Int. J. Math. And Appl., 8(1)(2020), 169–186
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Meir Keeler Type Contraction in Soft Cone Ball Metric Space

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Abstract: In this paper, we define a new cone ball-metric and get fixed points and common fixed points for the Meir-Keeler type functions in cone ball-metric spaces.

MSC: Primary 47H10; Secondary 54H25, 55M20.

Keywords: Meir-Keeler type mapping, cone ball-metric space, common fixed point theorem, fixed point theorem.

1. Introduction and preliminaries

In this section we gives some basic definitions and previous known results which are used to prove our main results.

Definition 1.1. *Let X be an initial universe set and E be a set of parameters. A pair (F, E) is called a soft set over X if and only if F is a mapping from E into the set of all subsets of the set X, i.e., $F : E \to P(X)$ where $P(X)$ is the power set of X.*

Definition 1.2. *The intersection of two soft sets (F, A) and (G, B) over X is the soft set (H, C), where $C = A \cap B$ and $\forall e \in C$, $H(e) = F(e) \cap G(e)$. This is denoted by $(F, A)\tilde{\cap}(G, B) = (H, C)$.*

Definition 1.3. *The union of two soft sets (F, A) and (G, B) over X is the soft set, where $C = A \cup B$ and $\forall e \in C$,*

$$H(e) = \begin{cases} F(e), & \text{if } e \in A - B \\ G(e), & \text{if } e \in B - A \\ F(e) \cup G(e), & \text{if } e \in A \cap B \end{cases}$$

This relationship is denoted by $(F, A)\tilde{\cup}(G, B) = (H, C)$.

Definition 1.4. *A soft set (F, A) over X is said to be a null soft set denoted by ϕ, if for all $e \in A, F(e) = \phi$, (null set).*

Definition 1.5. *A soft set (F, A) over X is said to be an absolute soft set denoted by $\tilde{X}$ if for all $e \in A$, $F(e) = X$.*

Definition 1.6. *The difference (H, E) of two soft sets (F, E) and (G, E) over X, denoted by $(F, E) \setminus (G, E)$, is defined as $H(e) = F(e) \setminus G(e)$ for all $e \in E$.*

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Definition 1.7. *The complement of a soft set (F, A) is denoted by $(F, A)^c$ and is defined by $(F, A)^c = (F^c, A)$, where $F^c : A \to P(X)$ is a mapping given by $F^c(e) = X - F(e)$, for all $e \in A$.*

Definition 1.8. *Let $\mathbb{R}$ be the set of real numbers and $B(\mathbb{R})$ be the collection of all non-empty bounded subsets of $\mathbb{R}$ and E be taken as a set of parameters. Then a mapping $F : E \to B(\mathbb{R})$ is called a soft real set. If a real soft set is a singleton soft set, it will be called a soft real number and denoted by $VisionRes.$, $\tilde{s}$, $\tilde{t}$ etc. $\tilde{0}$ and $\tilde{1}$ are the soft real numbers where $\tilde{0}(e) = 0$, $\tilde{1}(e) = 1$ for all $e \in E$ respectively.*

Definition 1.9. *Let $VisionRes.$, $\tilde{s}$ be two soft real numbers. Then the following statements hold:*

(i). $VisionRes. \leq \tilde{s}$ if $VisionRes.(e) \leq \tilde{s}(e)$ for all $e \in E$,

(ii). $VisionRes. \geq \tilde{s}$ if $VisionRes.(e) \geq \tilde{s}(e)$ for all $e \in E$,

(iii). $VisionRes. < \tilde{s}$ if $VisionRes.(e) < \tilde{s}(e)$ for all $e \in E$,

(iv). $VisionRes. > \tilde{s}$ if $VisionRes.(e) > \tilde{s}(e)$ for all $e \in E$.

Definition 1.10. *A soft set (F, E) over X is said to be a soft point, denoted by $\tilde{x}_e$, if for the element $e \in E$, $F(e) = \{x\}$ and $F(\tilde{e}) = \phi$, for all $\tilde{e} \in E \setminus \{e\}$.*

Definition 1.11. *Two soft points $\tilde{x}_e, \tilde{y}_e$ are said to be equal if $e = \tilde{e}$ and $x = y$. Thus $\tilde{x}_e \neq \tilde{y}_e$ or $e \neq \tilde{e}$.*

Proposition 1.12. *Every soft set can be expressed as a union of all soft points belonging to it as $(F, E) = \cup_{\tilde{x}_e \in (F,E)} \tilde{x}_e$. Conversely, any set of soft points can be considered as a soft set.*

Definition 1.13. *Let τ be a collection of soft sets over X. Then τ is said to be a soft topology on X if*

(1). $\phi, \tilde{X}$ belong to τ.

(2). The union of any number of soft sets in τ belongs to τ.

(3). The intersection of any two soft sets in τ belongs to τ.

The triplet (X, τ, E) is called a soft topological space over X.

Definition 1.14. *Let (X, τ, E) be a soft topological space over X. Then soft interior of (F, E), denoted by $(F, E)^\circ$, is defined as the union of all soft open sets contained in (F, E).*

Definition 1.15. *Let (X, τ, E) be a soft topological space over X. Then soft closure of (F, E), denoted by $\overline{(F, E)}$, is defined as the intersection of all soft closed super sets of (F, E).*

Definition 1.16. *Let (X, τ, E) be a soft topological space over X. Then soft boundary of soft set (F, E) over X ,denoted by $\partial(F, E)$, is defined as $\partial(F, E) = \overline{(F, E)} \tilde{\cap} \overline{(F, E)^c}$.*

Definition 1.17. *Let (X, τ, E) and (Y, τ', E) be two soft topological spaces, $f : (X, \tau, E) \to (Y, \tau', E)$ be a mapping. For each soft neighborhood (H, E) of $(f(\tilde{x}_e), E)$, if there exists a soft neighborhood (F, E) of $(\tilde{x}_e, E)$ such that $f((F, E)) \subset (H, E)$, then f is said to be soft continuous mapping at $(\tilde{x}_e, E)$. If f is soft continuous mapping for all $(\tilde{x}_e, E)$, then f is called soft continuous mapping. Let $SP(\tilde{X})$ be the collection of all soft points of X and $\mathbb{R}(E)^*$ denote the set of all non-negative soft real numbers.*

Definition 1.18. *A mapping $\tilde{d} : SP(\tilde{X}) \times SP(\tilde{X}) \to \mathbb{R}(E)^*$ is said to be a soft metric on the soft set $\tilde{X}$ if $\tilde{d}$ satisfies the following conditions:*

(M1) $\tilde{d}(\tilde{x}_e, \tilde{y_{e'}}) \tilde{\geq} \tilde{0}$ *for all* $\tilde{x}_e, \tilde{y_{e'}} \tilde{\in} \tilde{X}$,

(M2) $\tilde{d}(\tilde{x}_e, \tilde{y_{e'}}) \tilde{\geq} \tilde{0}$ *if and only if* $\tilde{x}_e = \tilde{y_{e'}}$,

(M3) $\tilde{d}(\tilde{x}_e, \tilde{y_{e'}}) = \tilde{d}(\tilde{y_{e'}}, \tilde{x}_e)$ *for all* $\tilde{x}_e, \tilde{y_{e'}} \tilde{\in} \tilde{X}$,

(M4) For all $\tilde{x}_e, \tilde{y_{e'}}, \tilde{z_{e''}} \tilde{\in} \tilde{X}$, $\tilde{d}(\tilde{x}_e, \tilde{z_{e''}}) \tilde{\leq} \tilde{d}(\tilde{x}_e, \tilde{y_{e'}}) + \tilde{d}(\tilde{y_{e'}}, \tilde{z_{e''}})$.

The soft set $\tilde{X}$ *with a soft metric* $\tilde{d}$ *on* $\tilde{X}$ *is called a soft metric space and denoted by* $(\tilde{X}, \tilde{d}, E)$.

Definition 1.19. *Let* $(\tilde{X}, \tilde{B}, E)$ *be a soft metric space and* ϵ *be a non-negative soft real number.*

$$B(\tilde{x}_e, \tilde{\epsilon}) = \{\tilde{y_{e'}} \in \tilde{X} : \tilde{d}(\tilde{x}_e, \tilde{y_{e'}}) \tilde{<} \tilde{\epsilon}\} \subset SP(\tilde{X})$$

is called the soft open ball with center $\tilde{x}_e$ *and radius* $\tilde{\epsilon}$.

$$B(\tilde{x}_e, \tilde{\epsilon}) = \{\tilde{y_{e'}} \in \tilde{X} : \tilde{d}(\tilde{x}_e, \tilde{y_{e'}}) \tilde{\leq} \tilde{\epsilon}\} \subseteq SP(\tilde{X})$$

is called the soft closed ball with center $\tilde{x}_e$ *and radius* $\tilde{\epsilon}$.

Definition 1.20. *Let* $(\tilde{X}, \tilde{d}, E)$ *be a soft metric space and* (F, E) *be a non-null soft subset of* $\tilde{X}$ *in* $(\tilde{X}, \tilde{d}, E)$. *Then* (F, E) *is said to be a soft open set in* $\tilde{X}$ *with respect to* $\tilde{d}$ *if and only if all soft points of* (F, E) *is soft interior points of* (F, E).

Definition 1.21. *Let* $\{x^{\tilde{n}}_{e_n}\}$ *be a sequence of soft points in a soft metric space* $(\tilde{X}, \tilde{d}, E)$. *Then the sequence* $\{x^{\tilde{n}}_{e_n}\}$ *is said to be convergent in* $(\tilde{X}, \tilde{d}, E)$ *if there is a soft point* $x^{\tilde{0}}_{e_0} \tilde{\in} \tilde{X}$ *such that* $\tilde{d}(x^{\tilde{n}}_{e_n}, x^{\tilde{0}}_{e_0}) \to \tilde{0}$ *as* $n \to \infty$.
This means for every $\tilde{\epsilon} \tilde{>} \tilde{0}$, *chosen arbitrarily, there is a natural number* $N = N(\tilde{\epsilon})$ *such that* $\tilde{0} \tilde{<} \tilde{d}(x^{\tilde{n}}_{e_n}, x^{\tilde{0}}_{e_0}) \tilde{\leq} \tilde{\epsilon}$ *whenever* $n > N$.

Definition 1.22. *Limit of a sequence in a soft metric space, if exist, is unique.*

Definition 1.23 (Cauchy Sequence). *The sequence* $\{x^{\tilde{n}}_{e_n}\}$ *of soft points in* $(\tilde{X}, \tilde{B}, E)$ *is called a Cauchy sequence in* $\tilde{X}$ *if corresponding to every* $\tilde{\epsilon} \tilde{>} \tilde{0}$, *there is a* $m \in N$ *such that* $\tilde{d}(x^{\tilde{i}}_{e_i}, x^{\tilde{j}}_{e_j}) \tilde{\leq} \tilde{\epsilon}$ *for all* $i, j \geq m$ *i.e.* $\tilde{d}(x^{\tilde{i}}_{e_i}, x^{\tilde{j}}_{e_j}) \to \tilde{0}$ *as* $i, j \to \infty$.

Definition 1.24 (Complete Metric Space). *The soft metric space* $(\tilde{X}, \tilde{B}, E)$ *is called complete if every Cauchy Sequence in* $\tilde{X}$ *converges to some point of* $\tilde{X}$. *The soft metric space* $(\tilde{X}, \tilde{d}, E)$ *is called incomplete if it is not complete.*

Definition 1.25. *Let* $(\tilde{E}, \|.\|, A)$ *(here we denote* A *be parametric set) be a soft real Banach space and* $(P, A) \in S(\tilde{E})$ *be a soft subset of* $\tilde{E}$. *Then* (P, A) *is called a soft cone if and only if*

(1). (P, A) *is closed,* $(P, A) \neq \phi$ *and* $(P, A) \neq SS(\theta)$,

(2). $\tilde{a}, \tilde{b} \in R(A)$ $\tilde{x}, \tilde{y} \tilde{\epsilon} (P, A)$ *implies* $\tilde{a}\tilde{x} + \tilde{y}\tilde{y} \tilde{\epsilon} (P, A)$,

(3). $\tilde{x} \tilde{\in} (P, A)$ *and* $-\tilde{x} \tilde{\in} (P, A)$ *implies*$\tilde{x} = \theta$.

