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A Local Existence Theorem for Nonlinear Perturbed Abstract Measure Integrodifferential Equations

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Abstract: In this paper, we prove the relevance and the local existence theorems for a class of nonlinear perturbed abstract measure integrodifferential equations via classical hybrid fixed point theorem of Dhage (2003) under weaker Lipschitz and Carathéodory conditions. Our natural hypotheses and claims have also been illustrated with a numerical example.

MSC: 34K10, 47H10.

Keywords: Abstract measure integrodifferential equation; Relevance theorem; Dhage fixed point principle; Existence theorem.

1. Statement of the Problem

Let X be a real Banach space with a convenient norm $\|\cdot\|$ and let $x, y \in X$ be any two elements. Then the line segment $\overline{xy}$ in X is defined by

$$\overline{xy} = \{z \in X \mid z = x + r(y - x), 0 \leq r \leq 1\}. \tag{1}$$

Let $x_0 \in X$ be a fixed point and $z \in X$. Then for any $x \in \overline{x_0 z}$, we define the sets S_x and $\overline{S}_x$ in X by

$$S_x = \{rx \mid -\infty < r < 1\}, \tag{2}$$

and

$$\overline{S}_x = \{rx \mid -\infty < r \leq 1\}. \tag{3}$$

Let $x_1, x_2 \in \overline{xy}$ be arbitrary. We say $x_1 < x_2$ if $S_{x_1} \subset S_{x_2}$, or equivalently, $\overline{x_0 x_1} \subset \overline{x_0 x_2}$. In this case we also write $x_2 > x_1$. Let M denote the σ-algebra of all subsets of X such that (X, M) is a measurable space. Let $\mathrm{ca}(X, M)$ be the space of all vector measures (real signed measures) and define a norm $|\cdot|$ on $\mathrm{ca}(X, M)$ by

$$\|p\| = |p|(X), \tag{4}$$

where $|p|$ is a total variation measure of p and is given by

$$|p|(X) = \sup_{\sigma} \sum_{i=1}^{\infty} |\, p(E_i)|,\ E_i \subset X, \tag{5}$$

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where the supremum is taken over all possible partitions $\sigma = \{E_i : i \in \mathbb{N}\}$ of measurable subsets of X. It is known that $\mathrm{ca}(X, M)$ is a Banach space with respect to the norm $\|\cdot\|$ given by (4).

Let μ be a σ-finite positive measure on X, and let $p \in \mathrm{ca}(X, M)$. We say p is absolutely continuous with respect to the measure μ if $\mu(E) = 0$ implies $p(E) = 0$ for some $E \in M$. In this case we also write $p << \mu$.

Let $x_0 \in X$ be fixed and let M_0 denote the σ- algebra on S_{x_0}. Let $z \in X$ be such that $z > x_0$ and let M_z denote the σ-algebra of all sets containing M_0 and the sets of the form S_x, $x \in \overline{x_0 z}$.

Given a vector measure $p \in \mathrm{ca}(X, M)$ with $p << \mu$, consider the nonlinear abstract measure integrodifferential equation (in short AMIGDE) of the form

$$\frac{dp}{d\mu} = \int\limits_{\overline{S}_x - S_{x_0}} f\big(\tau, p(\overline{S}_\tau)\big)\, d\mu + \int\limits_{\overline{S}_x - S_{x_0}} g\big(\tau, p(\overline{S}_\tau)\big)\, d\mu \quad \text{a.e. } [\mu] \text{ on } \overline{x_0 z}. \tag{6}$$

and

$$p(E) = q(E), \quad E \in M_0, \tag{7}$$

where q is a given known vector measure, $\dfrac{dp}{d\mu}$ is a Radon-Nikodym derivative of p with respect to μ, $f, g : S_z \times \mathbb{R} \to \mathbb{R}$ and $x \mapsto f\big(x, p(\overline{S}_x)\big)$ and $x \mapsto g\big(x, p(\overline{S}_x)\big)$ are μ-integrable for each $p \in \mathrm{ca}(S_z, M_z)$.

Definition 1.1. *Given an initial real measure q on M_0, a vector $p \in \mathrm{ca}(S_z, M_z)$ $(z > x_0)$ is said to be a solution of the perturbed AMIGDE (6)-(7) if*

(1). $p(E) = q(E)$, $E \in M_0$,

(2). $p << \mu$ on $\overline{x_0 z}$, and

(3). p satisfies (6)-(7) a.e. $[\mu]$ on $\overline{x_0 z}$.

Remark 1.2. *The AMIGDE (6)-(7) is equivalent to the abstract measure integral equation (in short AMIGDE)*

$$p(E) = \int\limits_E \Bigg(\int\limits_{\overline{S}_x - S_{x_0}} f\big(\tau, p(\overline{S}_\tau)\big)\, d\mu \Bigg) d\mu + \int\limits_E \Bigg(\int\limits_{\overline{S}_x - S_{x_0}} g\big(\tau, p(\overline{S}_\tau)\big)\, d\mu \Bigg) d\mu, \tag{8}$$

if $E \in M_z$, $E \subset \overline{x_0 z}$. and

$$p(E) = q(E) \quad \text{if } E \in M_0. \tag{9}$$

A solution p of the AMIGDE (6)-(7) on $\overline{x_0 z}$ will be denoted by $p(\overline{S}_{x_0}, q)$.

