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Some Aspects of Statistical Convergence and Cesàro Summability of Order α in Metric Spaces

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Abstract

In the present paper, we extend the notions of ρ-statistically convergence and ρ-statistically boundedness of order α of scaler valued sequence to that of metric valued sequences from an arbitrary metric space (X, d), where $\rho = (\rho_n)$ is a non decreasing sequence of positive real numbers. Beside this, we also generalize and study the notion of ρ-strongly Cesàro summability of order α for metric valued sequences. Finally we also demonstrate various inclusion relations among these concepts by varying the sequence $\rho = (\rho_n)$.

Keywords: Statistical convergence, Cesàro summability, ρ-statistical convergence, metric space, statistical boundedness.

2020 Mathematics Subject Classification: 46A45, 40A05, 40A35.

1. Introduction

Statistical convergence, which is in fact, a generalizations of usual convergence of real sequences was introduced in short by Fast [9] in 1951. Later on this concept was studied as "convergence in density" by Buck [4] in 1953. It is also a part of monograph by Zygmund [25] and referred as almost convergence. Steinhaus [22] and Schoenberg [20] introduced and studied this concept independently in connection with summability of sequences. Later on this concept of statistical convergence and its various extensions have been explored by many more mathematicians [3,10,12,15,19]. The concept of statistical convergence depends on the density of subsets of $\mathbb{N}$ of natural numbers. The natural density of $D \subset \mathbb{N}$ is denoted by $\delta(D)$ and defined as [17]

$$\delta(D) = \lim_{n\to\infty} \frac{1}{n} |\{m \in D : m \leq n\}|,$$

provided the limit exists. It is easily verify that $\delta(D) = 0$ for finite $D \subseteq \mathbb{N}$ and $\delta(D) + \delta(\mathbb{N} - D) = 1$ for every $A \subseteq \mathbb{N}$.

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A real valued sequence (z_m) is said to be statistically convergent to $\ell \in \mathbb{R}$ if for each $\varepsilon > 0$,

$$\delta(\{|z_m - \ell| \geq \varepsilon\}) = 0,$$
$$i.e., \lim_{n\to\infty} \frac{1}{n} |\{m \leq n : |z_m - \ell| \geq \varepsilon\}| = 0$$

and we write $z_m \to \ell(S)$. The set of all statistically convergent real sequences is denoted by S.
With the passage of time, various generalization of this notion, that is, $\lambda-$statistical convergence, $A-$statistical convergence, $I-$convergence, rough statistical convergence, $\mu-$statistical convergence, lacunary statistical convergence, deferred statistical convergence, convergence of order α etc. have been appeared and also these generalizations have been studied for vector valued sequences by many more mathematicians. One may refer to [3,6,7,11,16,24].

Definition 1.1 ([1]). *A real valued sequence (z_m) is said to be ρ-statistical convergent to ℓ if for every $\varepsilon > 0$,*

$$\lim_{n\to\infty} \frac{1}{\rho_n} |\{m \leq n : |z_m - \ell| \geq \varepsilon\}| = 0$$

where, $\rho = (\rho_n)$ is a non-decreasing sequence of positive real numbers tending to infinity, such that

$$\limsup_n \frac{\rho_n}{n} < \infty,\ \Delta\rho_n = \rho_{n+1} - \rho_n = O(1) \tag{1}$$

for each positive integer n. In this case, we write $z_m \xrightarrow{S_\rho} \ell$ The set of all ρ-statistical convergent sequences of reals is denoted by S_ρ, i.e.,

$$S_\rho = \{(z_m) : z_m \to \ell(S_\rho)\ for\ some\ \ell\}.$$

As statistical convergence has its deep root in the notion of natural density, similar is the relation of ρ-statistical convergence with ρ-density.

