Int. J. Math. And Appl., 8(1)(2020), 101–105
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of *Mathematics And its Applications*

Generalized Method to Find the Generators of Matrix Algebras when its Dimension 2 and 3

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Abstract: Let A be an algebraically closed field of characteristic zero and consider a set of 2×2 or 3×3 matrices. Using a theorem of Shemesh, we give conditions for when the matrices in the set generate the full matrix algebra.

Keywords: Generator, Matrix, Common left eigen vector, Sub Algebra.

1. Introduction

Let A be an algebraically closed field of characteristic zero, and let $N_m = N_m(A)$ be the algebra of $m \times m$ matrices over A. Given a set $K = \{B_1, \ldots, B_t\}$ of $m \times m$ matrices, we would like to have conditions for when the A_i generate the algebra M_n. In other words, determine whether every matrix in N_m can be written in the form $T(B_1, \ldots, B_t)$, where T is a non-commutative polynomial. (We identify scalars with scalar matrices so the constant polynomials give the scalar matrices.) The case $m = 1$ is of course trivial, and when $t = 1$, the single matrix B_1 generates a commutative sub algebra. We therefore assume that $m, t \geq 2$. This question has been studied by many authors, see for example the extensive bibliography in [7]. We will give some generalize in the case of $m = 2$ or 3.

2. General Observations

Let G be the algebra generated by K. If we could show that the dimension of G as a vector space is m^2, it would follow that $G = N_m$. This can sometimes be done when we know a linear spanning set $H = \{H_1, \ldots, H_q\}$ of G. Let N be the $m^2 \times q$ matrix obtained by writing the matrices in H as column vectors. We would like to show that rank $N = m^2$. Since N is an $m^2 \times m^2$ matrix and rank $N = rank(NN^*)$, it sufficient to show that $\det(NN^*) \neq 0$. Unfortunately, the size of H may be big [4]. In this paper we will combine this method with results of Shemesh and Spencer, Rivlin, Aslaksen and Sletsjoe to get some simple results for $m = 2$ or 3.

Lemma 2.1. *Let $\{B_1, \ldots, B_t\}$ be a set of matrices in N_m where $m = 2$ or 3. The b_i's generate N_m if and only if they do not have a common eigenvector.*

We can therefore use the following theorem due to Shemesh [5].

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Theorem 2.2. *Two $m \times m$ matrices, B and H, have a common eigenvector if and only if*

$$\sum_{u,v=1}^{m-1} [B^u, H^v]^* [B^u, H^v]$$

is singular.

Adding scalar matrices to the B_i's does not change the subalgebra they generate, so we some-times assume that our matrices lie in $W_M = \{N \in N_m | \; trace \; N = 0\}$. We also sometimes identify matrices in N_m with vectors in A_{m^2}, and if $M_1, \ldots, M_{m^2} \in N_m$, then $\det(M_1, \ldots, M_{m^2})$ denotes the determinant of the $m^2 \times m^2$ matrix whose j$^{\text{th}}$ column is M_j, written as $(M_{j1}, \ldots, M_{jm})^t$, where M_{jk} is the k$^{\text{th}}$ row of M_j for $k = 1, 2, \ldots, n$. We write the scalar matrix aI as a. When we say that a set of matrices generate N_m, we are talking about N_m as an algebra, while when we say that a set of matrices form a basis of N_m, we are talking about N_m as a vector space.

3. The 2×2 Case

The following theorem is well-known, but we include a proof since it illustrated a technique we will use in the 3×3 case. Notice that the proof gives us an explicit basis for N_2.

Theorem 3.1. *Let $B, H \in N_2$. B and H generate N_2 if and only if $[B, H]$ is invertible.*

Proof. We know that in matrix $BH = -HB$, then a direct computation shows that

$$\det(I, B, H, BH) = -\det(I, B, H, HB) = \det[B, H].$$

Hence

$$\det(I, B, H, [B, H]) = 2 \det[B, H] \tag{1}$$

But if $I, B, H, [B, H]$ are linearly independent, then the dimension of G as a vector space is 4, so B and H generate N_2. We call $[N, M, T] = [N, [M, T]]$ a double commutator. The characteristic polynomial of A can be written as

$$\lambda^2 - (trace\ B)\lambda + ((trace\ B)^2 - trace\ B^2)/2.$$

It follows that the discriminant of the characteristic polynomial of A can be written as discriminant $(B) = 2$ trace $B^2 - (trace\ B)^2$. □

Lemma 3.2. *Let $B, H, G \in N_2$ and suppose that no two of them generate N_2. Then B, H, G generate N_2 if and only if the double commutator $[B, H, G] = [B, [H, G]]$ is invertible.*

Proof. A direct computation shows that

$$\det(I, B, H, G)^2 = -\det[B, [H, G]] - discriminant(B) \det[H, G] \tag{2}$$

But if I, B, H, G are linearly independent, then B, H and G generate N_2. □

Notice that the above proof gives us an explicit basis for N_2. We can now give a complete solution for the case $m = 2$.