The existence theorem for the AMIGDE (6)-(7) is an open problem raised in Dhage and Ram [12] and this paper we prove a local existence result under some generalized natural Lipschitz and compactness type conditions. The study of abstract measure differential equations (in short AMDEs) is initiated by Sharma [15, 16] as the generalizations of the ordinary differential equations in which ordinary derivative is replaced with the Radon-Nykodym derivative of vector measures in an abstract space, whereas the study of nonlinear AMIGDEs as the generalization of the ordinary integrodifferential equations is initiated by Dhage [2–4]. The work contained in Bellale *et. al* [1] and the related other papers of these authors on the AMIGDEs is a fraud and incorrect duplication of the previously known results. In the present paper we discuss the relevance and existence theorems to the AMIGDE (6)-(7) under suitable natural conditions via a Dhage's hybrid fixed point technique from nonlinear functional analysis. In the following section 2 we prove the relevance theorem for the AMIGDE (6)-(7) by relating it to an ordinary integrodifferential equations. Section 3 deals with the fixed point results needed in the subsequent sections of the paper. The main existence and uniqueness results along with a couple of examples are presented in section 4.

2. Relevance Theorem

In this section we prove the relevance theorem for the AMIGDE (6)-(7) and it is shown that the AMIGDE (6)-(7) reduces to an ordinary integrodifferential equation, viz.,

$$\left.\begin{aligned} y'(x) &= \int_{x_0}^{x} f(\tau, y(\tau))\,d\tau \int_{x_0}^{x} g(\tau, y(\tau))\,d\tau, \quad x \geq x_0, \\ y(x_0) &= y_0, \end{aligned}\right\} \tag{10}$$

under certain suitable natural conditions, where f is Carathéodory real-valued function on $[x_0, x_0 + T] \times \mathbb{R}$ into $\mathbb{R}$.
Let $X = \mathbb{R}$, $\mu = m$, the Lebesgue measure on $\mathbb{R}$, $\overline{S}_x = (-\infty, x]$, $x \in \mathbb{R}$, and q a given real Borel measure on M_0. Then equations (6)-(7) take the form

$$\left.\begin{aligned} \frac{d}{dm} p((-\infty, x]) &= \int\limits_{[x_0,x]} f\big(\tau, p(-\infty,\tau]\big)\,dm + \int\limits_{[x_0,x]} g\big(\tau, p(-\infty,\tau]\big)\,dm, \\ p(E) &= q(E), \quad E \in M_0. \end{aligned}\right\} \tag{11}$$

It will now be shown that the equations (10) and (11) are equivalent in the sense of the following theorem.

Theorem 2.1. *Let $q : M_0 \to \mathbb{R}$ be a given initial measure such that $q(E) = 0$ for all $E \in M_0$ and $q(\{x_0\}) = 0$. Then,*

(a). to each solution $p = p(\overline{S}_{x_0}, q)$ of (11) existing on $[x_0, x_1)$, there corresponds a solution y of (10) satisfying $y(x_0) = y_0$.

(b). Conversely, to every solution $y(x)$ of (10), there corresponds a solution $p(\overline{S}_{x_0}, q)$, of (11) existing on $[x_0, x_1)$ with a suitable initial measure q provided f satisfies the relation $f(x_0, 0) = 0$.

Proof.

(a). Let $p = p(\overline{S}_{x_0}, q)$ be a solution of (11), existing on $[x_0, x_1)$. Define a real Borel measure p_1 on $\mathbb{R}$ as follows.

$$p_1((-\infty, x)) = \begin{cases} 0, & \text{if } x \leq x_0, \\ p((-\infty, x]) - p((-\infty, x_0]), & \text{if } x_0 < x < x_1 \\ p((-\infty, x_1)), & \text{if } x \geq x_1, \end{cases} \tag{12}$$

and

$$p_1(-\infty, x_0]) = p(-\infty, x_0]).$$

Define the functions $y_1(x)$ and $y(x)$ by

$$\begin{aligned} y_1(x) &= p_1((-\infty, x)), & x \in \mathbb{R} \\ y(x) &= y_1(x) + p((-\infty, x_0]), & x \in [x_0, x_1). \end{aligned} \tag{13}$$

The condition $q(\{x_0\}) = 0$, the definition of the solution p, and the definitions of $y(x)$ together imply that

$$p_1(\{x_0\}) = p(\{x_0\}) = 0.$$

Now for each $x \in [x_0, x_1)$ we obtain from (11) and the definition of $y(x)$

$$\begin{aligned} y(x) &= y_1(x) + p((-\infty, x_0]) \\ &= p_1((-\infty, x)) + p((-\infty, x_0]) \\ &= p(\overline{S}_x). \end{aligned} \tag{14}$$

Since p is a solution of (11) we have $p << m$ on $[x_0, x_1)$. Hence $y(x)$ is absolutely continuous on $[x_0, x_1)$. The details concerning these arguments appear in Rudin [14, pages 163-165]. This shows that $y'(x)$ exists a.e. on $[x_0, x_1)$. Now for each $x \in [x_0, x_1)$, we have, by virtue of (12) and (13)

$$p([x_0, x]) = \int\limits_{[x_0,x]} \frac{d}{dm} p((-\infty, t])\, dm.$$

Therefore,

$$p((-\infty, x]) - p((-\infty, x_0]) = \int\limits_{[x_0,x]} \frac{d}{dm} p((-\infty, t])\, dm.$$

This further implies that

$$p(\overline{S}_x) = p(\overline{S}_{x_0}) + \int_{x_0}^{x} \left(\int_{x_0}^{t} f\big(\tau, p(\overline{S}_\tau)\big)\, dm \right) dm + \int_{x_0}^{x} \left(\int_{x_0}^{t} g\big(\tau, p(\overline{S}_\tau)\big)\, dm \right) dm.$$

That is,

$$y(x) = y(x_0) + \int_{x_0}^{x} \left(\int_{x_0}^{t} f\big(\tau, y(\tau)\big)\, d\tau \right) dt + \int_{x_0}^{x} \left(\int_{x_0}^{t} g\big(\tau, y(\tau)\big)\, d\tau \right) dt.$$