Definition 1.2. *Let $\rho = (\rho_n)$ be a given sequence as defined in equation (1), then ρ-density of $D \subseteq \mathbb{N}$ is defined as*

$$\delta_\rho(D) = \lim_{n\to\infty} \frac{1}{\rho_n} |\{m \leq n : m \in D\}|.$$

Definition 1.3. *For a given sequence $\rho = (\rho_n)$ as defined above, if the set of indicies m 's for which (z_m) does not satisfy property P has zero ρ-density, then we say (z_m) satisfies P for "almost all m with respect to ρ" abbreviated as "a.a. m w.r.t. ρ"*

In the light of above definition, ρ-statistical convergence may also be defined as

Definition 1.4. *A real valued sequence (z_m) is said to be ρ-statistical convergent to $\ell \in \mathbb{R}$ if for given $\varepsilon > 0$,*

$$\delta_\rho(\{m \in \mathbb{N} : |z_m - \ell| \geq \varepsilon\}) = 0,\ \ i.e.,\ |z_m - \ell| < \varepsilon \quad a.a.\ m\ w.r.t\ \rho.$$

Definition 1.5. *Let X be a non-empty set. A function $d : X \times X \to \mathbb{R}$ satisfying the following*

(d_1) $d(x,y) \geq 0$

(d_2) $d(x,y) = 0 \Longleftrightarrow x = y$

(d_3) $d(x,y) = d(y,x)$

(d_4) $d(x,y) \leq d(x,z) + d(z,y)$ *for all* $x, y, z \in X$,

is said to be a metric on X *and* (X,d) *is called a metric space.*

Küçükaslan *et al*. [15] and Kayan *et al*. [14] introduced the notion of statistical convergence in metric space and we call this notion as statistical d-convergence.

Definition 1.6. *A sequence* (z_m) *in metric space* (X,p) *is said to be statistically d-convergent to some* $z_0 \in X$ *if for given* $\varepsilon > 0$,

$$\delta(\{m \in \mathbb{N} : d(z_m, z_0) \geq \varepsilon\}) = 0$$

and in this case we write $z_m \xrightarrow{d} u_0(S)$. *The class of all statistically convergent sequence in* (X,d) *is denoted by* S_d.

Definition 1.7. *A sequence* (z_m) *in a metric space* (X,d) *is said to be d-bounded if there exists* $M > 0$ *such that* $d(z_m, z_0) < M$ *for all* $m \geq 1$ *and* $z_0 \in X$.

In the present study, we extend the notion of ρ-statistical convergence of order α in metric space (X,d) by introducing the notion of ρ-statistical d-convergence and strongly $d\rho$-Cesàro summability of order α. Apart from this, we also establish various inclusion relations between ρ-statistical d-convergence of order α and ρ-statistical d-boundedness of order α in a metric space (X,d).
Let E be a sequence space in metric space (X,d). Motivating from [3,8,13], we have the following

(i) A sequence space E with metric d is said to be *solid (normal)* if $(\xi_m) \in E$ and (η_m) be a sequence such that $d(\eta_m, a) \leq d(\xi_m, a)$, $m \geq 1$ for all $a \in X$ implies $(\eta_m) \in E$,

(ii) *Monotone*, if it contains the canonical preimages of all its stepspaces,

(iii) *Symmetric*, if $(\eta_m) \in E$ implies $(\eta_{\sigma_m}) \in E$, where (σ_m) is permutation on m.

In this paper, we study the sequence space by using sequences from an arbitrary non-empty set X, equipped with a metric d and (X,d) will denote the metric space throughout the paper.

2. ρ-Statistical d-Convergence of Order α in (X,d)

We start this section by introducing the notion of ρ-statistical d-convergence of order α and ρ-statistical d-boundedness of order α for sequences from an arbitrary metric space (X,d) and establish various inclusion relations between these two concepts.