Given a soft cone $(P, A) \tilde{\in} S(\tilde{E})$, *we define a soft partial ordering* $\tilde{\preceq}$ *with respect to* (P, A) *by* $\tilde{x} \tilde{\preceq} \tilde{y}$ *if and only if* $\tilde{y} - \tilde{x} \tilde{\in} (P, A)$. *We write* $\tilde{x} \prec \tilde{y}$ *to indicate that* $\tilde{x} \tilde{\preceq} \tilde{y}$ *but* $\tilde{x} \neq \tilde{y}$, *while* $\tilde{x} \ll \tilde{y}$ *will stand for* $\tilde{y} - \tilde{x} \tilde{\in} Int(P, A)$, $Int(P, A)$ *denotes the interior of* (P, A).

Definition 1.26. *The soft cone* (P, A) *in soft real Banach space* $\tilde{E}$ *is called* *(a). normal, if there is a soft real number $\tilde{\alpha}\tilde{>}\tilde{0}$ such that for all $\tilde{x},\tilde{y}\tilde{\in}\tilde{E}$, $\theta\tilde{\preceq}\tilde{x}\tilde{\preceq}\tilde{y}$ implies $\|\tilde{x}\|\tilde{\preceq}\tilde{\alpha}\|\tilde{y}\|$, where $\tilde{\alpha}$ is called soft constant of (P,A).*

(b). minihedral, if $\sup(\tilde{x},\tilde{y})$ exists for all $\tilde{x},\tilde{y}\tilde{\in}\tilde{E}$.

(c). strongly minihedral, if every soft set in $\tilde{E}$ which is bounded from above has a supremum.

(d). solid, if $Int(P,A)\neq\phi$.

(e). regular, if every increasing sequence of soft elements in $\tilde{E}$ which is bounded from above is convergent. That is, if $\{\tilde{x}_n\}$ is a sequence of soft elements in $\tilde{E}$ such that $\tilde{x}_1\tilde{\preceq}\tilde{x}_2\tilde{\preceq}\dots\tilde{\preceq}\tilde{x}_n\tilde{\preceq}\dots$

for some soft elements in $\tilde{E}$ then there is $\tilde{x}\tilde{\in}\tilde{E}$ such that $\|\tilde{x}_n-\tilde{x}\|\to\tilde{0}$ as $n\to\infty$. Equivalently, the soft cone (P,A) is regular if and only if every decreasing sequence of soft elements in $\tilde{E}$ which is bounded from below is convergent.

Example 1.27. *Let $\mathbb{R}(A)$ be all soft real number, where A is a finite set of parameters. Let $\mathbb{R}^n(A)=\mathbb{R}(A)\times\mathbb{R}(A)\times\cdots\times\mathbb{R}(A)$. Then, $\mathbb{R}^n(A)$ is a soft Banach space. Let $\tilde{E}=\mathbb{R}^n(A)$ with $(P,A)=SS\{(\tilde{x}_1,\tilde{x}_2,\dots,\tilde{x}_n):\tilde{x}_i\tilde{\succeq}\tilde{0},\ \ \forall\ \ i=1,2,\dots,n\}$. Then the soft cone (P,A) is normal, minihedral, strongly minihedral and solid.*

In fact Chen and Tsai [3] introduced the following notion of the cone ball-metric $\mathcal{B}$.

Definition 1.28. *Let X be a non-empty set and $\tilde{X}$ be absolute soft set. Then the mapping $\tilde{\mathcal{B}}:SE(\tilde{X})\times SE(\tilde{X})\times SE(\tilde{X})\to SE(\tilde{X})$ is soft cone ball-metric on $\tilde{X}$ if $\tilde{\mathcal{B}}$ satisfy the following properties:*

(1). $\tilde{\mathcal{B}}(\tilde{x},\tilde{y},\tilde{z})=\tilde{\theta}$ if and only if $\tilde{x}=\tilde{y}=\tilde{z}$;

(2). $\tilde{\mathcal{B}}(\tilde{x},\tilde{x},\tilde{y})\succ\tilde{\theta}$, for all $\tilde{x}\neq\tilde{y}$;

(3). $\tilde{\mathcal{B}}(\tilde{x},\tilde{x},\tilde{y})\tilde{\preceq}\tilde{\mathcal{B}}(\tilde{x},\tilde{y},\tilde{z})$, for all $\tilde{x},\tilde{y},\tilde{z}\tilde{\in}\tilde{X}$;

(4). $\tilde{\mathcal{B}}(\tilde{x},\tilde{y},\tilde{z})=\tilde{\mathcal{B}}(\tilde{x},\tilde{z},\tilde{y})=\tilde{\mathcal{B}}(\tilde{z},\tilde{y},\tilde{x})=\cdots$ (symmetric in all three variables);

(5). $\tilde{\mathcal{B}}(\tilde{x},\tilde{y},\tilde{z})\tilde{\preceq}\tilde{\mathcal{B}}(\tilde{x},\tilde{w},\tilde{w})+\tilde{\mathcal{B}}(\tilde{w},\tilde{y},\tilde{z})$, for all $\tilde{x},\tilde{y},\tilde{z},\tilde{w}\tilde{\in}\tilde{X}$;

(6). $\tilde{\mathcal{B}}(\tilde{x},\tilde{y},\tilde{z})\tilde{\preceq}\tilde{\mathcal{B}}(\tilde{x},\tilde{w},\tilde{w})+\tilde{\mathcal{B}}(\tilde{y},\tilde{w},\tilde{w})+\tilde{\mathcal{B}}(\tilde{z},\tilde{w},\tilde{w})$, for all $\tilde{x},\tilde{y},\tilde{z},\tilde{w}\tilde{\in}\tilde{X}$.

Definition 1.29. *Let $(\tilde{X},\tilde{d})$ be a soft cone metric space, $\tilde{\mathcal{B}}:SE(\tilde{X})\times SE(\tilde{X})\times SE(\tilde{X})\to SE(\tilde{X})$, $\tilde{x},\tilde{y},\tilde{z}\in\tilde{X}$ and we denote*

$$\delta(B)=\sup\{\tilde{d}(\tilde{a},\tilde{b}):\tilde{a},\tilde{b}\in\tilde{\mathcal{B}}\},$$

and

$$\tilde{\mathcal{B}}(\tilde{x},\tilde{y},\tilde{z})=\delta(\tilde{\mathcal{B}}),$$

where $\tilde{\mathcal{B}}=\tilde{\cap}\{F\tilde{\subset}\tilde{X}|F$ is a soft closed ball and $\{\tilde{x},\tilde{y},\tilde{z}\}\tilde{\subset}F\}$. Then we call $\tilde{\mathcal{B}}$ a ball-metric with respect to the cone metric $\tilde{d}$, and $(\tilde{X},\tilde{\mathcal{B}})$ a soft cone ball-metric space. It is clear that $\tilde{\mathcal{B}}(\tilde{x},\tilde{x},\tilde{y})=\tilde{d}(\tilde{x},\tilde{y})$.

Definition 1.30. *Let $(\tilde{X},\tilde{\mathcal{B}})$ be a soft cone ball-metric space and $\{\tilde{x}_n^n\}$ be a sequence in $\tilde{X}$. We say that $\{\tilde{x}_n\}$ is*

(1). Cauchy sequence if for every $\tilde{\epsilon}\tilde{\in}\tilde{E}$ with $\theta\tilde{\ll}\tilde{\epsilon}$, there exists $n_0\tilde{\in}\mathbb{N}$ such that for all $n,m,l>n_0$, $\tilde{\mathcal{B}}(\tilde{x}_n^n,\tilde{x}_m^m,\tilde{x}_l^l)\tilde{\ll}\tilde{\epsilon}$.

(2). Convergent sequence if for every $\tilde{\epsilon}\tilde{\in}E$ with $\theta\tilde{\ll}\tilde{\epsilon}$, there exists $n_0\tilde{\cap}\mathbb{N}$ such that for all $n,m>n_0$, $\tilde{\mathcal{B}}(\tilde{x}_n^n,\tilde{x}_m^m,\tilde{x})\tilde{\ll}\tilde{\epsilon}$ for some $\tilde{x}\tilde{\in}\tilde{X}$. Here $\tilde{x}$ is called the limit of the sequence $\{\tilde{x}_n^n\}$ and is denoted by $\lim_{n\to\infty}\tilde{x}_n^n=\tilde{x}$ or $\tilde{x}_n^n\to\tilde{x}$ as $n\to\infty$.

Definition 1.31. *Let $(\tilde{X}, \tilde{\mathcal{B}})$ be a soft cone ball-metric space. Then $\tilde{X}$ is said to be complete if every Cauchy sequence is convergent in $\tilde{X}$.*

Proposition 1.32. *Let $(\tilde{X}, \tilde{\mathcal{B}})$ be a soft cone ball-metric space and $\{\tilde{x}_n^n\}$ be a sequence in $\tilde{X}$. Then the following are equivalent:*

(1). $\{\tilde{x}_n^n\}$ converges to $\tilde{x}$;

(2). $\tilde{\mathcal{B}}(\tilde{x}_n^n, \tilde{x}_n^n, \tilde{x}) \to \tilde{\theta}$ as $n \to \infty$;

(3). $\tilde{\mathcal{B}}(\tilde{x}_n^n, \tilde{x}, \tilde{x}) \to \tilde{\theta}$ as $n \to \infty$;

(4). $\tilde{\mathcal{B}}(\tilde{x}_n^n, \tilde{x}_m^m, \tilde{x}) \to \tilde{\theta}$ as $n, m \to \infty$.

Proposition 1.33. *Let $(\tilde{X}, \tilde{\mathcal{B}})$ be a soft cone ball-metric space and $\{\tilde{x}_n^n\}$ be a sequence in $\tilde{X}$, $\tilde{x}, \tilde{y} \tilde{\in} \tilde{X}$. If $\tilde{x}_n^n \to \tilde{x}$ and $\tilde{x}_n^n \to \tilde{y}$ as $n \to \infty$, then $\tilde{x} = \tilde{y}$.*

Proof. Let $\tilde{\epsilon} \tilde{\in} E$ with $\theta \tilde{\ll} \tilde{\epsilon}$ be given. Since $\tilde{x}_n^n \to \tilde{x}$ and $\tilde{x}_n^n \to \tilde{y}$ as $n \to \infty$, there exists $n_0 \tilde{\in} \mathbb{N}$ such that for all $m, n > n_0$,

$$\tilde{\mathcal{B}}(\tilde{x}_n^n, \tilde{x}_m^m, \tilde{x}) \tilde{\ll} \frac{\tilde{\epsilon}}{3} \text{ and } \tilde{\mathcal{B}}(\tilde{x}_n^n, \tilde{x}_m^m, y) \tilde{\ll} \frac{\tilde{\epsilon}}{3}.$$

Therefore,

$$\begin{aligned} \tilde{\mathcal{B}}(\tilde{x}, \tilde{x}, \tilde{y}) &\tilde{\preceq} \tilde{\mathcal{B}}(\tilde{x}, \tilde{x}_n^n, \tilde{x}_n^n) + \tilde{\mathcal{B}}(\tilde{x}_n^n, \tilde{x}, \tilde{y}) \\ &= \tilde{\mathcal{B}}(\tilde{x}, \tilde{x}_n^n, \tilde{x}_n^n) + \tilde{\mathcal{B}}(\tilde{y}, \tilde{x}_n^n, \tilde{x}) \\ &\tilde{\preceq} \tilde{\mathcal{B}}(\tilde{x}, \tilde{x}_n^n, \tilde{x}_n^n) + \tilde{\mathcal{B}}(\tilde{y}, \tilde{x}_m^m, \tilde{x}_m^m) + \tilde{\mathcal{B}}(\tilde{x}_m^m, \tilde{x}_n^n, \tilde{x}) \\ &\tilde{\ll} \frac{\tilde{\epsilon}}{3} + \frac{\tilde{\epsilon}}{3} + \frac{\tilde{\epsilon}}{3} = \tilde{\epsilon}. \end{aligned}$$