Hence,

$$y'(x) = \int_{x_0}^{x} f\big(\tau, y(\tau)\big)\, d\tau + \int_{x_0}^{x} g\big(\tau, y(\tau)\big)\, d\tau \quad \text{a.e on } [x_0, x_1).$$

This proves that $y(x)$ is a solution of (10) on $[x_0, x_1)$ satisfying

$$y(x_0) = y_0.$$

(b). Conversely, suppose that $y(x)$ be a solution of (10) existing on $[x_0, x_1]$. Then, y is absolutely continuous on $[x_0, x_1]$. Now, corresponding to the absolutely continuous function $y(x)$ which is a solution of (10) on $[x_0, x_1)$, we can construct a absolutely continuous real Borel measure p on M_{x_1} such that,

$$\begin{aligned} &p(E) = 0 \ \text{ for all } \ E \in M_0, \\ &p(\overline{S}_x) = y(x), \ \text{ if } \ x \in [x_0, x_1). \end{aligned} \tag{15}$$

The details concerning these arguments appear in Rudin [14, pages 163-165]. Since $y(x)$ is a solution of (10) we have for $x \in [x_0, x_1)$,

$$y(x) = y(x_0) + \int_{x_0}^{x} \left(\int_{x_0}^{t} f\big(\tau, y(\tau)\big)\, d\tau \right) dt + \int_{x_0}^{x} \left(\int_{x_0}^{t} g\big(\tau, y(\tau)\big)\, d\tau \right) dt.$$

Now, $y(x_0) = p\big(S_{x_0}\big) = 0$ and so, by (15) we obtain that

$$[p(\overline{S}_x) - p(\overline{S}_{x_0})] = \int\limits_{[x_0,x]} \left(\int\limits_{[x_0,t]} f\big(\tau, p(\overline{S}_\tau)\big)\, dm \right) dm + \int\limits_{[x_0,x]} \left(\int\limits_{[x_0,t]} g\big(\tau, p(\overline{S}_\tau)\big)\, dm \right) dm.$$

That is,

$$p([x_0, x]) = \int\limits_{[x_0,x]} \left(\int\limits_{[x_0,t]} f\big(\tau, p(\overline{S}_\tau)\big)\, dm \right) dm + \int\limits_{[x_0,x]} \left(\int\limits_{[x_0,t]} f\big(\tau, p(\overline{S}_\tau)\big)\, dm \right) dm.$$

In general, if $E \in M_{x_1}$, $E \subset \overline{x_0 x_1}$, then

$$p(E) = \int\limits_{E} \left(\int\limits_{\overline{S}_x - S_{x_0}} f\big(\tau, p((-\infty, \tau])\, dm \right) dm + \int\limits_{E} \left(\int\limits_{\overline{S}_x - S_{x_0}} g\big(\tau, p((-\infty, \tau])\, dm \right) dm.$$

By definition of Radon-Nykodym derivative, we obtain

$$\frac{d}{dm}\Big[p((-\infty,x])\Big] = \int\limits_{\overline{S}_x - S_{x_0}} f\big(\tau, p((-\infty,\tau])\big)\,dm + \int\limits_{\overline{S}_x - S_{x_0}} g\big(\tau, p((-\infty,\tau])\big)\,dm \;\; a.e.\ [\mu] \text{ on } \overline{x_0 z},$$

$$p(E) = 0 \text{ for } E \in M_0.$$

This shows that p is a solution of (11) on $[x_0, x_1)$ and the proof of (b) is complete. □

3. Fixed Point Results

To state the required fixed point techniques that will be used in the proofs of main results, we need the following definitions in what follows.

Definition 3.1 (Dhage [6, 8]). *An upper semi-continuous and nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a $\mathcal{D}$-function if $\psi(0) = 0$. The class of all $\mathcal{D}$-functions on $\mathbb{R}+$ is denoted by $\mathfrak{D}$.*

Definition 3.2 (Dhage [6, 8]). *Let $\mathfrak{X}$ be a Banach space with a norm $\|\cdot\|$. An operator $\mathcal{T} : \mathfrak{X} \to \mathfrak{X}$ is called $\mathcal{D}$-Lipschitz if there exists a $\mathcal{D}$-function $\psi_{\mathcal{T}} \in \mathfrak{D}$ such that*

$$\|\mathcal{T}x - \mathcal{T}y\| \le \psi_{\mathcal{T}}\big(\|x - y\|\big) \tag{16}$$

for all elements $x, y \in \mathfrak{X}$. If $\psi_{\mathcal{T}}(r) = k\,r$, $k > 0$, $\mathcal{T}$ is called a Lipschitz operator on $\mathfrak{X}$ with the Lipschitz constant k. Again, if $0 \le k < 1$, then $\mathcal{T}$ is called a contraction on $\mathfrak{X}$ with contraction constant k. Furthermore, if $\psi_{\mathcal{T}}(r) < r$ for $r > 0$, then $\mathcal{T}$ is called a nonlinear $\mathcal{D}$-contraction on $\mathfrak{X}$. The class of all $\mathcal{D}$-functions satisfying the condition of nonlinear $\mathcal{D}$-contraction is denoted by $\mathfrak{D}\mathfrak{N}$.