Definition 2.1. *Let (X,d) be a metric space and $\rho = (\rho_n)$ be the sequence satisfying (1) conditions. Let $0 < \alpha \leq 1$, then a metric valued sequence (z_m) is said to be ρ-statistically d-convergent of order α to $z_0 \in X$ if for $\varepsilon > 0$,*

$$\lim_{n\to\infty} \frac{1}{\rho_n^\alpha} |\{m \leq n : d(z_m, z_0) \geq \varepsilon\}| = 0$$

and in this case we write $z_m \xrightarrow{S_{d\rho}^\alpha} z_0$. By $S_{d\rho}^\alpha$ we shall denote the class of all ρ-statistically d-convergent sequences of order α.

Definition 2.2. *Let $\rho = (\rho_n)$ be the sequence defined above, then (α,ρ)-density (or $\rho_\alpha-$density) of $D \subseteq \mathbb{N}$ is defined as $\delta_\rho^\alpha(D) = \lim_{n\to\infty} \frac{1}{\rho_n^\alpha} |\{m \in \mathbb{N} : m \in D\}|$.*

Definition 2.3. *Let $\rho = (\rho_n)$ be the sequence defined above, if the set of indicies $m^{'}s$ for which (z_m) does not satisfy property P has zero (α,ρ)-density, then we say (z_m) satisfies P for "almost all m with respect to ρ_α" abbreviated as "a.a. m w.r.t. ρ_α."*

In view of above definition ρ-statistical d-convergence of order α is redefined as

Definition 2.4. *Let $\rho = (\rho_n)$ be the sequence satisfying (1) condition and $0 < \alpha \leq 1$. A metric valued sequence $(z_m) \in X$ is said to be ρ-statistically d-convergent of order α to $z_0 \in X$ if for $\varepsilon > 0$,*

$$\delta_\rho^\alpha(\{m \leq n : d(z_m, z_0) \geq \varepsilon\}) = 0, \quad i.e., d(z_m, z_0) < \varepsilon \quad a.a.\ m\ w.r.t. \quad \rho_\alpha.$$

Remark 2.5. *For particular choice of parameter α and the sequence $\rho = (\rho_n)$, we obtained the following special cases*

(i) If $(\rho_n) = n$, then $S_{d\rho}^\alpha$ is reduced to S_d^α, which was studied and defined by Kayan et al. [14].

(ii) If $(\rho_n) = n$ and $\alpha = 1$, then we write S_d instead of $S_{d\rho}^\alpha$, which was studied and defined by Küçükaslan et al. [15]

(iii) If we take $X = \mathbb{R}$ with usual metric $d(x,y) = |x-y|$ and $(\rho_n) = n$, then we write S^α instead of $S_{d\rho}^\alpha$, which was defined and studied by Çolak [5].

(iv) If $(\rho_n) = n$, $\alpha = 1$ and we take $X = \mathbb{R}$ with usual metric $d(x,y) = |x-y|$, then we write S instead of $S_{d\rho}^\alpha$, which was defined and studied and defined by Šalát [19].

Definition 2.6. *Let $\rho = (\rho_n)$ be the sequence defined above and $0 < \alpha \leq 1$. A sequence (z_m) in (X,d) is said to be ρ-statistically d-bounded of order α if there exist a real number M and $z_0 \in X$ such that*

$$\lim_{n\to\infty} \frac{1}{\rho_n^\alpha} |\{m \leq n : d(z_m, z_0) \geq M\}| = 0, \quad i.e., \quad d(z_m, z_0) < M \quad a.a.\ m\ w.r.t.\ \rho_\alpha$$

and the class of all ρ-statistically d-bounded sequences of order α is denoted by $BS_{d\rho}^\alpha$.