Theorem 3.3. *The matrices $B_1, \ldots, B_t \in N_2$ generate N_2 if and only if at least one of the commutators $[B_i, B_j]$ or double commutators $[B_i, B_j, B_k] = [B_i, [B_j, B_k]]$ is invertible.*

Proof. If $t > 4$, the matrices are linearly dependent, so we can assume that $t \leq 4$. Suppose that B_1, B_2, B_3, B_4 generate N_2, but that no proper subset of them generates N_2. Then the four matrices are linearly independent, and we can write the identity I as a linear combination of them. If the coefficient of B_4 in this expression is nonzero, then B_1, B_2, B_3, I span and therefore generate N_2, so B_1, B_2, B_3 generate N_2. Thus, if $B_1, \ldots, B_t$ generate N_2, we can always find a subset of three of these matrices that generate N_2. □

4. Two 3×3 Matrices

In the case of two 3×3 matrices, we have the following well-known theorem.

Theorem 4.1. *Let $B, H \in N_3$. If $[B, H]$ is invertible, then B and H generate N_3.*

For $N \in N_3$, we define $L(N)$ to be the linear term in the characteristic polynomial of N. Hence $L(N) = ((trace\ N)^2 - trace\ N^2)/2$, which is equal to the sum of the three principal minors of degree two of N. Notice that $L(N)$ is invariant under conjugation, and that if $[B, H]$ is singular, then $[B, H]$ is nilpotent if and only if $L([B, H]) = 0$. The following theorem shows that if $[B, H]$ is invertible and $L([B, H]) \neq 0$, then we can give an explicit basis for N_3.

Theorem 4.2. *Let $B, H \in N_3$. Then*

$$\det(I, B, B^2, H, H^2, BH, HB, [B, [B, H]], [H, [H, B]]) = 9 \det[B, H] L([B, H]), \tag{3}$$

so if $\det[B, H] \neq 0$ *and* $L([B, H]) \neq 0$, *then* $\{I, B, B^2, H, H^2, BH, HB, [B, [B, H]], [H, [H, B]]\}$ *form a basis for* N_3.

The proof of (3) is by direct computation. Notice that this can be thought of as a generalization of (1) and (2). We can also use Shemesh's Theorem to characterize pairs of generators for N_3.

Theorem 4.3. *The two 3×3 matrices B and H generate N_3 if and only if both*

$$\sum_{u,v=1}^{m-1} [B^u, H^v]^* [B^u, H^v] \quad \text{and} \quad \sum_{u,v=1}^{m-1} [B^u, H^v][B^u, H^v]^*$$

are invertible.

5. Three or More 3×3 Matrices

We start with the following theorem due to Laffey [6].

Theorem 5.1. *Let K be a set of generators for N_3. If K has more than four elements, then N_3 can be generated by a proper subset of K.*

It is therefore sufficient to consider the cases $t = 3$ or 4. Following the approach outlined earlier, we start by finding a linear spanning set. Using the polarized Cayley-Hamilton Theorem, Spencer and Rivlin [1, 2] deduced the following theorem.

Theorem 5.2. *Let $B, H, G \in N_3$. Define*

$$K(B) = \{B, B^2\}$$
$$S(B, H) = \{H, B^2H, BH^2, B^2H^2, B^2HB, B^2H^2A\}$$
$$K(B_1, B_2) = S(B_1, B_2) \cup S(B_2, B_1)$$

$$S(B,H,G) = \{BHG, B^2HG, HB^2G, HGB^2, B^2H^2G, GB^2H^2, BHGB^2\}$$

$$K(B_1,B_2,B_3) = \bigcup_{\alpha\in K_3} T(B_\sigma(1), B_\sigma(2), B_\sigma(3)).$$

1. *The sub algebra generated by B and H is spanned by* $I \cup K(B) \cup K(H) \cup K(B,H)$.