Hence, $\tilde{\mathcal{B}}(\tilde{x}, \tilde{x}, \tilde{y}) \tilde{\ll} \frac{\tilde{\epsilon}}{\alpha}$ for all $\alpha \geq 1$, and so $\frac{\tilde{\epsilon}}{\alpha} - \tilde{\mathcal{B}}(\tilde{x}, \tilde{x}, \tilde{y}) \tilde{\in} P$ for all $\alpha \geq 1$. Since $\frac{\tilde{\epsilon}}{\alpha} \to \theta$ as $\alpha \to \infty$ and P is closed, we have that $-\tilde{\mathcal{B}}(\tilde{x}, \tilde{x}, \tilde{y}) \tilde{\in} P$. This implies that $\tilde{\mathcal{B}}(\tilde{x}, \tilde{x}, \tilde{y}) = \tilde{\theta}$, since $\tilde{\mathcal{B}}(\tilde{x}, \tilde{x}, \tilde{y}) \tilde{\in} P$. So $\tilde{x} = \tilde{y}$. □

Proposition 1.34. *Let $(\tilde{X}, \tilde{\mathcal{B}})$ be a soft cone ball-metric space and $\{\tilde{x}_n^n\}, \{\tilde{y}_m^m\}, \{\tilde{z}_l^l\}$ be three sequences in $\tilde{X}$. If $\tilde{x}_n^n \to \tilde{x}$, $\tilde{y}_m^m \to \tilde{y}$, $\tilde{z}_l^l \to \tilde{z}$ as $n \to \infty$, then $\tilde{\mathcal{B}}(\tilde{x}_n^n, \tilde{y}_m^m, \tilde{z}_l^l) \to \tilde{\mathcal{B}}(\tilde{x}, \tilde{y}, \tilde{z})$ as $n \to \infty$.*

Proof. Let $\tilde{\epsilon} \tilde{\in} E$ with $\theta \tilde{\ll} \tilde{\epsilon}$ be given. Since $\tilde{x}_n^n \to \tilde{x}$, $\tilde{y}_m^m \to \tilde{y}$, $\tilde{z}_l^l \to \tilde{z}$ as $n \to \infty$, there exists $n_0 \tilde{\in} \mathbb{N}$ such that for all $n, m, l > n_0$,

$$\tilde{\mathcal{B}}(\tilde{x}_n^n, \tilde{x}, \tilde{x}) \tilde{\ll} \frac{\tilde{\epsilon}}{3}, \quad \tilde{\mathcal{B}}(\tilde{y}_m^m, \tilde{y}, \tilde{y}) \tilde{\ll} \frac{\tilde{\epsilon}}{3}, \quad \tilde{\mathcal{B}}(\tilde{z}_l^l, \tilde{z}, \tilde{z}) \tilde{\ll} \frac{\tilde{\epsilon}}{3},$$

Therefore,

$$\begin{aligned} \tilde{\mathcal{B}}(\tilde{x}_n^n, \tilde{y}_m^m, \tilde{z}_l^l) &\tilde{\preceq} \tilde{\mathcal{B}}(\tilde{x}_n^n, \tilde{x}, \tilde{x}) + \tilde{\mathcal{B}}(\tilde{x}, \tilde{y}_m^m, \tilde{z}_l^l) \\ &\tilde{\preceq} \tilde{\mathcal{B}}(\tilde{x}_n^n, \tilde{x}, \tilde{x}) + \tilde{\mathcal{B}}(\tilde{y}_m^m, \tilde{y}, \tilde{y}) + \tilde{\mathcal{B}}(\tilde{y}, \tilde{x}, \tilde{z}_l^l) \\ &\tilde{\preceq} \tilde{\mathcal{B}}(\tilde{x}_n^n, \tilde{x}, \tilde{x}) + \tilde{\mathcal{B}}(\tilde{y}_m^m, \tilde{y}, \tilde{y}) + \tilde{\mathcal{B}}(\tilde{z}_l^l, \tilde{z}, \tilde{z}) + \tilde{\mathcal{B}}(\tilde{z}, \tilde{x}, \tilde{y}) \\ &\tilde{\ll} \frac{\tilde{\epsilon}}{3} + \frac{\tilde{\epsilon}}{3} + \frac{\tilde{\epsilon}}{3} + \tilde{\mathcal{B}}(\tilde{x}, \tilde{y}, \tilde{z}), \end{aligned}$$

that is,

$$\tilde{\mathcal{B}}(\tilde{x}_n^n, \tilde{y}_m^m, \tilde{z}_l^l) - \tilde{\mathcal{B}}(\tilde{x}, \tilde{y}, \tilde{z}) \tilde{\ll} \tilde{\epsilon}.$$

Similarly,

$$\tilde{\mathcal{B}}(\tilde{x},\tilde{y},\tilde{z})-\tilde{\mathcal{B}}(\tilde{x}_n^n,\tilde{y}_m^m,\tilde{z}_l^l)\tilde{\ll}\tilde{\epsilon}.$$

Therefore, for all $\alpha\geq 1$, we have

$$\tilde{\mathcal{B}}(\tilde{x}_n^n,\tilde{y}_m^m,\tilde{z}_l^l)-\tilde{\mathcal{B}}(\tilde{x},\tilde{y},\tilde{z})\tilde{\ll}\frac{\tilde{\epsilon}}{\alpha},$$

and

$$\tilde{\mathcal{B}}(\tilde{x},\tilde{y},\tilde{z})-\tilde{\mathcal{B}}(\tilde{x}_n^n,\tilde{y}_m^m,\tilde{z}_l^l)\tilde{\ll}\frac{\tilde{\epsilon}}{\alpha}.$$

These imply that

$$\frac{\tilde{\epsilon}}{\alpha}-\tilde{\mathcal{B}}(\tilde{x}_n^n,\tilde{y}_m^m,\tilde{z}_l^l)+\tilde{\mathcal{B}}(\tilde{x},\tilde{y},\tilde{z})\tilde{\in}P,$$
$$\frac{\tilde{\epsilon}}{\alpha}+\tilde{\mathcal{B}}(\tilde{x}_n^n,\tilde{y}_m^m,\tilde{z}_l^l)-\tilde{\mathcal{B}}(\tilde{x},\tilde{y},\tilde{z})\tilde{\in}P.$$

Since P is closed and $\frac{\tilde{\epsilon}}{\alpha}\to\theta$ as $\alpha\to\infty$, we have that

$$\lim_{n,m,l\to\infty}[-\tilde{\mathcal{B}}(\tilde{x}_n^n,\tilde{y}_m^m,\tilde{z}_l^l)+\tilde{\mathcal{B}}(\tilde{x},\tilde{y},\tilde{z})]\tilde{\in}P,$$

$$\lim_{n,m,l\to\infty}[\tilde{\mathcal{B}}(\tilde{x}_n^n,\tilde{y}_m^m,\tilde{z}_l^l)-\tilde{\mathcal{B}}(\tilde{x},\tilde{y},\tilde{z})]\tilde{\in}P.$$

These show that

$$\lim_{n,m,l\to\infty}\tilde{\mathcal{B}}(\tilde{x}_n^n,\tilde{y}_m^m,\tilde{z}_l^l)=\tilde{\mathcal{B}}(\tilde{x},\tilde{y},\tilde{z}).$$

So we complete the proof. □

2. Main results

In the section, we first recall the notion of the Meir-Keeler type function. A function $\psi:\mathbb{R}^+\to\mathbb{R}^+$ is said to be a Meir-Keeler type function (see [5]), if for each $\eta\in\mathbb{R}^+$, there exists $\delta>0$ such that for $t\in\mathbb{R}^+$ with $\eta\leq t<\eta+\delta$, we have $\psi(t)<\eta$. We now define a new weaker Meir-Keeler type function in a soft cone ball-metric space $(\tilde{X},\tilde{\mathcal{B}})$, as follows:

Definition 2.1. *Let $(\tilde{X},\tilde{\mathcal{B}})$ be a cone ball-metric space with cone P, and let $\psi:intP\cup\{\tilde{\theta}\}\to intP\tilde{\cup}\{\tilde{\theta}\}$. Then the function ψ is called a weaker Meir-Keeler type function in $\tilde{X}$, if for each $\tilde{\eta}$, $\tilde{\theta}\tilde{\ll}\tilde{\eta}$, there exists $\tilde{\delta}$, $\tilde{\theta}\tilde{\ll}\tilde{\delta}$ such that for $(\tilde{x},\tilde{y},\tilde{z})\tilde{\in}\tilde{X}$ with $\tilde{\eta}\tilde{\preceq}\tilde{\mathcal{B}}(\tilde{x},\tilde{y},\tilde{z})\tilde{\ll}\tilde{\delta}+\tilde{\eta}$, there exists $n_0\tilde{\in}\mathbb{N}$ such that $\psi^{n_0}(\tilde{\mathcal{B}}(\tilde{x},\tilde{y},\tilde{z}))\tilde{\ll}\tilde{\eta}$. Further, we let the function $\psi:intP\tilde{\cup}\{\tilde{\theta}\}\to intP\tilde{\cup}\{\tilde{\theta}\}$ satisfying the following conditions:*

(i) ψ be a weaker Meir-Keeler type function;

(ii) for each $t\in intP$, we have $\tilde{\theta}\tilde{\ll}\psi(t)\tilde{\ll}t$ and $\psi(\tilde{\theta})=\tilde{\theta}$;

(iii) for $t_n\in intP\tilde{\cup}\{\tilde{\theta}\}$, if $\lim_{n\to\infty}t_n=\gamma\gg\tilde{\theta}$, then $\lim_{n\to\infty}\psi(t_n)\tilde{\ll}\tilde{\gamma}$;

(iv) $\{\psi^n(t)\}_{n\in\mathbb{N}}$ is non-increasing.

Then we call this mapping a ψ-function.

We now state our main common fixed point result for the weaker Meir-Keeler type function in a soft cone ball-metric space $(\tilde{X},\tilde{\mathcal{B}})$, as follows:

Theorem 2.2. *Let $(\tilde{X}, \tilde{\mathcal{B}})$ be a complete soft cone ball-metric space, P be a regular cone in $\tilde{E}$ and f, g be two self-mapplings of $\tilde{X}$ such that $f\tilde{X} \subset g\tilde{X}$. Suppose that there exists a ψ-function such that*

$$\tilde{\mathcal{B}}(f\tilde{x}, f\tilde{y}, f\tilde{z}) \preccurlyeq \psi(L(\tilde{x}, \tilde{y}, \tilde{z})), \tag{1}$$

where

$$L(\tilde{x}, \tilde{y}, \tilde{z}) = \max\{\tilde{\mathcal{B}}(g\tilde{x}, g\tilde{y}, g\tilde{z}), \tilde{\mathcal{B}}(g\tilde{x}, f\tilde{x}, f\tilde{x}), \tilde{\mathcal{B}}(g\tilde{y}, f\tilde{y}, f\tilde{y}), \tilde{\mathcal{B}}(g\tilde{z}, f\tilde{z}, f\tilde{z})\}.$$

If $g\tilde{X}$ is closed, then f and g have a coincidence point in $\tilde{X}$. Moreover, if f and g commute at their coincidence points, then f and g have a unique common fixed point in $\tilde{X}$

Proof. Given $\tilde{x}_0^0 \tilde{\in} \tilde{X}$. Since $f\tilde{X} \subset g\tilde{X}$, we can choose $\tilde{x}_1^1 \tilde{\in} \tilde{X}$ such that $g\tilde{x}_1^1 = f\tilde{x}_0^0$. Continuing this process, we define the sequence $\{\tilde{x}_n^n\}$ in $\tilde{X}$ recursively as follows:

$$f\tilde{x}_n^n = g\tilde{x}_{n+1}^{n+1} \text{ for each } n \in \mathbb{N}\tilde{\cup}\{0\}.$$