Theorem 2.7. *Every convergent sequence is ρ-statistically d-convergent of order α in a metric space, converse need not be true.*

Proof. Let (z_m) be a convergent sequence, then for given $\varepsilon > 0$ and $z_0 \in X$ there exist $n_0 \in \mathbb{N}$ such that $d(z_m, z_0) < \varepsilon$ for all $m \geq n_0$. Hence

$$\lim_{n\to\infty} \frac{1}{\rho_n^\alpha} |\{m \leq n : d(z_m, z_0) \geq \varepsilon\}| = 0$$

since $\{m \leq n : d(z_m, z_0) \geq \varepsilon\}$ is a finite set and hence has zero ρ_α-density. To show that inclusion is strict, choose $(\rho_n) = n$, $X = \mathbb{R}$, $d(x, y) = |x - y|$, $\alpha = 1$ and define a sequence (z_m) as

$$z_m = \begin{cases} n & \text{for } m = n^2 \\ 0 & \text{for } m \neq n^2 \end{cases} \qquad n \in \mathbb{N}. .$$

Clearly (z_m) is not convergent but it is ρ-statistically d-convergent of order α. □

Theorem 2.8. *Every bounded sequence is ρ-statistically d-bounded of order α in a metric space, but converse may not be true.*

Proof. Following the same technique as in Theorem 2.7, the proof follows, hence omitted. □

Theorem 2.9. $S_{d\rho}^{\alpha} \subseteq BS_{d\rho}^{\alpha}$, *i.e., ρ-statistically d-convergence of order α implies ρ-statistically d-boundedness of order α. Converse may not be true in general.*

Proof. Let $\rho = (\rho_n)$ be the sequence defined above and (z_m) be a ρ-statistically d-convergent of order α to some $z_0 \in X$. Then for given $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{\rho_n^\alpha} |\{m \leq n : d(z_m, z_0) \geq \varepsilon\}| = 0.$$

Now for sufficiently large M, we have,

$$|\{m \leq n : d(z_m, z_0) > M\}| \leq |\{m \leq n : d(z_m, z_0) \geq \varepsilon\}|$$

and taking the limit $n \to \infty$ the result follows.
To show the containment is proper, let $\rho = (\rho_n) = n$, $X = \mathbb{R}$ with $d(x, y) = |x - y|$; $x, y \in \mathbb{R}$ and $\alpha = 1$. Define the sequence (z_m) in X as

$$z_m = \begin{cases} -1 & \text{for } m = 2n \\ 1 & \text{for } m \neq 2n \end{cases} \qquad n \in \mathbb{N}.$$

It is clear that (z_m) is ρ-statistically d-bounded of order α but not ρ-statistically d-convergent of order α. □

Theorem 2.10. *Let α, β are the real number such that $0 < \alpha \leq \beta \leq 1$. Then $S_{d\rho}^{\alpha} \subset S_{d\rho}^{\beta}$, converse need not be true in general.*

Proof. The results follows from the inequality

$$0 \leq \frac{1}{\rho_n^{\beta}} \left|\{m \leq n : d(z_m, z_0) \geq \varepsilon\}\right| \leq \frac{1}{\rho_n^{\alpha}} \left|\{m \leq n : d(z_m, z_0) \geq \varepsilon\}\right|.$$

In order to show that inclusion is proper, we consider the following example:
Let $\rho_n = n$ and $X = \mathbb{R}$ with metric d defined as $d(x, y) = |x - y| \;\; ; \;\; x, y \in \mathbb{R}$. Consider a sequence (z_m) as

$$z_m = \begin{cases} n & \text{for } m = n^2 \\ 0 & \text{otherwise,} \end{cases}$$

implies, $\left|\{m \leq n : d(z_m, z_0) \geq \varepsilon\}\right| \leq \sqrt{n}$. Thus if we consider $\beta \in \left(\frac{1}{2}, 1\right]$, then we have

$$\lim_{n\to\infty} \frac{1}{\rho_n^{\beta}} \left|\{m \leq n : d(z_m, 0) \geq \varepsilon\}\right| \leq \lim_{n\to\infty} \frac{\sqrt{n}}{\rho_n^{\beta}} \longrightarrow 0.$$