2. *The sub algebra generated by B, H and G is spanned by* $I \cup K(B) \cup K(H) \cup K(B,H) \cup K(B,H,G)$.

These spanning sets are not optimal. They include words of length 5. Paz [3] has proved that N_m can be generated by words of length $[m^2+2]/3$. For N_3 this gives words of length 4. The general bound has been improved by Pappacena [4]. We next give a version of Shemesh's Theorem for three 3×3 matrices.

Theorem 5.3. *The matrices* $B,H,G \in N_3$ *have a common eigenvector if and only the matrix*

$$N(B,H,G) = \sum_{\substack{N\in K(B),\\ M\in K(H)}} [N,M]^*[N,M] + \sum_{\substack{N\in K(B),\\ M\in K(G)}} [N,M]^*[N,M] + \sum_{\substack{N\in K(H),\\ M\in K(G)}} [N,M]^*[N,M] + \sum_{\substack{N\in K(B,H),\\ M\in K(G)}} [N,M]^*[N,M]$$

is singular.

Proof. Let G be the algebra generated by B,H,G. Set

$$X = \bigcap_{\substack{N\in K(B),\\ M\in K(H)}} ker[N,M] \bigcap_{\substack{N\in K(B),\\ M\in K(G)}} ker[N,M] \bigcap_{\substack{N\in K(H),\\ M\in K(G)}} ker[N,M] \bigcap_{\substack{N\in K(B,H),\\ M\in K(G)}} ker[N,M]$$

We claim that X is invariant under G. Let $x\in X$ and consider Gx. We know from Theorem 5.1 that any element of G is a linear combination of terms of the form $t(B,H)G^iu(B,H)G^jv(B,H)$ with $t(B,H), u(B,H), v(B,H) \in I\cup K(B)\cup K(H)\cup K(B,H)$. Since $x \in ker[K(B,H),K(G)] \cap ker[K(B),K(G)] \cap ker[K(H),K(G)]$, we get

$$\begin{aligned} t(B,H)G^iu(B,H)G^jv(B,H)x &= t(B,H)G^iu(B,H)v(B,H)G^jx \\ &= t(B,H)G^{i+j}u(B,H)v(B,H)x \\ &= t(B,H)u(B,H)v(B,H)G^{i+j}x \\ &= G^{i+j}t(B,H)u(B,H)v(B,H)x. \end{aligned}$$

In the same way we use the fact that $x\in[K(B),K(H)]$ to sort the terms of the form $t(B,H)u(B,H)v(B,H)x$, so that we finally get

$$G_x = \{a_{ijk}G^iH^jB^kx | 0\le i,j,k\le 2, a_{ijk}\in A\}$$

Using the above technique, it follows easily that $G_x \subset X$ and that X is G invariant. Hence we can restrict G to X, but since the elements of G commute on X, they have a common eigenvector, and we can finish as in the proof of Theorem 2.2. □

From this we deduce the following theorem.

Theorem 5.4. *Let* $B,H,G\in N_3$. *Then* B,H,G *generate* N_3 *if and only if both* $N(B,H,G)$ *and* $N(B^t,H^t,G^t)$ *are invertible.*

For the case of four matrices, we can prove the following theorem.

Theorem 5.5. *The matrices* $B_1, B_2, B_3, B_4 \in N_3$ *have a common eigenvector if and only the matrix*

$$N(B_1, B_2, B_3, B_4) = \sum_{\substack{i,j=1 \\ i<j}}^{4} \left(\sum_{\substack{N \in K(B_i) \\ M \in K(B_j)}} [N, M]^*[N, M] \right) + \sum_{\substack{i,j=1 \\ i<j}}^{3} \left(\sum_{\substack{N \in K(B_i, B_j) \\ M \in K(B_4)}} [N, M]^*[N, M] \right)$$
$$+ \sum_{\substack{N \in K(B_1, B_2) \\ M \in K(B_3)}} [N, M]^*[N, M] + \sum_{\substack{N \in K(B_1, B_2, B_3) \\ M \in K(B_4)}} [N, M]^*[N, M]$$

is singular.

Proof. Similar to the proof of Theorem 5.3. □

From this we deduce the following theorem.

Theorem 5.6. *Let* $B, H, G, J \in M_3$. *Then* B, H, G, J *generate* N_3 *if and only if both* $N(B, H, G, J)$ *and* $N(B^t, H^t, G^t, J^t)$ *are invertible.*

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