In what follows we will suppose that $f\tilde{x}_{n+1}^{n+1} \neq f\tilde{x}_n^n$ for all $n \in \mathbb{N}$, since if $f\tilde{x}_{n+1}^{n+1} = f\tilde{x}_n^n$ for some n, then $f\tilde{x}_{n+1}^{n+1} = g\tilde{x}_{n+1}^{n+1}$, that is , f, g have a coincidence point $\tilde{x}_{n+1}^{n+1}$, and so we complete the proof. By (1), we have

$$\tilde{\mathcal{B}}(f\tilde{x}_n^n, f\tilde{x}_{n+1}^{n+1}, f\tilde{x}_{n+1}^{n+1}) \preccurlyeq \psi(L(\tilde{x}_n^n, \tilde{x}_{n+1}^{n+1}, \tilde{x}_{n+1}^{n+1})),$$

where

$$\begin{aligned} L(\tilde{x}_n^n, \tilde{x}_{n+1}^{n+1}, \tilde{x}_{n+1}^{n+1}) &= \max\{\tilde{\mathcal{B}}(g\tilde{x}_n^n, g\tilde{x}_{n+1}^{n+1}, g\tilde{x}_{n+1}^{n+1}), \tilde{\mathcal{B}}(g\tilde{x}_n^n, f\tilde{x}_n^n, f\tilde{x}_n^n), \\ &\quad \tilde{\mathcal{B}}(g\tilde{x}_{n+1}^{n+1}, f\tilde{x}_{n+1}^{n+1}, f\tilde{x}_{n+1}^{n+1}), \tilde{\mathcal{B}}(g\tilde{x}_{n+1}^{n+1}, f\tilde{x}_{n+1}^{n+1}, f\tilde{x}_{n+1}^{n+1})\} \\ &= \max\{\tilde{\mathcal{B}}(f\tilde{x}_{n-1}^{n-1}, f\tilde{x}_n^n, f\tilde{x}_n^n), \tilde{\mathcal{B}}(f\tilde{x}_{n-1}^{n-1}, f\tilde{x}_n^n, f\tilde{x}_n^n), \\ &\quad \tilde{\mathcal{B}}(f\tilde{x}_n^n, f\tilde{x}_{n+1}^{n+1}, f\tilde{x}_{n+1}^{n+1}), \tilde{\mathcal{B}}(f\tilde{x}_n^n, f\tilde{x}_{n+1}^{n+1}, f\tilde{x}_{n+1}^{n+1})\} \\ &= \max\{\tilde{\mathcal{B}}(f\tilde{x}_{n-1}^{n-1}, f\tilde{x}_n^n, f\tilde{x}_n^n), \tilde{\mathcal{B}}(f\tilde{x}_n^n, f\tilde{x}_{n+1}^{n+1}, f\tilde{x}_{n+1}^{n+1})\}. \end{aligned}$$

Therefore, by the condition (ii) of ψ, we conclude that for each $n \in \mathbb{N}$,

$$\tilde{\mathcal{B}}(f\tilde{x}_n^n, f\tilde{x}_{n+1}^{n+1}, f\tilde{x}_{n+1}^{n+1}) \;\tilde{\ll}\; \tilde{\mathcal{B}}(f\tilde{x}_{n-1}^{n-1}, f\tilde{x}_n^n, f\tilde{x}_n^n),$$

and

$$\begin{aligned} \tilde{\mathcal{B}}(f\tilde{x}_n^n, f\tilde{x}_{n+1}^{n+1}, f\tilde{x}_{n+1}^{n+1}) &\tilde{\preceq} \psi(\tilde{\mathcal{B}}(f\tilde{x}_{n-1}^{n-1}, f\tilde{x}_n^n, f\tilde{x}_n^n)) \\ &\tilde{\preceq} \cdots \\ &\tilde{\preceq} \psi^n(\tilde{\mathcal{B}}(f\tilde{x}_0^0, f\tilde{x}_1^1, f\tilde{x}_1^1)). \end{aligned}$$

Since $\{\psi^n(\tilde{\mathcal{B}}(f\tilde{x}_0^0, f\tilde{x}_1^1, f\tilde{x}_1^1))\}_{n \in \mathbb{N}}$ is non-incresing, it must converge to some $\tilde{\eta}$, $\tilde{\theta} \preccurlyeq \tilde{\eta}$. We claim that $\tilde{\eta} = \tilde{\theta}$. On the contrary, assume that $\tilde{\theta} \tilde{\ll} \tilde{\eta}$. Then by the definition of the ψ-function, there exists δ, $\tilde{\theta} \tilde{\ll} \delta$ such that for $\tilde{\theta} \tilde{\ll} \tilde{\mathcal{B}}(f\tilde{x}_0^0, f\tilde{x}_1^1, f\tilde{x}_1^1)$ with $\tilde{\eta} \preccurlyeq$

$\tilde{\mathcal{B}}(f\tilde{x}_0^0, f\tilde{x}_1^1, f\tilde{x}_1^1) \lll \delta + \tilde{\eta}$, there exists $n_0 \in \mathbb{N}$ such that $\psi^{n_0}(\tilde{\mathcal{B}}(f\tilde{x}_0^0, f\tilde{x}_1^1, f\tilde{x}_1^1)) \lll \tilde{\eta}$. Since $\lim_{n\to\infty} \psi^n(\tilde{\mathcal{B}}(f\tilde{x}_0^0, f\tilde{x}_1^1, f\tilde{x}_1^1)) = \tilde{\eta}$, there exists $m_0 \in \mathbb{N}$ such that $\tilde{\eta} \preccurlyeq \psi^m \tilde{\mathcal{B}}(f\tilde{x}_0^0, f\tilde{x}_1^1, f\tilde{x}_1^1) \lll \tilde{\delta} + \tilde{\eta}$, for all $m \geq m_0$. Thus, we conclude that

$$\psi^{m_0+n_0}(\tilde{\mathcal{B}}(f\tilde{x}_0^0, f\tilde{x}_1^1, f\tilde{x}_1^1)) \lll \tilde{\eta}.$$

So we get a contradiction. So $\lim_{n\to\infty} \psi^n(\tilde{\mathcal{B}}(f\tilde{x}_0^0, f\tilde{x}_1^1, f\tilde{x}_1^1)) = \tilde{\theta}$, and so we have $\lim_{n\to\infty} \tilde{\mathcal{B}}(f\tilde{x}_n^n, f\tilde{x}_{n+1}^{n+1}, f\tilde{x}_{n+1}^{n+1}) = \tilde{\theta}$. Next, we claim that the sequence $\{f\tilde{x}_n^n\}$ is a Cauchy sequence. Suppose that $\{f\tilde{x}_n^n\}$ is not a Cauchy sequence. Then there exists $\tilde{\gamma} \tilde{\in} \tilde{E}$ with $\tilde{\theta} \lll \tilde{\gamma}$ such that for all $k \tilde{\in} \mathbb{N}$, there are $m_k, n_k \tilde{\in} \mathbb{N}$ with $m_k > n_k \geq k$ satisfying:

(1) m_k is even and n_k is odd,

(2) $\tilde{\mathcal{B}}(f\tilde{x}_{n_k}^{n_k}, f\tilde{x}_{m_k}^{m_k}, f\tilde{x}_{m_k}^{m_k}) \tilde{\succeq} \tilde{\gamma}$, and

(3) m_k is the smallest even number such that the conditions (1), (2) hold.

Since $\lim_{n\to\infty} \tilde{\mathcal{B}}(f\tilde{x}_n^n, f\tilde{x}_{n+1}^{n+1}, f\tilde{x}_{n+1}^{n+1}) = \tilde{\theta}$ and by (2), (3), we have that

$$\begin{aligned}
\tilde{\gamma} &\tilde{\preceq} \tilde{\mathcal{B}}(f\tilde{x}_{n_k}^{n_k}, f\tilde{x}_{m_k}^{m_k}, f\tilde{x}_{m_k}^{m_k}) \\
&\tilde{\preceq} \tilde{\mathcal{B}}(f\tilde{x}_{n_k}^{n_k}, f\tilde{x}_{m_k-1}^{m_k-1}, f\tilde{x}_{m_k-1}^{m_k-1}) + \tilde{\mathcal{B}}(f\tilde{x}_{m_k-1}^{m_k-1}, f\tilde{x}_{m_k}^{m_k}, f\tilde{x}_{m_k}^{m_k}) \\
&\tilde{\preceq} \tilde{\mathcal{B}}(f\tilde{x}_{n_k}^{n_k}, f\tilde{x}_{m_k-2}^{m_k-2}, f\tilde{x}_{m_k-2}^{m_k-2}) + \tilde{\mathcal{B}}(f\tilde{x}_{m_k-2}^{m_k-2}, f\tilde{x}_{m_k-1}^{m_k-1}, f\tilde{x}_{m_k-1}^{m_k-1}) \\
&\quad + \tilde{\mathcal{B}}(f\tilde{x}_{m_k-1}^{m_k-1}, f\tilde{x}_{m_k-1}^{m_k-1}, f\tilde{x}_{m_k-1}^{m_k-1}) \\
&\tilde{\preceq} \gamma + \tilde{\mathcal{B}}(f\tilde{x}_{m_k-2}^{m_k-2}, f\tilde{x}_{m_k-1}^{m_k-1}, f\tilde{x}_{m_k-1}^{m_k-1}) + \tilde{\mathcal{B}}(f\tilde{x}_{m_k-1}^{m_k-1}, f\tilde{x}_{m_k}^{m_k}, f\tilde{x}_{m_k}^{m_k}).
\end{aligned}$$

Taking $\lim_{k\to\infty}$, we deduce

$$\lim_{k\to\infty} \tilde{\mathcal{B}}(f\tilde{x}_{n_k}^{n_k}, f\tilde{x}_{m_k}^{m_k}, f\tilde{x}_{m_k}^{m_k}) = \tilde{\gamma}.$$

Since

$$\begin{aligned}
\tilde{\mathcal{B}}(f\tilde{x}_{n_k-1}^{n_k-1}, f\tilde{x}_{m_k-1}^{m_k-1}, f\tilde{x}_{m_k-1}^{m_k-1}) &\tilde{\preceq} \tilde{\mathcal{B}}(f\tilde{x}_{n_k-1}^{n_k-1}, f\tilde{x}_{n_k}^{n_k}, f\tilde{x}_{n_k}^{n_k}) + \tilde{\mathcal{B}}(f\tilde{x}_{n_k}^{n_k}, f\tilde{x}_{m_k}^{m_k}, f\tilde{x}_{m_k}^{m_k}) \\
&\quad + \tilde{\mathcal{B}}(f\tilde{x}_{m_k}^{m_k}, f\tilde{x}_{m_k-1}^{m_k-1}, f\tilde{x}_{m_k-1}^{m_k-1}).
\end{aligned}$$

Taking $\lim_{k\to\infty}$, we deduce

$$\lim_{k\to\infty} \tilde{\mathcal{B}}(f\tilde{x}_{n_k-1}^{n_k-1}, f\tilde{x}_{m_k-1}^{m_k-1}, f\tilde{x}_{m_k-1}^{m_k-1}) \preccurlyeq \gamma. \tag{2}$$

On the other hand,

$$\begin{aligned}
\tilde{\gamma} &\tilde{\preceq} \tilde{\mathcal{B}}(f\tilde{x}_{n_k}^{n_k}, f\tilde{x}_{m_k}^{m_k}, f\tilde{x}_{m_k}^{m_k}) \\
&\tilde{\preceq} \tilde{\mathcal{B}}(f\tilde{x}_{n_k}^{n_k}, f\tilde{x}_{n_k-1}^{n_k-1}, f\tilde{x}_{n_k-1}^{n_k-1}) + \tilde{\mathcal{B}}(f\tilde{x}_{n_k-1}^{n_k-1}, f\tilde{x}_{m_k}^{m_k}, f\tilde{x}_{m_k}^{m_k}) \\
&\tilde{\preceq} \tilde{\mathcal{B}}(f\tilde{x}_{n_k}^{n_k}, f\tilde{x}_{n_k-1}^{n_k-1}, f\tilde{x}_{n_k-1}^{n_k-1}) + \tilde{\mathcal{B}}(f\tilde{x}_{n_k-1}^{n_k-1}, f\tilde{x}_{m_k-1}^{m_k-1}, f\tilde{x}_{m_k-1}^{m_k-1}) \\
&\quad + \tilde{\mathcal{B}}(f\tilde{x}_{m_k-1}^{m_k-1}, f\tilde{x}_{m_k}^{m_k}, f\tilde{x}_{m_k}^{m_k}).
\end{aligned}$$