While on other side for $0 < \alpha < \frac{1}{2}$, we have

$$\lim_{n\to\infty} \frac{1}{\rho_n^{\alpha}} \left|\{m \leq n : d(z_m, 0) \geq \varepsilon\}\right| \leq \lim_{n\to\infty} \frac{\sqrt{n}}{\rho_n^{\alpha}} \nrightarrow 0.$$

Implies inclusion is strict for $\alpha < \beta$ with $0 < \alpha < \frac{1}{2}$ and $\frac{1}{2} < \beta \leq 1$. □

Corollary 2.11. *Let $\rho = (\rho_n)$ be the sequence defined above and $0 < \alpha \leq \beta \leq 1$. Then we have following*

(i) $S_{d\rho}^{\alpha} = S_{d\rho}^{\beta}$ *iff* $\alpha = \beta$.

(ii) $S_{d\rho}^{\alpha} = S_{d\rho}$ *iff* $\alpha = 1$.

Theorem 2.12. *Let $\rho = (\rho_n)$ be the sequence satisfying* (1) *conditions and $0 < \alpha \leq 1$. Then we have following*

(i) $BS_{d\rho}^{\alpha}$ *is not symmetric.*

(ii) $BS_{d\rho}^{\alpha}$ *is normal.*

(iii) $BS_{d\rho}^{\alpha}$ *is monotone.*

Proof. (i) In order to show that $BS_{d\rho}^{\alpha}$ is not symmetric, let $\rho = (\rho_n) = n$, $X = \mathbb{R}$ with $d(\xi, \eta) = |\xi - \eta|$ and $\alpha = 1$.

Take the sequence $(z_m) = (1, 0, 0, 2, 0, 0, 0, 0, 3, 0, 0, 0, 0, 0, 0, 4, \ldots)$. Then $(z_m) \in BS_{d\rho}^{\alpha}$. Let (z'_m) be a sequence obtained by rearrangement of (z_m) as follows

$$(z'_m) = (z_1, z_2, z_4, z_3, z_9, z_5, z_{16}, z_6, z_{25}, z_7, z_{36}, z_8, z_{49}, z_{10}, \ldots)$$

$$= (1, 0, 2, 0, 3, 0, 4, 0, 5, 0, 6, 0, 7, 0, 8, 0, \ldots).$$

Then for any $M > 0$, we have

$$\lim_{n\to\infty} \frac{1}{\rho_n^\alpha} |\{m \leq n : d(z'_m, z_0) \geq M\}| \neq 0$$

so $(z'_m) \notin BS_{d\rho}^\alpha$ and hence $BS_{d\rho}^\alpha$ is not symmetric.

(ii) Let $(z_m) \in BS_{d\rho}^\alpha$ so there exists $M > 0$ such that

$$\lim_{n\to\infty} \frac{1}{\rho_n^\alpha} |\{m \leq n : d(z_m, z_0) \geq M\}| = 0 \quad \text{for all} \quad z_0 \in X.$$

Let (z'_m) be a sequence such that $d(z'_m, a) \leq d(z_m, a)$ for all $m \in \mathbb{N}$ and for all $a \in X$. Then clearly

$$|\{m : |d(z'_m, z_0) \geq M\}| \leq |\{m \ : d(z_m, z_0) \geq M\}|$$

so $(z'_m) \in BS_{d\rho}^\alpha$. Hence $BS_{d\rho}^\alpha$ is normal.

(iii) It is well known that every normal sequence space is monotone, hence $BS_{d\rho}^\alpha$ is monotone.

