-1}} < \frac{1}{j}$ and if $r\neq r_j$, $\tau_r = 0$. This implies $(u_i)\in N_\theta(c_\sigma)$ and hence in view of Theorem 3.2, $(u_i)\in S_\theta(c_\sigma)$. Now

$$\begin{aligned}\frac{1}{2m_{r_j-1}}\left|\left\{m\leq 2m_{r_j-1} : |\sigma(u_i,0)-\sigma(0,0)|\geq\varepsilon\right\}\right| &= \frac{1}{2m_{r_j-1}}\left|\left\{m\leq 2m_{r_j-1} : |u_i|\geq\varepsilon\right\}\right| \\ &\geq \frac{1}{2m_{r_j-1}}m_{r_j-1} = \frac{1}{2}\ and\ so\ (u_i)\notin S(c_\sigma).\end{aligned}$$

□

In view of Theorem 3.3 and Theorem 3.4, we have the following

Corollary 3.5. *For a lacunary sequence $\theta=(m_r)$, we have $S_\theta(c_\sigma)=S(c_\sigma)$ if and only if* $1<\liminf q_r<\limsup_r q_r<\infty$.

4. Result on Lacunary Refinement

The present section contains the results related to various inclusion relations which arises with the change of lacunary sequences.

Definition 4.1. *A lacunary sequence $\theta^*=(m_r^*)$ is said to be lacunary refinement of lacunary sequence $\theta=(m_r)$ if $(m_r^*)\supseteq(m_r)$.*

Throughout the section, we use $h_r^*=m_r^*-m_{r-1}^*$.

Theorem 4.2. *If θ^* is lacunary refinement of θ, then $(u_m)\notin N_\theta(c_\sigma)$ implies $(u_m)\notin N_{\theta^*}(c_\sigma)$.*

Proof. Let $(u_m)\notin N_\theta(c_\sigma)$. Then for any real number u_0,

$$\lim_{n\to\infty}\frac{1}{h_r}\sum_{m\in I_r}|\sigma(u_m,u_0)-\sigma(u_0,u_0)|\neq 0.$$

Then there exists $\varepsilon>0$ and a subsequence $\left(m_{r_j}\right)$ of (m_r) such that

$$\frac{1}{h_{r_j}}\sum_{m\in I_{r_j}}|\sigma(u_m,u_0)-\sigma(u_0,u_0)|\geq\varepsilon$$

Writing $I_{r_j}=I_{s+1}^*\cup I_{s+2}^*\cup...\cup I_{s+p}^*$, where $m_{r_j-1}=m_s^*<m_{s+1}^*<...<m_{s+p}^*=m_{r_j}$ we have

$$\frac{\sum_{I_{s+1}^*}|\sigma(u_i,u_0)-\sigma(u_0,u_0)|+\ldots+\sum_{I_{s+p}^*}|\sigma(u_i,u_0)-\sigma(u_0,u_0)|}{h_{s+1}^*+\ldots+h_{s+p}^*}\geq\varepsilon$$

This implies for some j, we have,

$$\frac{1}{h^*_{s+j}} \sum_{I^*_{s+j}} |\sigma(u_i, u_0) - \sigma(u_0, u_0)| \geq \varepsilon$$

and as a result $(u_i) \notin N_{\theta^*}(c_\sigma)$. □

Theorem 4.3. *If θ^* is lacunary refinement of θ and if $\lim_{r\to\infty} \inf \frac{h_r}{h^*_r} > 0$. Then $u_m \xrightarrow{\sigma} u_0\,(S_{\theta^*})$ implies $u_m \xrightarrow{\sigma} u_0\,(S_\theta)$.*

Proof. As $I^*_r \supseteq I_r$ for all $r \in \mathbb{N}$, so for $\varepsilon > 0$, we have

$$\{m \in I^*_r : |\sigma(u_m, u_0) - \sigma(u_0, u_0)| \geq \varepsilon\} \supseteq \{m \in I_r : |\sigma(u_m, u_0) - \sigma(u_0, u_0)| \geq \varepsilon\}.$$

This implies

$$\begin{aligned}\frac{1}{h^*_r} |\{m \in I^*_r : |\sigma(u_m, u_0) - \sigma(u_0, u_0)| \geq \varepsilon\}| &\geq \frac{1}{h^*_r} |\{m \in I_r : |\sigma(u_m, u_0) - \sigma(u_0, u_0)| \geq \varepsilon\}| \\ &= \left(\frac{h_r}{h^*_r}\right) \frac{1}{h_r} |\{m \in I_r : |\sigma(u_m, u_0) - \sigma(u_0, u_0)| \geq \varepsilon\}|.\end{aligned}$$

Hence the proof. □

Theorem 4.4. *If $\lim_{r\to\infty} \inf \frac{h_r}{h^*_r} > 0$, then $N_{\theta^*}(c_\sigma) \subseteq N_\theta(c_\sigma)$.*

Proof. Let $(u_m) \in N_{\theta^*}(c_\sigma)$. Then for given $\varepsilon > 0$ and $u_0 \in X$, we have

$$\begin{aligned}\frac{1}{h^*_r}\left(\sum_{m\in I^*_r} |\sigma(u_m, u_0) - \sigma(u_0, u_0)|\right) &= \frac{1}{h^*_r}\left(\sum_{m\in I^*_r - I_r} |\sigma(u_m, u_0) - \sigma(u_0, u_0)| + \sum_{m\in I_r} |\sigma(u_m, u_0) - \sigma(u_0, u_0)|\right) \\ &\geq \left(\frac{h_r}{h^*_r}\right) \frac{1}{h_r}\left(\sum_{m\in I_r} |\sigma(u_m, u_0) - \sigma(u_0, u_0)|\right)\end{aligned}$$

and hence the result follows from above inequality by taking limit $r \to \infty$. □

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