An operator $\mathcal{T} : \mathfrak{X} \to \mathfrak{X}$ is called compact if $\overline{\mathcal{T}(\mathfrak{X})}$ is a compact subset of $\mathfrak{X}$. $\mathcal{T}$ is called totally bounded if for any bounded subset S of $\mathfrak{X}$, $\mathcal{T}(S)$ is a totally bounded subset of $\mathfrak{X}$. T is called completely continuous if $\mathcal{T}$ is continuous and totally bounded on $\mathfrak{X}$. Every compact operator is totally bounded, but the converse may not be true, however, two notions are equivalent on bounded subsets of $\mathfrak{X}$. The details of different types of nonlinear contraction, compact and completely continuous operators appear in Granas and Dugundji [13].

To prove the main existence results of next section, we need the following hybrid fixed point principle of Dhage [8] involving the sum of two operators in a Banach space $\mathfrak{X}$.

Theorem 3.3 (Dhage [8]). *Let S be a closed convex and bounded subset of a Banach space $\mathfrak{X}$ and let $\mathcal{A} : \mathfrak{X} \to \mathfrak{X}$ and $\mathcal{B} : S \to \mathfrak{X}$ be two operators satisfying the following conditions.*

(a). $\mathcal{A}$ is nonlinear $\mathcal{D}$-contraction,

(b). $\mathcal{B}$ is completely continuous, and

(c). $\mathcal{A}x + \mathcal{B}y = x \implies x \in S$ for all $y \in S$.

Then the operator equation $\mathcal{A}x + \mathcal{B}x = x$ has a solution in S.

4. Existence Theorem

We need the following definition in the sequel.

Definition 4.1. *A function $\beta : S_z \times \mathbb{R} \to \mathbb{R}$ is called Carathéodory if*

(i). $x \mapsto \beta(x, u)$ is μ-measurable for each $u \in \mathbb{R}$, and

(ii). $u \mapsto \beta(x, u)$ is continuous almost everywhere $[\mu]$ on $\overline{x_0 z}$.
Further a Carathéodory function $\beta(x, u)$ is called $L^{\mu}_{\mathbb{R}}$-Carathéodory if

(iii). there exists a μ-integrable function $h : S_z \to \mathbb{R}$ such that

$$|\beta(x, u)| \le h(x) \quad a.e.\ [\mu] \ \ on\ x \in \overline{x_0 z},$$

for all $u \in \mathbb{R}$.

We consider the following set of assumptions.

(H$_0$) For any $z > x_0$, the σ-algebra M_z is compact with respect to the topology generated by the Pseudo-metric d defined on M_z by

$$d(E_1, E_2) = \mu(E_1 \Delta E_2)$$

for all $E_1, E_2 \in M_z$.

(H$_1$) $\mu(\{x_0\}) = 0$.

(H$_2$) There exists a $\mathcal{D}$-function $\psi_f \in \mathfrak{D}$ such that

$$|f(x, u) - f(x, v)| \le \psi_f\big(|u - v|\big) \ \text{a.e.}\ [\mu] \text{ on } x \in \overline{x_0 z},$$

for all $u, v \in \mathbb{R}$. Moreover, $\psi_f(r) < r$ for each $r > 0$.

(H$_3$) q is continuous on M_0 with respect to the Pseudo-metric d defined in (H_0).

(H$_4$) The function $g(x, u)$ is $L^{\mu}_{\mathbb{R}}$-Carathéodory.

Theorem 4.2. *Suppose that the hypotheses (H$_0$)-(H$_4$) hold. Then the AMIGDE (6)-(7) has a solution.*

Proof. By expressions (2) and (3), we have a real number $r(> 1)$ such that $r \to 1$ and $S_{rx_0} \supset S_{x_0}$. Then, from hypothesis (H$_1$), it follows that

$$\bigcap_{r \to 1} \left(\overline{S}_{rx_0} - S_{x_0}\right) = \{x_0\}$$

and

$$\mu\left(\overline{S}_{rx_0} - S_{x_0}\right) = \mu(\{x_0\}) = 0$$

whenever $r \to 1$.

Therefore, we can choose a real number r^* such that $S_{r^* x_0} \supset S_{x_0}$ and satisfying

$$\mu\big(\overline{S}_{r^* x_0} - S_{x_0}\big) < 1 \quad \text{and} \quad \int\limits_{\overline{S}_{r^* x_0} - S_{x_0}} h(x)\, d\mu < 1.$$

Let $z^* = r^* x_0$. Consider the vector measure p_0 on M_{z^*} which is a continuous extension of the measure q on M_0 defined by

$$p_0(E) = \begin{cases} q(E) & \text{if} \quad E \in M_0, \\ 0 & \text{if} \quad E \notin M_0. \end{cases}$$

Now, we define a subset $S(\rho)$ of $\mathrm{ca}(S_{z^*}, M_{z^*})$ by

$$S(\rho) = \{p \in \mathrm{ca}(S_{z^*}, M_{z^*}) \mid \|p - p_0\| \leq \rho\} \tag{17}$$

where $\rho = M_f + 1$. Clearly, $S(\rho)$ is a closed convex ball in $\mathrm{ca}(S_{z^*}, M_{z^*})$ centered at p_0 of radius ρ and $q \in S(\rho)$.