Taking $\lim_{k\to\infty}$, we also deduce

$$\tilde{\gamma} \tilde{\preceq} \lim_{k\to\infty} \tilde{\mathcal{B}}(f\tilde{x}_{n_k-1}^{n_k-1}, f\tilde{x}_{m_k-1}^{m_k-1}, f\tilde{x}_{m_k-1}^{m_k-1}). \tag{3}$$

By (2) and (3), we get

$$\lim_{k\to\infty} \tilde{\mathcal{B}}(f\tilde{x}^{n_k-1}_{n_k-1}, f\tilde{x}^{m_k-1}_{m_k-1}, f\tilde{x}^{m_k-1}_{m_k-1}) = \tilde{\gamma}.$$

And, by (1), we have that

$$\tilde{\mathcal{B}}(f\tilde{x}^{n_k}_{n_k}, f\tilde{x}^{m_k}_{m_k}, f\tilde{x}^{m_k}_{m_k}) \preccurlyeq \psi(L(\tilde{x}^{n_k}_{n_k}, \tilde{x}^{m_k}_{m_k}, \tilde{x}^{m_k}_{m_k}))$$

where

$$\begin{aligned} L(\tilde{x}^{n_k}_{n_k}, \tilde{x}^{m_k}_{m_k}, \tilde{x}^{m_k}_{m_k}) &= \max\{\tilde{\mathcal{B}}(g\tilde{x}^{n_k}_{n_k}, g\tilde{x}^{m_k}_{m_k}, g\tilde{x}^{m_k}_{m_k}), \tilde{\mathcal{B}}(g\tilde{x}^{n_k}_{n_k}, f\tilde{x}^{n_k}_{n_k}, f\tilde{x}^{n_k}_{n_k}), \\ &\quad \tilde{\mathcal{B}}(g\tilde{x}^{m_k}_{m_k}, f\tilde{x}^{m_k}_{m_k}, f\tilde{x}^{m_k}_{m_k}), \tilde{\mathcal{B}}(g\tilde{x}^{m_k}_{m_k}, f\tilde{x}^{m_k}_{m_k}, f\tilde{x}^{m_k}_{m_k})\} \\ &= \max\{\tilde{\mathcal{B}}(f\tilde{x}^{n_k-1}_{n_k-1}, f\tilde{x}^{m_k-1}_{m_k-1}, f\tilde{x}^{m_k-1}_{m_k-1}), \tilde{\mathcal{B}}(f\tilde{x}^{n_k-1}_{n_k-1}, f\tilde{x}^{n_k}_{n_k}, f\tilde{x}^{n_k}_{n_k}), \\ &\quad \tilde{\mathcal{B}}(f\tilde{x}^{m_k-1}_{m_k-1}, f\tilde{x}^{m_k}_{m_k}, f\tilde{x}^{m_k}_{m_k}), \tilde{\mathcal{B}}(f\tilde{x}^{m_k-1}_{m_k-1}, f\tilde{x}^{m_k}_{m_k}, f\tilde{x}^{m_k}_{m_k})\}. \end{aligned}$$

(I) If

$$L(\tilde{x}^{n_k}_{n_k}, \tilde{x}^{m_k}_{m_k}, \tilde{x}^{m_k}_{m_k}) = \tilde{\mathcal{B}}(f\tilde{x}^{n_k-1}_{n_k-1}, f\tilde{x}^{m_k-2}_{m_k-2}, f\tilde{x}^{m_k-1}_{m_k-1}),$$

then taking $lim_{k\to\infty}$, we deduce

$$\lim_{k\to\infty} \tilde{\mathcal{B}}(f\tilde{x}^{n_k-1}_{n_k-1}, f\tilde{x}^{m_k-1}_{m_k-1}, f\tilde{x}^{m_k-1}_{m_k-1}) = \tilde{\gamma},$$

and

$$\tilde{\gamma}\tilde{\preceq} \lim_{k\to\infty} \tilde{\mathcal{B}}(f\tilde{x}^{n_k}_{n_k}, f\tilde{x}^{m_k}_{m_k}, f\tilde{x}^{m_k}_{m_k})\tilde{\ll}\tilde{\gamma},$$

a contradiction. (II) If

$$L(\tilde{x}^{n_k}_{n_k}, \tilde{x}^{m_k}_{m_k}, \tilde{x}^{m_k}_{m_k}) = \tilde{\mathcal{B}}(f\tilde{x}^{n_k-1}_{n_k-1}, f\tilde{x}^{n_k}_{n_k}, f\tilde{x}^{n_k}_{n_k}),$$

or

$$L(\tilde{x}^{n_k}_{n_k}, \tilde{x}^{m_k}_{m_k}, \tilde{x}^{m_k}_{m_k}) = \tilde{\mathcal{B}}(f\tilde{x}^{m_k-1}_{m_k-1}, f\tilde{x}^{m_k}_{m_k}, f\tilde{x}^{m_k}_{m_k}),$$

then taking $lim_{k\to\infty}$, we deduce

$$\lim_{k\to\infty} \tilde{\mathcal{B}}(f\tilde{x}^{n_k-1}_{n_k-1}, f\tilde{x}^{n_k}_{n_k}, f\tilde{x}^{n_k}_{n_k}) = \tilde{\theta},$$

$$\lim_{k\to\infty} \tilde{\mathcal{B}}(f\tilde{x}^{m_k-1}_{m_k-1}, f\tilde{x}^{m_k}_{m_k}, f\tilde{x}^{m_k}_{m_k}) = \tilde{\theta},$$

and

$$\gamma\tilde{\preceq} \lim_{k\to\infty} \tilde{\mathcal{B}}(f\tilde{x}^{n_k}_{n_k}, f\tilde{x}^{m_k}_{m_k}, f\tilde{x}^{m_k}_{m_k})\tilde{\preceq}\tilde{\theta},$$

a contradiction. Follow (I) and (II), we get the sequence $\{f\tilde{x}^n_n\}$ is a Cauchy sequence. Since $\tilde{X}$ is complete and $g\tilde{X}$ is closed, there exist $\tilde{\nu}, \tilde{\mu}\tilde{\in}\tilde{X}$ such that

$$\lim_{n\to\infty} g(\tilde{x}^n_n) = \lim_{n\to\infty} f(\tilde{x}^n_n) = g(\tilde{\mu}) = \tilde{\nu}.$$

We shall show that $\tilde{\mu}$ is a coincidence point of f and g, that is, we claim that

$$\tilde{\mathcal{B}}(g\tilde{\mu}, f\tilde{\mu}, f\tilde{\mu}) = \tilde{\theta}.$$

If not, assume that $\tilde{\mathcal{B}}(g\tilde{\mu}, f\tilde{\mu}, f\tilde{\mu}) \neq \tilde{\theta}$, then by (1), we have

$$\begin{aligned}\tilde{\mathcal{B}}(g\tilde{\mu}, f\tilde{\mu}, f\tilde{\mu}) &\tilde{\preceq} \tilde{\mathcal{B}}(g\tilde{\mu}, f\tilde{x}_n^n, f\tilde{x}_n^n) + \tilde{\mathcal{B}}(f\tilde{x}_n^n, f\tilde{\mu}, f\tilde{\mu}) \\ &\tilde{\preceq} \tilde{\mathcal{B}}(g\tilde{\mu}, f\tilde{x}_n^n, f\tilde{x}_n^n) + \psi(L(\tilde{x}_n^n, \tilde{\mu}, \tilde{\mu})),\end{aligned}$$

where

$$L(\tilde{x}_n^n, \tilde{\mu}, \tilde{\mu}) \in \{\tilde{\mathcal{B}}(g\tilde{x}_n^n, g\tilde{\mu}, g\tilde{\mu}), \tilde{\mathcal{B}}(g\tilde{x}_n^n, f\tilde{x}_n^n, f\tilde{x}_n^n), \tilde{\mathcal{B}}(g\tilde{\mu}, f\tilde{\mu}, f\tilde{\mu}), \tilde{\mathcal{B}}(g\tilde{\mu}, f\tilde{\mu}, f\tilde{\mu})\}.$$

(III) If

$$L(\tilde{x}_n^n, \tilde{\mu}, \tilde{\mu}) = \tilde{\mathcal{B}}(g\tilde{x}_n^n, g\tilde{\mu}, g\tilde{\mu}),$$

then taking $\lim_{n\to\infty}$, we deduce

$$\lim_{n\to\infty} \tilde{\mathcal{B}}(g\tilde{x}_n^n, g\tilde{\mu}, g\tilde{\mu}) = \tilde{\mathcal{B}}(g\tilde{\mu}, g\tilde{\mu}, g\tilde{\mu}) = \tilde{\theta},$$

and

$$\begin{aligned}\tilde{\mathcal{B}}(g\tilde{\mu}, f\tilde{\mu}, f\tilde{\mu}) &= \lim_{n\to\infty} \tilde{\mathcal{B}}(g\tilde{\mu}, f\tilde{x}_n^n, f\tilde{x}_n^n) + \lim_{n\to\infty} \psi(\tilde{\mathcal{B}}(g\tilde{x}_n^n, g\tilde{\mu}, g\tilde{\mu})) \\ &\tilde{\preceq} \tilde{\theta},\end{aligned}$$

a contradiction. (IV) If

$$L(\tilde{x}_n^n, \tilde{\mu}, \tilde{\mu}) = \tilde{\mathcal{B}}(g\tilde{x}_n^n, f\tilde{x}_n^n, f\tilde{x}_n^n),$$

then taking $\lim_{n\to\infty}$, we deduce

$$\lim_{n\to\infty} \tilde{\mathcal{B}}(g\tilde{x}_n^n, f\tilde{x}_n^n, f\tilde{x}_n^n) = \tilde{\mathcal{B}}(g\tilde{\mu}, g\tilde{\mu}, g\tilde{\mu}) = \tilde{\theta},$$

and

$$\tilde{\mathcal{B}}(g\tilde{\mu}, f\tilde{\mu}, f\tilde{\mu}) = \lim_{n\to\infty} \tilde{\mathcal{B}}(g\tilde{\mu}, f\tilde{x}_n^n, f\tilde{x}_n^n) + \lim_{n\to\infty} \psi(\tilde{\mathcal{B}}(g\tilde{x}_n^n, f\tilde{x}_n^n, f\tilde{x}_n^n)\tilde{\preceq}\tilde{\theta},$$

a contradiction. (V) If

$$L(\tilde{x}_n^n, \tilde{\mu}, \tilde{\mu}) = \tilde{\mathcal{B}}(g\tilde{\mu}, f\tilde{\mu}, f\tilde{\mu}),$$

then

$$\tilde{\mathcal{B}}(g\tilde{\mu}, f\tilde{\mu}, f\tilde{\mu}) = \psi(\tilde{\mathcal{B}}(g\tilde{\mu}, f\tilde{\mu}, f\tilde{\mu}))\ll\tilde{\mathcal{B}}(g\tilde{\mu}, f\tilde{\mu}, f\tilde{\mu}),$$

a contradiction. Follow (III)-(V), we obtain that $\tilde{\mathcal{B}}(g\tilde{\mu}, f\tilde{\mu}, f\tilde{\mu}) = \tilde{\theta}$, that is, $g\tilde{\mu} = f\tilde{\mu} = \tilde{\nu}$, and so $\tilde{\mu}$ is a coincidence point of f and g. Suppose that f and g commute at $\tilde{\mu}$. Then

$$f\tilde{\nu} = fg\tilde{\mu} = gf\tilde{\mu} = g\tilde{\nu}.$$

Later, we claim that $\tilde{\mathcal{B}}(f\tilde{\mu}, f\tilde{\nu}, f\tilde{\nu}) = \tilde{\theta}$. By (1), we have

$$\tilde{\mathcal{B}}(f\tilde{\mu}, f\tilde{\nu}, f\tilde{\nu})\tilde{\preceq}\psi(L(\tilde{\mu}, \tilde{\nu}, \tilde{\nu})),$$

where

$$\begin{aligned} L(\tilde{x},\tilde{y},\tilde{z}) &= \max\{\tilde{\mathcal{B}}(g\tilde{\mu},g\tilde{\nu},g\tilde{\nu}),\tilde{\mathcal{B}}(g\tilde{\mu},f\tilde{\mu},f\tilde{\mu}),\tilde{\mathcal{B}}(g\tilde{\nu},f\tilde{\nu},f\tilde{\nu}),\tilde{\mathcal{B}}(g\tilde{\nu},f\tilde{\nu},f\tilde{\nu})\} \\ &= \max\{\tilde{\mathcal{B}}(f\tilde{\mu},f\tilde{\nu},f\tilde{\nu}),\tilde{\mathcal{B}}(f\tilde{\mu},f\tilde{\mu},f\tilde{\mu}),\tilde{\mathcal{B}}(f\tilde{\nu},f\tilde{\nu},f\tilde{\nu}),\tilde{\mathcal{B}}(f\tilde{\nu},f\tilde{\nu},f\tilde{\nu})\} \\ &= \max\{\tilde{\mathcal{B}}(f\tilde{\mu},f\tilde{\nu},f\tilde{\nu}),\tilde{\theta}\}. \end{aligned}$$