□

Theorem 2.13. *A metric valued sequence (z_m) is ρ-statistically d-convergent of order α to some $z_0 \in X$ iff there exists a sequence (y_m), convergent to z_0 such that $y_m = z_m$ a.a. m w.r.t. ρ_α.*

Proof. Let $z_m \xrightarrow{S_{d\rho}^\alpha} z_0$, so for given $\varepsilon > 0$, we have $\delta_\rho^\alpha(K) = 0$, where $K = \{m \in \mathbb{N} : d(z_m, z_0) \geq \varepsilon\}$. Let

$$y_m = \begin{cases} z_m & \text{for } m \in \mathbb{N} - K \\ z_0 & \text{for } m \in K. \end{cases}$$

Now $\{m \in \mathbb{N} : y_m \neq z_m\} \subseteq K$ and so $y_m = z_m$ *a.a. m w.r.t.* ρ_α. Also

$$d(y_m, z_0) = \begin{cases} d(z_m, z_0) & \text{for } m \in \mathbb{N} - K \\ d(z_0, z_0) & \text{for } m \in K \end{cases}$$

implies $d(y_m, z_0) < \varepsilon$ for all $m \geq 1$. Thus (y_m) is convergent sequence and $y_m = z_m$ *a.a. m w.r.t.* ρ_α. Conversely, let there exists $m_0 \in \mathbb{N}$ such that $d(y_m, z_0) < \varepsilon$ for all $m \geq m_0$. Now

$$\{m : d(z_m, z_0) \geq \varepsilon\} \subseteq K \cup \{1, 2, 3, \ldots, m_0 - 1\}.$$

Hence the result follows in view of above inclusion relation. □

Following on the similar lines, we have

Theorem 2.14. *A metric valued sequence (z_m) is ρ-statistically d-bounded of order α iff there exists a bounded sequence (y_m) such that $y_m = z_m$ a.a. m w.r.t ρ_α.*

Definition 2.15. *A sequence (z_{m_n}), $n \in \mathbb{N}$ is said to be ρ_α-statistical dense if $\delta_\rho^\alpha(B) = 1$, where $B = \{m_1 < m_2 < m_3 < \ldots\}$, i.e., $\lim_{n\to\infty} \frac{1}{\rho_n^\alpha} |\{m \leq n : m \in B\}| = 1$.*

In order to characterizes ρ-statistically d-convergent sequence of order α in term of ρ_α-statistically dense convergent subsequences, we have the following

Theorem 2.16. *A sequence (z_m) is ρ-statistically d-convergent of order α iff every ρ_α-statistically dense subsequence of (z_m) is ρ-statistically d-convergent of order α.*

Proof. Let $z_m \xrightarrow{S_{d\rho}^\alpha} z_0$, so for given $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \frac{1}{\rho_n^\alpha} |\{m \leq n : d(z_m, z_0) \geq \varepsilon\}| = 0,$$

i.e., $\delta_\rho^\alpha(A) = 0$ where $A = \{m \in \mathbb{N} : d(z_m, z_0) \geq \varepsilon\}$ and suppose $(z_{m_n})_{n\in\mathbb{N}}$ is ρ-statistically dense subsequence of (z_m) which is not ρ-statistically d-convergent of order α, i.e.,

$$\liminf_{n\to\infty} \frac{1}{\rho_n^\alpha} |\{m_n \leq n : d(z_{m_n}, z_0) \geq \varepsilon\}| = \lambda, \text{ where } \lambda \in (0, 1).$$

Now

$$|\{m \leq n : d(z_m, z_0) \geq \varepsilon\}| \geq |\{m_n \leq n : d(z_{m_n}, z_0) \geq \varepsilon\}|.$$

So $\liminf_{n\to\infty} \frac{1}{\rho_n^\alpha} |\{m \leq n : d(z_m, z_0) \geq \varepsilon\}| \geq \lambda \neq 0$, a contradiction to given.
Converse part follows from the fact that every sequence is a α-lacunary statistical dense subsequence of itself. □

Theorem 2.17. *A sequence (z_m) is ρ-statistically d-bounded of order α iff every ρ_α-statistically dense subsequence of (z_m) is ρ-statistically d-bounded of order α.*

Proof. The proof follows by using the same technique as in Theorem 2.16 and hence omitted. □

3. Strongly $d\rho$-Cesàro Summability of Order α in (X, d)

In the present section, we introduce and study the notion of Strongly $d\rho$-Cesàro summability of order α in a metric space (X, d). Apart this some inclusion relations are also established by varying the sequence $\rho = (\rho_n)$ in metric space.