Define the two operators $\mathcal{A} : \mathrm{ca}(S_{z^*}, M_{z^*}) \to \mathrm{ca}(S_{z^*}, M_{z^*})$ and $\mathcal{B} : S(\rho) \to \mathrm{ca}(S_{z^*}, M_{z^*})$ by

$$\mathcal{A}p(E) = \begin{cases} \displaystyle\int_E \Bigg(\int\limits_{\overline{S}_x - S_{x_0}} f\big(\tau, p(\overline{S}_\tau)\big)\, d\mu \Bigg)\, d\mu & \text{if } E \in M_{z^*},\ E \subset \overline{x_0 z^*}, \\ 0 & \text{if } E \in M_0. \end{cases} \tag{18}$$

and

$$\mathcal{B}p(E) = \Bigg\{ \int_E \Bigg(\int\limits_{\overline{S}_x - S_{x_0}} f\big(\tau, p(\overline{S}_\tau)\big)\, d\mu \Bigg)\, d\mu \quad \text{if } E \in M_{z^*},\ E \subset \overline{x_0 z^*},\ q(E) \quad \text{if } E \in M_0. \tag{19}$$

Then the AMIGDE (6)-(7) is equivalent to the operator equation

$$\mathcal{A}p(E) + \mathcal{B}p(E) = p(E), \quad E \in M_z. \tag{20}$$

We shall show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of the hybrid fixed point theorem, Theorem 2.1 on $S(\rho)$. This will be done in a series of following steps.

Step I: Firstly, we show that $\mathcal{A}$ is bounded on $\mathfrak{X} = \mathrm{ca}(S_{z^*}, M_{z^*})$. Let $p \in \mathrm{ca}(S_{z^*}, M_{z^*})$ be arbitrary element. Then for any $E \in M_{z^*}$, there exist subsets $F \in M_0$ and $G \in M_{z^*}$, $G \subset \overline{x_0 z^*}$ such that $E = F \cup G$ and $F \cap G \neq \emptyset$. Now, by definition of the operator $\mathcal{A}$, we obtain $\mathcal{A}p(F) = 0$. Therefore, we have

$$|\mathcal{A}p(E)| \leq \int_G \Bigg(\int\limits_{\overline{S}_x - S_{x_0}} |f(x, p(\overline{S}_\tau))|\, d\mu \Bigg)\, d\mu \leq M_f$$

for all $E \in M_{z^*}$. Therefore, by definition of the norm,

$$\|\mathcal{A}p\| = |\mathcal{A}p|(E) = \sup_\sigma \sum_{i=1}^{\infty} |\mathcal{T}p(E_i)| \leq M_f$$

for all $p \in \mathfrak{X}$. As a result, $\mathcal{A}$ is a bounded operator on $\mathrm{ca}(S_{z^*}, M_{z^*})$ into itself.

Step II: First we show that $\mathcal{A}$ is a nonlinear $\mathcal{D}$- contraction on $\mathrm{ca}(S_{z^*}, M_{z^*})$. Let $p_1, p_2 \in \mathrm{ca}(S_{z^*}, M_{z^*})$ be any two elements. Then, by definition of the operator $\mathcal{T}$, we obtain

$$\mathcal{A}p_1(E) - \mathcal{A}p_2(E) = 0 \quad \text{if } E \in M_0,$$

and

$$\mathcal{A}p_1(E) - \mathcal{A}p_2(E) = \int_E \Bigg[\Bigg(\int\limits_{\overline{S}_x - S_{x_0}} \big[f\big(\tau, p_1(\overline{S}_\tau)\big) - f\big(\tau, p_2(\overline{S}_\tau)\big) \big]\, d\mu \Bigg) \Bigg]\, d\mu$$

for all $E \in M_{z^*}$, $E \subset \overline{x_0 z^*}$.

Therefore, by hypotheses (H_4), we obtain

$$\begin{aligned}
|\mathcal{A}p_1(E) - \mathcal{A}p_2(E)| &\leq \int_E \Big| \int\limits_{\overline{S}_x - S_{x_0}} \big|f\big(\tau, p_1(\overline{S}_\tau)\big) - f\big(\tau, p_2(\overline{S}_\tau)\big)\big| \, d\mu \Big| \, d\mu \\
&\leq \int_E \Bigg(\int\limits_{\overline{x_0 x}} \psi_f \left(\left|p_1(\overline{S}_\tau) - p_2(\overline{S}_\tau)\right|\right) \, d\mu \Bigg) \, d\mu \\
&\leq \int_E \Bigg(\int\limits_{\overline{x_0 x}} \psi_f \left(\left|p_1 - p_2\right|(\overline{S}_\tau)\right) \, d\mu \Bigg) \, d\mu \\
&\leq \int_E \Bigg(\int\limits_{\overline{x_0 z^*}} \psi_f \left(\left\|p_1 - p_2\right\|\right) \, d\mu \Bigg) \, d\mu \\
&\leq \int\limits_{\overline{x_0 z^*}} \psi_f \Big(\left\|p_1 - p_2\right\|\Big) \, d\mu \\
&\leq \psi_f \Big(\left\|p_1 - p_2\right\|\Big)
\end{aligned}$$

for all $E \in M_{z^*}$, $E \subset \overline{x_0 z^*}$. This further in view of definition of the norm in $\mathrm{ca}(S_{z^*}, M_{z^*})$ implies that

$$\|\mathcal{A}p_1 - \mathcal{A}p_2\| \leq \psi_f \Big(\left\|p_1 - p_2\right\|\Big)$$

for all $E \in M_{z^*}$, $E \subset \overline{x_0 z^*}$. Hence, $\mathcal{T}$ is a nonlinear $\mathcal{D}$-contraction on $\mathrm{ca}(S_{z^*}, M_{z^*})$.