Therefore, if

$$\tilde{\mathcal{B}}(f\tilde{\mu},f\tilde{\nu},f\tilde{\nu})\tilde{\preceq}\psi(\tilde{\mathcal{B}}(f\tilde{\mu},f\tilde{\nu},f\tilde{\nu}))\tilde{\ll}\tilde{\mathcal{B}}(f\tilde{\mu},f\tilde{\nu},f\tilde{\nu}),$$

then we get a contradition, which implies that $\tilde{\mathcal{B}}(f\tilde{\mu},f\tilde{\nu},f\tilde{\nu})=\tilde{\theta}$, $\tilde{\mathcal{B}}(\tilde{\nu},f\tilde{\nu},f\tilde{\nu})=\tilde{\theta}$, that is, $\tilde{\nu}=f\tilde{\nu}=g\tilde{\nu}$. So $\tilde{\nu}$ is a common fixed point of f and g. Let $\overline{\tilde{\nu}}$ be another common fixed point of f and g. By (1),

$$\tilde{\mathcal{B}}(\overline{\tilde{\nu}},\tilde{\nu},\tilde{\nu})=\tilde{\mathcal{B}}(f\overline{\tilde{\nu}},f\tilde{\nu},f\tilde{\nu})\tilde{\preceq}\psi(L(\overline{\tilde{\nu}},\tilde{\nu},\tilde{\nu})),$$

where

$$\begin{aligned} L(\tilde{x},\tilde{y},\tilde{z}) &= \max\{\tilde{\mathcal{B}}(g\overline{\tilde{\nu}},g\tilde{\nu},g\tilde{\nu}),\tilde{\mathcal{B}}(g\overline{\tilde{\nu}},f\overline{\tilde{\nu}},f\overline{\tilde{\nu}}),\tilde{\mathcal{B}}(g\tilde{\nu},f\tilde{\nu},f\tilde{\nu}),\tilde{\mathcal{B}}(g\tilde{\nu},f\tilde{\nu},f\tilde{\nu})\} \\ &= \max\{\tilde{\mathcal{B}}(f\overline{\tilde{\nu}},f\tilde{\nu},f\tilde{\nu}),\tilde{\mathcal{B}}(f\overline{\tilde{\nu}},f\overline{\tilde{\nu}},f\overline{\tilde{\nu}}),\tilde{\mathcal{B}}(f\tilde{\nu},f\tilde{\nu},f\tilde{\nu}),\tilde{\mathcal{B}}(f\tilde{\nu},f\tilde{\nu},f\tilde{\nu})\} \\ &= \max\{\tilde{\mathcal{B}}(f\overline{\tilde{\nu}},f\tilde{\nu},f\tilde{\nu}),\tilde{\theta}\} \\ &= \{\tilde{\mathcal{B}}(\overline{\tilde{\nu}},\tilde{\nu},\tilde{\nu}),\tilde{\theta}\}. \end{aligned}$$

Therefore, we also conclude that $\tilde{\mathcal{B}}(\overline{\tilde{\nu}},\tilde{\nu},\tilde{\nu})=\tilde{\theta}$, that is $\overline{\tilde{\nu}}=\tilde{\nu}$. So we show that $\tilde{\nu}$ is the unique common fixed point of g and f.

□

Next, we state the following fixed point results for the weaker Meir-Keeler type functions in ball-metric spaces.

Theorem 2.3. *Let $(X,\tilde{\mathcal{B}})$ be a complete soft cone ball -metric space, P be a regular cone in E and $f:X\to X$. Suppose that there exists a ψ-function such that*

$$\tilde{\mathcal{B}}(f\tilde{x},f\tilde{y},f\tilde{z})\tilde{\preceq}\psi(\mathcal{L}(\tilde{x},\tilde{y},\tilde{z}))\ \textit{for all}\ \tilde{x},\tilde{y},\tilde{z}\tilde{\in}\tilde{X}, \tag{4}$$

where

$$\mathcal{L}(\tilde{x},\tilde{y},\tilde{z})=\max\{\tilde{\mathcal{B}}(\tilde{x},\tilde{y},\tilde{z}),\tilde{\mathcal{B}}(\tilde{x},f\tilde{x},f\tilde{x}),\tilde{\mathcal{B}}(\tilde{y},f\tilde{y},f\tilde{y}),\tilde{\mathcal{B}}(f\tilde{x},\tilde{y},\tilde{z})\}$$

Then f has a unique fixed point (say $\tilde{\mu}$) in $\tilde{X}$ and f is continuous at $\tilde{\mu}$.

Proof. Given $\tilde{x}_0^0\in X$. Define the sequence $\{\tilde{x}_n^n\}$ in X recursively as follows:

$$f\tilde{x}_{n-1}^{n-1}=\tilde{x}_n^n\ \text{ for each }\ n\in\mathbb{N}.$$

In what follows we will suppose that $\tilde{x}_{n+1}^{n+1}\neq\tilde{x}_n^n$ for all $n\in\mathbb{N}$, since if $\tilde{x}_{n+1}^{n+1}=\tilde{x}_n^n$ for some n, then $\tilde{x}_{n+1}^{n+1}=f\tilde{x}_n^n=\tilde{x}_n^n$, and so we complete the proof. By (4), we deduce

$$\tilde{\mathcal{B}}(\tilde{x}_n^n,\tilde{x}_{n+1}^{n+1},\tilde{x}_{n+1}^{n+1}) = \tilde{\mathcal{B}}(f\tilde{x}_{n-1}^{n-1},f\tilde{x}_n^n,f\tilde{x}_n^n)$$

$$\tilde{\preceq} \ \psi(\mathcal{L}(\tilde{x}_{n-1}^{n-1}, \tilde{x}_n^n, \tilde{x}_n^n)),$$

where

$$\begin{aligned}\mathcal{L}(\tilde{x}_{n-1}^{n-1}, \tilde{x}_n^n, \tilde{x}_n^n) &= \max\{\tilde{\mathcal{B}}(\tilde{x}_{n-1}^{n-1}, \tilde{x}_n^n, \tilde{x}_n^n), \tilde{\mathcal{B}}(\tilde{x}_{n-1}^{n-1}, f\tilde{x}_{n-1}^{n-1}, f\tilde{x}_{n-1}^{n-1}), \\ &\quad \tilde{\mathcal{B}}(\tilde{x}_n^n, f\tilde{x}_n^n, f\tilde{x}_n^n), \tilde{\mathcal{B}}(f\tilde{x}_{n-1}^{n-1}, \tilde{x}_n^n, \tilde{x}_n^n)\} \\ &= \max\{\tilde{\mathcal{B}}(\tilde{x}_{n-1}^{n-1}, \tilde{x}_n^n, \tilde{x}_n^n), \tilde{\mathcal{B}}(\tilde{x}_{n-1}^{n-1}, \tilde{x}_n^n, \tilde{x}_n^n), \\ &\quad \tilde{\mathcal{B}}(\tilde{x}_n^n, \tilde{x}_{n+1}^{n+1}, \tilde{x}_{n+1}^{n+1}), \tilde{\mathcal{B}}(\tilde{x}_n^n, \tilde{x}_n^n, \tilde{x}_n^n)\} \\ &= \max\{\tilde{\mathcal{B}}(\tilde{x}_{n-1}^{n-1}, \tilde{x}_n^n, \tilde{x}_n^n), \tilde{\mathcal{B}}(\tilde{x}_n^n, \tilde{x}_{n+1}^{n+1}, \tilde{x}_{n+1}^{n+1})\}.\end{aligned}$$

If

$$\mathcal{L}(\tilde{x}_{n-1}^{n-1}, \tilde{x}_n^n, \tilde{x}_n^n) = \tilde{\mathcal{B}}(\tilde{x}_n^n, \tilde{x}_{n+1}^{n+1}, \tilde{x}_{n+1}^{n+1}),$$

then

$$\begin{aligned}\tilde{\mathcal{B}}(\tilde{x}_n^n, \tilde{x}_{n+1}^{n+1}, \tilde{x}_{n+1}^{n+1}) &\tilde{\preceq} \ \psi(\mathcal{L}(\tilde{x}_{n-1}^{n-1}, \tilde{x}_n^n, \tilde{x}_n^n)) \\ &\tilde{\ll} \ \tilde{\mathcal{B}}(\tilde{x}_n^n, \tilde{x}_{n+1}^{n+1}, \tilde{x}_{n+1}^{n+1}),\end{aligned}$$

a contradiction. So we deduce that

$$\begin{aligned}\tilde{\mathcal{B}}(\tilde{x}_n^n, \tilde{x}_{n+1}^{n+1}, \tilde{x}_{n+1}^{n+1}) &\tilde{\preceq} \ \psi(\mathcal{L}(\tilde{x}_{n-1}^{n-1}, \tilde{x}_n^n, \tilde{x}_n^n)) \\ &\tilde{\ll} \ \tilde{\mathcal{B}}(\tilde{x}_{n-1}^{n-1}, \tilde{x}_n^n, \tilde{x}_n^n),\end{aligned}$$

and

$$\begin{aligned}\tilde{\mathcal{B}}(\tilde{x}_n^n, \tilde{x}_{n+1}^{n+1}, \tilde{x}_{n+1}^{n+1}) &\tilde{\preceq} \ \psi(\tilde{\mathcal{B}}(\tilde{x}_{n-1}^{n-1}, \tilde{x}_n^n, \tilde{x}_n^n)) \\ &\tilde{\preceq} \ \psi^2(\tilde{\mathcal{B}}(x_{n-2}, \tilde{x}_{n-1}^{n-1}, \tilde{x}_{n-1}^{n-1})) \\ &\tilde{\preceq} \ \cdots\cdots \\ &\tilde{\preceq} \ \psi^n(\tilde{\mathcal{B}}(\tilde{x}_0^0, \tilde{x}_1^1, \tilde{x}_1^1)).\end{aligned}$$