Definition 3.1. *Let $0 < \alpha \leq 1$. A metric sequence (z_m) is said to be strongly $d\rho$-Cesàro summable of order α to $z_0 \in X$ if*

$$\lim_{n\to\infty} \frac{1}{\rho_n^\alpha} \sum_{m=1}^{n} d(z_m, z_0) = 0$$

and we write $z_m \xrightarrow{W^{\alpha}_{d\rho}} z_0$. *The set of all strongly* $d\rho$*-Cesàro summable sequences of order* α *in metric space* (X,d) *is denoted by* $W^{\alpha}_{d\rho}$. *We write* $W^{\alpha}_{d\rho}$ *for* $\alpha = 1$ *as* $W_{d\rho}$.

Theorem 3.2. *Let* $\alpha \in (0,1]$, *then* $W^{\alpha}_{d\rho} \subset S^{\alpha}_{d\rho}$, *i.e., every strongly* $d\rho$*-Cesàro summable sequence of order* α *is* ρ*-statistically d-convergent of order* α *to same limit.*

Proof. Let $(z_m) \in W^{\alpha}_{d\rho}$, so for given $\varepsilon > 0$ and $z_0 \in X$, we have

$$\lim_{n\to\infty} \frac{1}{\rho_n^{\alpha}} \sum_{m=1}^{n} d(z_m, z_0) = 0.$$

Now for $0 < \alpha \leq 1$, we have the following

$$\begin{aligned}
\frac{1}{\rho_n^{\alpha}} \sum_{m=1}^{n} d(z_m, z_0) &= \frac{1}{\rho_n^{\alpha}} \sum_{\substack{m=1n \\ d(z_m,z_0)\geq\varepsilon}}^{n} d(z_m, z_0) + \frac{1}{\rho_n^{\alpha}} \sum_{\substack{m=1n \\ d(z_m,z_0)<\varepsilon}}^{n} d(z_m, z_0) \\
&\geq \frac{1}{\rho_n^{\alpha}} \sum_{\substack{m=1 \\ d(z_m,z_0)\geq\varepsilon}}^{n} d(z_m, z_0) \\
&\geq \varepsilon . \frac{1}{\rho_n^{\alpha}} |\{m \leq n : d(z_m, z_0) \geq \varepsilon\}|
\end{aligned}$$

and hence the result follows by applying the limit $n \to \infty$ in the above inequality. □

The following theorem show that the converse of Theorem 3.2 is also holds under some restrictions.

Theorem 3.3. *Every bounded* ρ*-statistically d-convergent sequence of order* α *is* $d\rho$*-Cesàro summable of order* α, *if* $\lim_{n\to\infty} \frac{\rho_n}{\rho_n^{\alpha}} = 1$.

Proof. Let $(z_m) \in X$ be a bounded sequence, so there exists $M > 0$ such that $d(z_m, a) \leq M$ for all $m \geq 1$ and for all $a \in X$. Let $(z_m) \in S^{\alpha}_{d\rho}$, then for $z_0 \in X$ and $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \frac{1}{\rho_n^{\alpha}} |\{m \leq n : d(z_m, z_0) \geq \varepsilon\}| = 0.$$