Step III : Thirdly, we show that $\mathcal{B}$ is continuous on $S(\rho)$. Let $\{p_n\}$ be a sequence of vector measures in $S(\rho)$ converging to a vector measure p. Then by dominated convergence theorem,

$$\begin{aligned}
\lim_{n\to\infty} \mathcal{B}p_n(E) &= \lim_{n\to\infty} \int_E \Bigg(\int\limits_{\overline{S}_x - S_{x_0}} g\big(\tau, p_n(\overline{S}_\tau)\big) \, d\mu \Bigg) \, d\mu \\
&= \int_E \Bigg(\int\limits_{\overline{S}_x - S_{x_0}} \Big[\lim_{n\to\infty} g\big(\tau, p_n(\overline{S}_\tau)\big)\Big] \, d\mu \Bigg) \, d\mu \\
&= \int_E \Bigg(\int\limits_{\overline{S}_x - S_{x_0}} g\big(\tau, p_n(\overline{S}_\tau)\big) \, d\mu \Bigg) \, d\mu \\
&= \mathcal{B}p(E)
\end{aligned}$$

for all $E \in M_{z^*}$, $E \subset \overline{x_0 z^*}$. Similarly, if $E \in M_0$, then

$$\lim_{n\to\infty} \mathcal{B}p_n(E) = q(E) = \mathcal{B}p(E),$$

and so $\mathcal{B}$ is a pointwise continuous operator on $S(\rho)$.

Next we show that $\{\mathcal{B}p_n : n \in \mathbb{N}\}$ is a equi-continuous sequence in $ca(S_{z^*}, M_{z^*})$. Let $E_1, E_2 \in M_{z^*}$. Then there exist subsets $F_1, F_2 \in M_0$ and $G_1, G_2 \in M_{z^*}$, $G_1 \subset \overline{x_0 z^*}$, $G_2 \subset \overline{x_0 z^*}$ such that

$$E_1 = F_1 \cup G_1 \quad \text{with} \quad F_1 \cap G_1 = \emptyset$$

and

$$E_2 = F_2 \cup G_2 \quad \text{with} \quad F_2 \cap G_2 = \emptyset.$$

We know the identities

$$G_1 = (G_1 - G_2) \cup (G_2 \cap G_1), \tag{21}$$

and

$$G_2 = (G_2 - G_1) \cup (G_1 \cap G_2). \tag{22}$$

Therefore, we have

$$\mathcal{B}p_n(E_1) - \mathcal{B}p_n(E_2) \leq q(F_1) - q(F_2) + \int\limits_{G_1 - G_2} \Big(\int\limits_{\overline{S}_x - S_{x_0}} g\big(\tau, p_n(\overline{S}_\tau)\big)\, d\mu \Big)\, d\mu + \int\limits_{G_2 - G_1} \Big(\int\limits_{\overline{S}_x - S_{x_0}} g\big(\tau, p_n(\overline{S}_\tau)\big)\, d\mu \Big)\, d\mu.$$

Since $f(x,y)$ is $L^{\mu}_{\mathbb{R}}$- Carathéodory, we have that

$$\begin{aligned} |\mathcal{B}p_n(E_1) - \mathcal{B}p_n(E_2)| &\leq |q(F_1) - q(F_2)| + \int\limits_{G_1 \Delta G_2} \Big| \Big(\int\limits_{\overline{S}_x - S_{x_0}} g\big(\tau, p(\overline{S}_\tau)\big)\, d\mu \Big) \Big|\, d\mu \\ &\leq |q(F_1) - q(F_2)| + \int\limits_{G_1 \Delta G_2} h(x)\, d\mu. \end{aligned}$$

Assume that

$$d(E_1, E_2) = \mu(E_1 \Delta E_2) \to 0.$$

Then we have that $E_1 \to E_2$. As a result $F_1 \to F_2$ and $\mu(G_1 \Delta G_2) \to 0$. As q is continuous on compact M_{z^*} , it is uniformly continuous and so

$$|\mathcal{B}p_n(E_1) - \mathcal{B}p_n(E_2)| \leq |q(F_1) - q(F_2)| + \int\limits_{G_1 \Delta G_2} h(x) d\mu \quad \to 0 \text{ as } E_1 \to E_2$$

uniformly for all $n \in \mathbb{N}$. This shows that $\{\mathcal{B}p_n : n \in \mathbb{N}\}$ is a equi-continuous set in ca(S_{z^*}, M_{z^*}). As a result, $\{\mathcal{B}p_n\}$ converges to $\mathcal{B}p$ uniformly on M_{z^*} and a fortiori $\mathcal{B}$ is a continuous operator on $S(\rho)$ into ca(S_{z^*}, M_{z^*}).

Step IV: Next we show that $\mathcal{T}(S(\rho))$ is a totally bounded set in ca(S_{z^*}, M_{z^*}). We shall show that the set is uniformly bounded and equi-continuous set in ca(S_{z^*}, M_{z^*}). Firstly, we show that $\mathcal{T}(S(\rho))$ is a uniformly bounded set in ca(S_{z^*}, M_{z^*}). Let $\lambda \in \mathcal{T}(S)$ be an arbitrary element. Then, there is a member $p \in S$ such that $\lambda(E) = \mathcal{B}p(E)$ for all $E \in M_{z^*}$. Let $E \in M_{z^*}$. Then there exists two subsets $F \in M_0$ and $G \in M_{z^*}$, $G \subset \overline{x_0 z^*}$ such that

$$E = F \cup G \qquad \text{and} \qquad F \cap G = \phi.$$

Hence by definition of $\mathcal{B}$,

$$\begin{aligned} |\lambda(E)| &= |\mathcal{B}p(E)| \\ &\leq |q(F)| + \int_G \Big| \Big(\int\limits_{\overline{S}_x - S_{x_0}} g\big(\tau, p(\overline{S}_\tau)\big)\, d\mu \Big) \Big|\, d\mu \\ &\leq \|q\| + \int_G h(x) d\mu \\ &\leq \|q\| + \int_E h(x) d\mu \\ &< \|q\| + 1 \end{aligned}$$

for all $E \in M_{z^*}$. From (3.5) it follows that

$$\|\lambda\| = \|\mathcal{B}p\| = |\mathcal{B}p|(E) = \sup_{\sigma} \sum_{i=1}^{\infty} |Tp(E_i)| \leq \|q\| + 1$$

for all $\lambda \in \mathcal{B}(S(\rho))$. As a result $\mathcal{B}$ defines a mapping $\mathcal{B} : S(\rho) \to S(\rho)$. Moreover, $\mathcal{B}(S(\rho))$ is a uniformly bounded set in $\text{ca}(S_{z^*}, M_{z^*})$.