Since $\{\psi^n(\mathbb{B}(\tilde{x}_0^0, \tilde{x}_1^1, \tilde{x}_1^1))\}_{n\tilde{\in}\mathbb{N}}$ is non-incresing, it must converge to some $\tilde{\eta}$, $\tilde{\eta} \succcurlyeq \tilde{\theta}$. We claim that $\tilde{\eta} = \tilde{\theta}$. On the contrary, assume that $\tilde{\eta} \gg \tilde{\theta}$. Then by the definition of the ψ-function, there exists $\delta \gg \tilde{\theta}$ such that for $\tilde{x}_0^0, \tilde{x}_1^1 \in X$ with $\tilde{\eta}\tilde{\preceq}\tilde{\mathcal{B}}(\tilde{x}_0^0, \tilde{x}_1^1, \tilde{x}_1^1)\tilde{\ll}\delta + \tilde{\eta}$, there exists $n_0 \in \mathbb{N}$ such that $\psi^{n_0}(\tilde{\mathcal{B}}(f\tilde{x}_0^0, f\tilde{x}_1^1, f\tilde{x}_1^1))\tilde{\ll}\tilde{\eta}$. Since $\lim_{n\to\infty} \psi^n(\tilde{\mathcal{B}}(\tilde{x}_0^0, \tilde{x}_1^1, \tilde{x}_1^1)) = \tilde{\eta}$, there exists $m_0 \in \mathbb{N}$ such that

$$\tilde{\eta}\tilde{\preceq}\psi^m\tilde{\mathcal{B}}(\tilde{x}_0^0, \tilde{x}_1^1, \tilde{x}_1^1)\tilde{\ll}\delta + \tilde{\eta},$$

for all $m \geq m_0$. Thus, we get

$$\psi^{m_0+n_0}(\tilde{\mathcal{B}}(\tilde{x}_0^0, \tilde{x}_1^1, \tilde{x}_1^1))\tilde{\ll}\tilde{\eta},$$

and we get a contradiction. So

$$\lim_{n\to\infty} \psi^n(\tilde{\mathcal{B}}(\tilde{x}_0^0, \tilde{x}_1^1, \tilde{x}_1^1)) = \tilde{\theta},$$

and so we have $\lim_{n\to\infty}\tilde{\mathcal{B}}(\tilde{x}_n^n,\tilde{x}_{n+1}^{n+1},\tilde{x}_{n+1}^{n+1})=\tilde{\theta}$. For $m,n\in\mathbb{N}$ with $m>n>\kappa_0$, we claim that the following result holds:

$$\tilde{\mathcal{B}}(\tilde{x}_n^n,\tilde{x}_m^m,\tilde{x}_m^m)\prec\tilde{\epsilon}\ \text{ for all }\ m>n>\kappa_0. \tag{5}$$

Let $\tilde{\epsilon}\tilde{\in}\tilde{E}$ with $\tilde{\epsilon}\gg 0$ be given. Since $\lim_{n\to\infty}\varphi^n(\tilde{\mathcal{B}}(\tilde{x}_0^0,\tilde{x}_1^1,\tilde{x}_1^1))=\tilde{\theta}$ and $\psi(\tilde{\epsilon})\tilde{\ll}\tilde{\epsilon}$, there exists $\kappa_0\in\mathbb{N}$ such that

$$\psi^n(\tilde{\mathcal{B}}(\tilde{x}_0^0,\tilde{x}_1^1,\tilde{x}_1^1))\tilde{\ll}\tilde{\epsilon}-\psi(\tilde{\epsilon})\ \text{ for all }\ n\geq\kappa_0,$$

that is,

$$\tilde{\mathcal{B}}(\tilde{x}_n^n,\tilde{x}_{n+1}^{n+1},\tilde{x}_{n+1}^{n+1})\tilde{\ll}\tilde{\epsilon}-\psi(\tilde{\epsilon})\ \text{ for all }\ n\geq. \tag{6}$$

We prove (5) by induction on m. Assume that the inequality (5) holds for $m=k$. Then by (6), we have that for $m=k+1$,

$$\begin{aligned}\tilde{\mathcal{B}}(\tilde{x}_n^n,\tilde{x}_{n_k+1}^{n_k+1},\tilde{x}_{n_k+1}^{n_k+1}) &\tilde{\preceq} \tilde{\mathcal{B}}(\tilde{x}_n^n,\tilde{x}_{n+1}^{n+1},\tilde{x}_{n+1}^{n+1})+\tilde{\mathcal{B}}(\tilde{x}_{n+1}^{n+1},\tilde{x}_{n_k+1}^{n_k+1},\tilde{x}_{n_k+1}^{n_k+1})\\ &\tilde{\ll} \tilde{\mathcal{B}}(\tilde{x}_n^n,\tilde{x}_{n+1}^{n+1},\tilde{x}_{n+1}^{n+1})+\psi(\tilde{\mathcal{B}}(\tilde{x}_n^n,\tilde{x}_{n_k}^{n_k},\tilde{x}_{n_k}^{n_k})\\ &\tilde{\ll} \tilde{\epsilon}-\psi(\tilde{\epsilon})+\psi(\tilde{\epsilon})\\ &= \tilde{\epsilon}.\end{aligned}$$

Thus, we conclude that $\tilde{\mathcal{B}}(\tilde{x}_n^n,x_m,x_m)\tilde{\ll}\tilde{\epsilon}$ for all $m>n>\kappa_0$. So $\{\tilde{x}_n^n\}$ is a Cauchy sequence in $\tilde{X}$. Since $(\tilde{X},\tilde{\mathcal{B}})$ is a complete soft cone ball -metric space, there exists $\tilde{\mu}\in\tilde{X}$ such that $\lim_{n\to\infty}\tilde{x}_n^n=\tilde{\mu}$, that is, $\tilde{\mathcal{B}}(\tilde{x}_n^n,\tilde{x}_n^n,\tilde{\mu})\to\tilde{\theta}$. For $n\in\mathbb{N}$, we have

$$\begin{aligned}\tilde{\mathcal{B}}(\tilde{\mu},\tilde{\mu},f\tilde{\mu}) &\tilde{\preceq} \tilde{\mathcal{B}}(\tilde{\mu},\tilde{\mu},\tilde{x}_n^n)+\tilde{\mathcal{B}}(\tilde{x}_n^n,\tilde{x}_n^n,f\tilde{\mu})\\ &\tilde{\preceq} \tilde{\mathcal{B}}(\tilde{\mu},\tilde{\mu},\tilde{x}_n^n)+\tilde{\mathcal{B}}(f\tilde{x}_{n-1}^{n-1},f\tilde{x}_{n-1}^{n-1},f\tilde{\mu})\\ &\tilde{\preceq} \tilde{\mathcal{B}}(\tilde{\mu},\tilde{\mu},\tilde{x}_n^n)+\psi(\mathcal{L}(\tilde{x}_{n-1}^{n-1},\tilde{x}_{n-1}^{n-1},\tilde{\mu})),\end{aligned}$$

where

$$\begin{aligned}\mathcal{L}(\tilde{x}_{n-1}^{n-1},\tilde{x}_{n-1}^{n-1},\tilde{\mu}) &= \max\{\tilde{\mathcal{B}}(\tilde{x}_{n-1}^{n-1},\tilde{x}_{n-1}^{n-1},\tilde{\mu}),\tilde{\mathcal{B}}(\tilde{x}_{n-1}^{n-1},f\tilde{x}_{n-1}^{n-1},f\tilde{x}_{n-1}^{n-1}),\\ &\quad\tilde{\mathcal{B}}(\tilde{x}_{n-1}^{n-1},f\tilde{x}_{n-1}^{n-1},f\tilde{x}_{n-1}^{n-1}),\tilde{\mathcal{B}}(f\tilde{x}_{n-1}^{n-1},\tilde{x}_{n-1}^{n-1},\tilde{\mu})\}\\ &= \max\{\tilde{\mathcal{B}}(\tilde{x}_{n-1}^{n-1},\tilde{x}_{n-1}^{n-1},\tilde{\mu}),\tilde{\mathcal{B}}(\tilde{x}_{n-1}^{n-1},\tilde{x}_n^n,\tilde{x}_n^n),\\ &\quad\tilde{\mathcal{B}}(\tilde{x}_{n-1}^{n-1},\tilde{x}_n^n,\tilde{x}_n^n),\tilde{\mathcal{B}}(\tilde{x}_n^n,\tilde{x}_{n-1}^{n-1},\tilde{\mu})\}.\end{aligned}$$

(I) If

$$\mathcal{L}(\tilde{x}_{n-1}^{n-1},\tilde{x}_{n-1}^{n-1},\tilde{\mu})=\tilde{\mathcal{B}}(\tilde{x}_{n-1}^{n-1},\tilde{x}_{n-1}^{n-1},\tilde{\mu}),$$

then

$$\tilde{\mathcal{B}}(\tilde{\mu},\tilde{\mu},f\tilde{\mu})\tilde{\ll}\tilde{\mathcal{B}}(\tilde{\mu},\tilde{\mu},\tilde{x}_n^n)+\tilde{\mathcal{B}}(\tilde{x}_{n-1}^{n-1},\tilde{x}_{n-1}^{n-1},\tilde{\mu}).$$

Letting $n\to\infty$, we conclude that $\tilde{\mathcal{B}}(\tilde{\mu},\tilde{\mu},f\tilde{\mu})=\tilde{\theta}$, and so $\tilde{\mu}=f\tilde{\mu}$. (II) If

$$\mathcal{L}(\tilde{x}_{n-1}^{n-1},\tilde{x}_{n-1}^{n-1},\tilde{\mu})=\tilde{\mathcal{B}}(\tilde{x}_{n-1}^{n-1},\tilde{x}_n^n,\tilde{x}_n^n),$$

then

$$\tilde{\mathcal{B}}(\tilde{\mu},\tilde{\mu},f\tilde{\mu})\tilde{\ll}\tilde{\mathcal{B}}(\tilde{\mu},\tilde{\mu},\tilde{x}_n^n)+\tilde{\mathcal{B}}(\tilde{x}_{n-1}^{n-1},\tilde{x}_n^n,\tilde{x}_n^n).$$

Letting $n\to\infty$, we conclude that $\tilde{\mathcal{B}}(\tilde{\mu},\tilde{\mu},f\tilde{\mu})=\tilde{\theta}$, and so $\tilde{\mu}=f\tilde{\mu}$. (III) If

$$\mathcal{L}(\tilde{x}_{n-1}^{n-1},\tilde{x}_{n-1}^{n-1},\tilde{\mu})=\tilde{\mathcal{B}}(\tilde{x}_n^n,\tilde{x}_{n-1}^{n-1},\tilde{\mu})\tilde{\preceq}\tilde{\mathcal{B}}(\tilde{x}_n^n,\tilde{x}_{n-1}^{n-1},\tilde{x}_{n-1}^{n-1})+\tilde{\mathcal{B}}(\tilde{x}_{n-1}^{n-1},\tilde{x}_{n-1}^{n-1},\tilde{\mu}),$$

then

$$\tilde{\mathcal{B}}(\tilde{\mu},\tilde{\mu},T\tilde{\mu})\tilde{\ll}\tilde{\mathcal{B}}(\tilde{\mu},\tilde{\mu},\tilde{x}_n^n)+\tilde{\mathcal{B}}(\tilde{x}_n^n,\tilde{x}_{n-1}^{n-1},\tilde{x}_{n-1}^{n-1})+\tilde{\mathcal{B}}(\tilde{x}_{n-1}^{n-1},\tilde{x}_{n-1}^{n-1},\tilde{\mu}).$$

Letting $n\to\infty$, we conclude that $\tilde{\mathcal{B}}(\tilde{\mu},\tilde{\mu},f\tilde{\mu})=\tilde{\theta}$, and so $\tilde{\mu}=f\tilde{\mu}$. Follow (I), (II) and (III), we have that $\tilde{\mu}$ is a fixed point of f. Let $\tilde{\nu}$ be another fixed point of f with $\tilde{\mu}\neq\tilde{\nu}$. Then

$$\tilde{\mathcal{B}}(\tilde{\mu},\tilde{\nu},\tilde{\nu})=\tilde{\mathcal{B}}(f\tilde{\mu},f\tilde{\nu},f\tilde{\nu})\tilde{\preceq}\psi(\mathcal{L}(\tilde{\mu},\tilde{\nu},\tilde{\nu})),$$

where

$$\begin{aligned}\mathcal{L}(\tilde{\mu},\tilde{\nu},\tilde{\nu}) &= \max\{\tilde{\mathcal{B}}(\tilde{\mu},\tilde{\nu},\tilde{\nu}),\tilde{\mathcal{B}}(\tilde{\mu},f\tilde{\mu},f\tilde{\mu}),\tilde{\mathcal{B}}(\tilde{\nu},f\tilde{\nu},f\tilde{\nu}),\tilde{\mathcal{B}}(f\tilde{\mu},\tilde{\nu},\tilde{\nu})\}\\ &= \max\{\tilde{\mathcal{B}}(\tilde{\mu},\tilde{\nu},\tilde{\nu}),\tilde{\mathcal{B}}(\tilde{\mu},\tilde{\mu},\tilde{\mu}),\tilde{\mathcal{B}}(\tilde{\nu},\tilde{\nu},\tilde{\nu}),\tilde{\mathcal{B}}(\tilde{\mu},\tilde{\nu},\tilde{\nu})\}\\ &= \max\{\tilde{\mathcal{B}}(\tilde{\mu},\tilde{\nu},\tilde{\nu})\},\tilde{\theta}\}.\end{aligned}$$