Now

$$\begin{aligned}
\frac{1}{\rho_n^{\alpha}} \sum_{m=1}^{n} d(z_m, z_0) &= \frac{1}{\rho_n^{\alpha}} \sum_{\substack{m=1 \\ d(z_m,z_0)\geq\varepsilon}}^{n} d(z_m, z_0) + \frac{1}{\rho_n^{\alpha}} \sum_{\substack{m=1 \\ d(z_m,z_0)<\varepsilon}}^{n} d(z_m, z_0) \\
&\leq \frac{M}{\rho_n^{\alpha}} \sum_{\substack{m=1 \\ d(z_m,z_0)\geq\varepsilon}}^{n} 1 + \frac{\varepsilon}{\rho_n^{\alpha}} \sum_{\substack{m=1 \\ d(z_m,z_0)<\varepsilon}}^{n} 1 \\
&\leq \frac{M}{\rho_n^{\alpha}} |\{m \leq n : d(z_m, z_0) \geq \varepsilon\}| + \varepsilon . \frac{n}{\rho_n^{\alpha}} \\
&\leq \frac{M}{\rho_n^{\alpha}} |\{m \leq n : d(z_m, z_0) \geq \varepsilon\}| + \varepsilon . \frac{\rho_n}{\rho_n^{\alpha}}
\end{aligned}$$

and hence the result follows from above inequality using limit $n \to \infty$. □

In view of Theorem 3.2 and Theorem 3.3, we have

Corollary 3.4. $\ell_\infty \bigcap S^\alpha_{d\rho} = \ell_\infty \bigcap W^\alpha_{d\rho}$.

Theorem 3.5. *Let $\rho = (\rho_n)$ and $\tau = (\tau_n)$ be two sequence such that $\rho_n \leq \tau_n$ for all $n \in \mathbb{N}$, and $0 < \alpha \leq \beta \leq 1$ if $\liminf_{n\to\infty} \frac{\rho_n}{\tau_n} > 0$, then $S^\alpha_{d\tau} \subseteq S^\beta_{d\rho}$.*

Proof. As $\rho \leq \tau_n$ for all $n \in \mathbb{N}$, so for $\varepsilon > 0$, we have

$$\begin{aligned}\frac{1}{\rho_n^\alpha} |\{m \leq n : d(z_m, z_0) \geq \varepsilon\}| &\geq \frac{1}{\tau_n^\alpha} |\{m \leq n : d(z_m, z_0) \geq \varepsilon\}| \\ &\geq \frac{1}{\tau_n^\beta} |\{m \leq n : d(z_m, z_0) \geq \varepsilon\}| \\ &= \left(\frac{\rho_n}{\tau_n}\right)^\beta \frac{1}{\rho_n^\beta} |\{m \leq n : d(z_m, z_0) \geq \varepsilon\}|.\end{aligned}$$

Hence the proof follows from above inequality. □

Corollary 3.6. *If $0 < \alpha \leq 1$ and $\liminf \frac{\rho_n}{\tau_n} > 0$, then*

(i) $S^\alpha_{d\tau} \subseteq S_{d\rho}$.

(ii) $S_{d\tau} \subseteq S_{d\rho}$.

Theorem 3.7. *Let $\rho = (\rho_n)$ and $\tau = (\tau_n)$ be two sequence such that $\rho_n \leq \tau_n$, $0 < \alpha \leq \beta \leq 1$ if $\lim_{n\to\infty} \inf \frac{\rho_n}{\tau_n} > 0$, then $W^\alpha_{d\tau} \subseteq W^\beta_{d\rho}$.*

Proof. The proof is obtained from the following inequality

$$\frac{1}{\rho_n^\alpha} \sum_{m=1}^{n} d(z_m, z_0) \leq \left(\frac{\rho_n}{\tau_n}\right)^\beta \frac{1}{\rho_n^\beta} \sum_{m=1}^{n} d(z_m, z_0)$$

□

Corollary 3.8. *If $0 < \alpha \leq 1$ and $\liminf \frac{\rho_n}{\tau_n} > 0$, then*

(i) $W^\alpha_{d\tau} \subseteq W_{d\rho}$.

(ii) $W_{d\tau} \subseteq W_{d\rho}$.

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