Next we show that $\mathcal{B}(S(\rho))$ is a equi-continuous set of measures in $ca(S_{z^*}, M_{z^*})$. Let $E_1, E_2 \in M_{z^*}$. Then there exist subsets $F_1, F_2 \in M_0$ and $G_1, G_2 \in M_{z^*}$, $G_1 \subset \overline{x_0 z^*}$, $G_2 \subset \overline{x_0 z^*}$ such that

$$E_1 = F_1 \cup G_1 \quad \text{with} \quad F_1 \cap G_1 = \emptyset$$

and

$$E_2 = F_2 \cup G_2 \quad \text{with} \quad F_2 \cap G_2 = \emptyset.$$

We know the identities

$$G_1 = (G_1 - G_2) \cup (G_2 \cap G_1), \tag{23}$$

and

$$G_2 = (G_2 - G_1) \cup (G_1 \cap G_2). \tag{24}$$

Therefore, we have

$$\begin{aligned}
|\lambda(E_1) - \lambda(E_2)| &= |\mathcal{B}p(E_1) - \mathcal{B}p(E_2)| \\
&\leq |q(F_1) - q(F_2)| + \int\limits_{G_1 - G_2} \left| \Bigg(\int\limits_{\overline{S}_x - S_{x_0}} g\big(\tau, p(\overline{S}_\tau)\big)\, d\mu \Bigg) \right| d\mu + \int\limits_{G_2 - G_1} \left| \Bigg(\int\limits_{\overline{S}_x - S_{x_0}} g\big(\tau, p(\overline{S}_\tau)\big)\, d\mu \Bigg) \right| d\mu.
\end{aligned}$$

Since $g(x, y)$ is $L^\mu_{\mathbb{R}}$- Carathéodory, we have that

$$\begin{aligned}
|\lambda(E_1) - \lambda(E_2)| &\leq |q(F_1) - q(F_2)| + \int\limits_{G_1 \Delta G_2} \left| \Bigg(\int\limits_{\overline{S}_x - S_{x_0}} g\big(\tau, p(\overline{S}_\tau)\big)\, d\mu \Bigg) \right| d\mu \\
&\leq |q(F_1) - q(F_2)| + \int\limits_{G_1 \Delta G_2} h(x)\, d\mu.
\end{aligned}$$

Assume that

$$d(E_1, E_2) = \mu(E_1 \Delta E_2) \to 0.$$

Then we have that $E_1 \to E_2$. As a result $F_1 \to F_2$ and $\mu(G_1 \Delta G_2) \to 0$. As q is continuous on compact M_0, it is uniformly continuous and so

$$|\lambda(E_1) - \lambda(E_2)| \leq |q(F_1) - q(F_2)| + \int\limits_{G_1 \Delta G_2} h(x)\, d\mu \quad \to 0 \text{ as } E_1 \to E_2$$

uniformly for all $\lambda \in \mathcal{B}(S)$. This shows that $\mathcal{T}(S(\rho))$ is a equi-continuous set in $\text{ca}(S_{z^*}, M_{z^*})$. Now an application of the Arzelà-Ascoli theorem yields that $\mathcal{B}$ is a totally bounded operator on $S(\rho)$. Now, $\mathcal{B}$ is continuous and totally bounded, it is completely continuous operator on $S(\rho)$ into itself.

Step V: Finally, we show that the hypothesis (c) of Theorem 3.3 is satisfied. Let $p \in S(\rho)$ be arbitrary and let there is an element $u \in \text{ca}(S_{z^*}, M_{z^*})$ such that $\mathcal{A}u + \mathcal{B}p = u$. We show that $u \in S$. Now, by definitions of the operators $\mathcal{A}$ and $\mathcal{B}$,

$$u(E) = \begin{cases} \int\limits_E \Bigg(\int\limits_{\overline{S}_x - S_{x_0}} f\big(\tau, p(\overline{S}_\tau)\big)\, d\mu \Bigg) d\mu + \int\limits_E \Bigg(\int\limits_{\overline{S}_x - S_{x_0}} g\big(\tau, p(\overline{S}_\tau)\big)\, d\mu \Bigg) d\mu, & \text{if } E \in M_z,\ E \subset \overline{x_0 z^*}, \\ q(E), & \text{if } E \in M_0. \end{cases}$$

for all $E \in M_z$. If $E \in M_{z^*}$, then there exist sets $F \in M_0$ and $G \in M_{z^*}$, $G \subset \overline{x_0 z^*}$ such that $E = F \cup G$ and $F \cap G = \emptyset$. Then we have

$$u(E) = q(F) + \int_E \Big(\int_{\overline{S}_x - S_{x_0}} f\big(\tau, p(\overline{S}_\tau)\big)\, d\mu \Big)\, d\mu + \int_E \Big(\int_{\overline{S}_x - S_{x_0}} g\big(\tau, p(\overline{S}_\tau)\big)\, d\mu \Big)\, d\mu.$$

which further yields

$$u(E) - p_0(E) = \int_E \Big(\int_{\overline{S}_x - S_{x_0}} f\big(\tau, p(\overline{S}_\tau)\big)\, d\mu \Big)\, d\mu + \int_E \Big(\int_{\overline{S}_x - S_{x_0}} g\big(\tau, p(\overline{S}_\tau)\big)\, d\mu \Big)\, d\mu.$$

Hence,

$$\begin{aligned} |u(E) - p_0(E)| &\leq \int_E \Big(\int_{\overline{S}_x - S_{x_0}} \big|f\big(\tau, p(\overline{S}_\tau)\big)\big|\, d\mu \Big)\, d\mu + \int_E \Big(\int_{\overline{S}_x - S_{x_0}} \big|g\big(\tau, p(\overline{S}_\tau)\big)\big|\, d\mu \Big)\, d\mu \\ &\leq M_f + \int_{\overline{x_0 z^*}} h(x)\, d\mu \\ &< M_f + 1 \end{aligned}$$

which further implies that

$$\|u - p_0\| \leq M_f + 1 = \rho.$$

As a result, we have $u \in S$ and so hypothesis (c) of Theorem 3.3 is satisfied. In consequence, the operator equation $\mathcal{A}p(E) + \mathcal{B}p(E) = p(E)$ has a solution $p(S_{x_0}, q)$ in $\mathrm{ca}(S_{z^*}, M_{z^*})$. This further implies that the AMIGDE (6)-(7) has a solution on $\overline{x_0 z}$. This completes the proof. □

Example 4.3. *Given a vector measure $p \in \mathrm{ca}(X, M)$ with $p << \mu$, consider the AMIGDE with a linear perturbation of second type of the form*

$$\frac{dp}{d\mu} = \int_{\overline{S}_x - S_{x_0}} \frac{|p(\overline{S}_\tau)|}{1 + |p(\overline{S}_\tau)|}\, d\mu + \int_{\overline{S}_x - S_{x_0}} \frac{\ln(1 + |p(\overline{S}_\tau)|)}{1 + |p(\overline{S}_\tau)|}\, d\mu \quad a.e.\ [\mu]\ on\ \overline{x_0 z}. \tag{25}$$

and

$$p(E)) = 0, \tag{26}$$

where $\dfrac{dp}{d\mu}$ is a Radon-Nikodym derivative of p with respect to μ.
Here, $f(x, u) = \dfrac{|u|)}{1 + |u|}$ and $g(x, u) = \dfrac{\ln(1 + |u|)}{1 + |u|}$ for all $x \in \overline{x_0 z}$ and $u \in \mathbb{R}$. Clearly, f is a continuous and bounded function on $S_z \times \mathbb{R}$ with bound $M_f = 1$. Again, the function f satisfies the hypothesis (H_3) on $S_z \times \mathbb{R}$ with $\mathcal{D}$-function $\psi_f(r) = \dfrac{r}{1 + r}$. Furthermore, g is a continuous and bounded function on $S_z \times \mathbb{R}$ with the growth or comparison function $h(x) = 1$ for all $x \in S_z$ and so, the hypotheses (H_3) and (H_4) are satisfied. Therefore, if the assumptions (H_0)-(H_1) hold, then the AMIGDE (25) - (26) has a solution $p(\overline{S}_{x_0}, q)$ defined on $\overline{x_0 z^*}$ provided $\mu(\overline{x_0 z^*}) < 1$.

References

[1] S. S. Bellale, S. B. Birajdar and D. S. Palimkar, *Existence theorem for abstract measure delay integrodifferential equations*, Appl. Comput. Math., 4(4)(2015), 225-231.

[2] B. C. Dhage, *On abstract measure integro-differential equations*, J. Math. Phy. Sci., 20(1986), 367-380.

[3] B. C. Dhage, *On system of abstract measure integro-differential inequalities and applications*, Bull. Inst. Math. Acad. Sinica, 18(1989), 65-75.

[4] B. C. Dhage, *Mixed monotonicity theorems for a system of abstract measure delay integro-differential equations*, An. Stint. Univ. "Al. I. Cuza" Iasi, XLII(1996), 355-366.

[5] B. C. Dhage, *On a fixed point theorem of Krasnoselskii-Schaefer type*, Electronic J. Qualitative Theory of Differential Equations, 6(2002), 19.

[6] B. C. Dhage, *Remarks on two fixed point theorems involving the sum and product of two operators*, Compt. Math. Appl., 46(12)(2003), 1779-1785.

[7] B. C. Dhage, *On some nonlinear alternatives of Leray-Schauder type and functional integral equations*, Arch. Math. (Brno), 42(2006), 11-23.

[8] B. C. Dhage, *Some variants of two basic hybrid fixed point theorems of Krasnoselskii and Dhage with applications*, *Nonlinear Studies* **25**(3) (2018), 559-573.

[9] B. C. Dhage, D. N. Chate and S. K. Ntouyas, *Abstract measure differential equations*, Dynamic Systems Appl., 13(2004), 105-108.

[10] B. C.Dhage and S. S. Bellale, *Existence theorem for perturbed abstract measure differential Equations*, Nonlinear Analysis: TMA, 71(2009), 319-328.

[11] B. C. Dhage and J. R. Graef, *On stability of abstract measure delay integrodifferential equations*, Dynamic Systems and Appl., 19(2010), 323-334.

[12] B. C. Dhage and P. R. M. Reddy, *On existence and uniqueness theorems for abstract measure integro-differential equations*, Jñānābha, 49(2)(2019), 82-93.

[13] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer Verlag, (2003).

[14] W. Rudin, *Real and Complex Analysis*, McGraw-hill Inc., New York, (1966).

[15] R. R. Sharma, *An abstract measure differential equation*, Proc. Amer. Math. Soc., 32(1972), 503-510.

[16] R. R. Sharma, *Existence of solution of abstract measure differential equation*, Proc. Amer. Math. Soc., 35(1)(1972), 129-136.