Therefore, if $\tilde{\mathcal{B}}(\tilde{\mu},\tilde{\nu},\tilde{\nu})\tilde{\ll}\tilde{\mathcal{B}}(\tilde{\mu},\tilde{\nu},\tilde{\nu})$, then we get a contradiction. So $\tilde{\mu}=\tilde{\nu}$, and we show that $\tilde{\mu}$ is a unique fixed point of f. To show that f is continuous at $\tilde{\mu}$. Let $\{\tilde{y}_n^n\}$ be any sequence in X such that $\{\tilde{y}_n^n\}$ convergent to $\tilde{\mu}$. Then

$$\begin{aligned}\tilde{\mathcal{B}}(\tilde{\mu},\tilde{\mu},f\tilde{y}_n^n) &= \tilde{\mathcal{B}}(f\tilde{\mu},f\tilde{\mu},f\tilde{y}_n^n)\\ &\tilde{\preceq} \varphi(\mathcal{L}(\tilde{\mu},\tilde{\mu},\tilde{y}_n^n)),\end{aligned}$$

where

$$\begin{aligned}\mathcal{L}(\tilde{\mu},\tilde{\mu},\tilde{y}_n^n) &= \max\{\tilde{\mathcal{B}}(\tilde{\mu},\tilde{\mu},\tilde{y}_n^n),\tilde{\mathcal{B}}(\tilde{\mu},f\tilde{\mu},f\tilde{\mu}),\tilde{\mathcal{B}}(\tilde{\mu},f\tilde{\mu},f\tilde{\mu}),\tilde{\mathcal{B}}(f\tilde{\mu},\tilde{\mu},\tilde{y}_n^n)\}\\ &= \max\{\tilde{\mathcal{B}}(\tilde{\mu},\tilde{\mu},\tilde{y}_n^n),\tilde{\theta}\}.\end{aligned}$$

Thus

$$\tilde{\mathcal{B}}(\tilde{\mu},\tilde{\mu},f\tilde{y}_n^n)\tilde{\ll}\tilde{\mathcal{B}}(\tilde{\mu},\tilde{\mu},\tilde{y}_n^n).$$

Letting $n\to\infty$. Then we deduce that $\{f\tilde{y}_n^n\}$ is convergent to $f\tilde{\mu}=\tilde{\mu}$. Hence f is continuous at $\tilde{\mu}$. □

By Theorem 2.3, we immediate get the following corollary.

Corollary 2.4. *Let $(\tilde{X},\tilde{\mathcal{B}})$ be a complete cone ball-metric space, P be a regular cone in $\tilde{E}$ and $f:X\to X$. Suppose that there exists a ψ-function such that*

$$\tilde{\mathcal{B}}(f\tilde{x},f\tilde{y},f\tilde{z})\tilde{\preceq}\psi(\tilde{\mathcal{B}}(\tilde{x},\tilde{y},\tilde{z})\qquad(\tilde{x},\tilde{y},\tilde{z}\tilde{\in}\tilde{X}).$$

Then f has a unique fixed point (say $\tilde{\mu}$) in $\tilde{X}$ and f is continuous at $\tilde{\mu}$.

In the sequel, we introduce the stronger Meir-Keeler cone-type function $\phi : intP \cup \{\tilde{\theta}\} \to [0,1)$ in cone ball-metric spaces, and prove the fixed point theorem for this type of function.

Definition 2.5. *Let $(\tilde{X}, \tilde{\mathcal{B}})$ be a cone ball-metric space with cone P, and let*

$$\phi : intP\tilde{\cup}\{\tilde{\theta}\} \to [0,1).$$

Then the function ϕ is called a stronger Meir-Keeler type function, if for each $\tilde{\eta}\tilde{\in}P$ with $\tilde{\eta} \gg \tilde{\theta}$, there exists $\delta \gg \tilde{\theta}$ such that for $\tilde{x}, \tilde{y}, \tilde{z}\tilde{\in}\tilde{X}$ with $\tilde{\eta}\tilde{\preceq}\tilde{\mathcal{B}}(\tilde{x}, \tilde{y}, \tilde{z})\tilde{\ll}\delta + \tilde{\eta}$, there exists $\gamma_{\tilde{\eta}} \in [0,1)$ such that $\phi(\tilde{\mathcal{B}}(\tilde{x}, \tilde{y}, \tilde{z})) < \gamma_{\tilde{\eta}}$.

Theorem 2.6. *Let $(\tilde{X}, \tilde{\mathcal{B}})$ be a complete cone ball-metric space, P be a regular cone in E and $f : X \to X$. Suppose that there exists a stronger Meir-Keeler type function $\phi : intP\tilde{\cup}\{0\} \to [0,1)$ such that*

$$\tilde{\mathcal{B}}(f\tilde{x}, f\tilde{y}, f\tilde{z})\tilde{\preceq}\phi(\mathcal{L}(\tilde{x}, \tilde{y}, \tilde{z})) \cdot \mathcal{L}(\tilde{x}, \tilde{y}, \tilde{z}) \text{ for all } \tilde{x}, \tilde{y}, \tilde{z} \in X, \tag{7}$$

where

$$\mathcal{L}(\tilde{x}, \tilde{y}, \tilde{z}) = \max\{\tilde{\mathcal{B}}(\tilde{x}, \tilde{y}, \tilde{z}), \tilde{\mathcal{B}}(\tilde{x}, f\tilde{x}, f\tilde{x}), \tilde{\mathcal{B}}(\tilde{y}, f\tilde{y}, f\tilde{y}), \tilde{\mathcal{B}}(f\tilde{x}, \tilde{y}, \tilde{z})\}$$

Then f has a unique fixed point (say $\tilde{\mu}$) in $\tilde{X}$ and f is continuous at $\tilde{\mu}$.

Proof. Given $\tilde{x}_0^0\tilde{\in}\tilde{X}$. Define the sequence $\{\tilde{x}_n^n\}$ in X recursively as follows:

$$f\tilde{x}_{n-1}^{n-1} = \tilde{x}_n^n \text{ for each } n \in \mathbb{N}.$$

In what follows, we will suppose that $\tilde{x}_{n+1}^{n+1} \neq \tilde{x}_n^n$ for all $n\tilde{\in}\mathbb{N}$, since if $\tilde{x}_{n+1}^{n+1} = \tilde{x}_n^n$ for some n, then $\tilde{x}_{n+1}^{n+1} = f\tilde{x}_n^n = \tilde{x}_n^n$, and so we complete the proof. By (7), we deduce

$$\begin{aligned}
\tilde{\mathcal{B}}(\tilde{x}_n^n, \tilde{x}_{n+1}^{n+1}, \tilde{x}_{n+1}^{n+1}) &= \tilde{\mathcal{B}}(f\tilde{x}_{n-1}^{n-1}, f\tilde{x}_n^n, f\tilde{x}_n^n) \\
&\tilde{\preceq} \phi(\mathcal{L}(\tilde{x}_{n-1}^{n-1}, \tilde{x}_n^n, \tilde{x}_n^n)) \cdot \mathcal{L}(\tilde{x}_{n-1}^{n-1}, \tilde{x}_n^n, \tilde{x}_n^n) \\
&\tilde{\ll} \gamma_{\tilde{\eta}} \cdot \mathcal{L}(\tilde{x}_{n-1}^{n-1}, \tilde{x}_n^n, \tilde{x}_n^n),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{L}(\tilde{x}_{n-1}^{n-1}, \tilde{x}_n^n, \tilde{x}_n^n) &= \max\{\tilde{\mathcal{B}}(\tilde{x}_{n-1}^{n-1}, \tilde{x}_n^n, \tilde{x}_n^n), \tilde{\mathcal{B}}(\tilde{x}_{n-1}^{n-1}, f\tilde{x}_{n-1}^{n-1}, f\tilde{x}_{n-1}^{n-1}), \\
&\quad \tilde{\mathcal{B}}(\tilde{x}_n^n, f\tilde{x}_n^n, f\tilde{x}_n^n), \tilde{\mathcal{B}}(f\tilde{x}_{n-1}^{n-1}, \tilde{x}_n^n, \tilde{x}_n^n)\} \\
&= \max\{\tilde{\mathcal{B}}(\tilde{x}_{n-1}^{n-1}, \tilde{x}_n^n, \tilde{x}_n^n), \tilde{\mathcal{B}}(\tilde{x}_{n-1}^{n-1}, \tilde{x}_n^n, \tilde{x}_n^n), \\
&\quad \tilde{\mathcal{B}}(\tilde{x}_n^n, \tilde{x}_{n+1}^{n+1}, \tilde{x}_{n+1}^{n+1}), \tilde{\mathcal{B}}(\tilde{x}_n^n, \tilde{x}_n^n, \tilde{x}_n^n)\} \\
&= \max\{\tilde{\mathcal{B}}(\tilde{x}_{n-1}^{n-1}, \tilde{x}_n^n, \tilde{x}_n^n), \tilde{\mathcal{B}}(\tilde{x}_n^n, \tilde{x}_{n+1}^{n+1}, \tilde{x}_{n+1}^{n+1})\}.
\end{aligned}$$

If

$$\mathcal{L}(\tilde{x}_{n-1}^{n-1}, \tilde{x}_n^n, \tilde{x}_n^n) = \tilde{\mathcal{B}}(\tilde{x}_n^n, \tilde{x}_{n+1}^{n+1}, \tilde{x}_{n+1}^{n+1}),$$

then

$$\tilde{\mathcal{B}}(\tilde{x}_n^n, \tilde{x}_{n+1}^{n+1}, \tilde{x}_{n+1}^{n+1}) \;\tilde{\ll}\; \gamma_{\tilde{\eta}} \cdot \tilde{\mathcal{B}}(\tilde{x}_n^n, \tilde{x}_{n+1}^{n+1}, \tilde{x}_{n+1}^{n+1}),$$

a contradiction. So we deduce that

$$\begin{aligned}\tilde{\mathcal{B}}(\tilde{x}_n^n, \tilde{x}_{n+1}^{n+1}, \tilde{x}_{n+1}^{n+1}) &\tilde{\preceq} \phi(\mathcal{L}(\tilde{x}_{n-1}^{n-1}, \tilde{x}_n^n, \tilde{x}_n^n)) \\ &\tilde{\ll} \tilde{\gamma}_{\tilde{\eta}} \cdot \tilde{\mathcal{B}}(\tilde{x}_{n-1}^{n-1}, \tilde{x}_n^n, \tilde{x}_n^n).\end{aligned}$$

Then the sequence $\{\tilde{\mathcal{B}}(\tilde{x}_n^n, \tilde{x}_{n+1}^{n+1}, \tilde{x}_{n+1}^{n+1})\}$ is decreasing and bounded below. Let

$$\lim_{n\to\infty} \tilde{\mathcal{B}}(\tilde{x}_n^n, \tilde{x}_{n+1}^{n+1}, \tilde{x}_{n+1}^{n+1}) = \tilde{\eta} \tilde{\succeq} \tilde{\theta}.$$

Then there exists $\kappa_0 \tilde{\in} \mathbb{N}$ and $\delta \gg \tilde{\theta}$ such that for all $n > \kappa_0$

$$\eta \tilde{\preceq} \tilde{\mathcal{B}}(\tilde{x}_n^n, \tilde{x}_{n+1}^{n+1}, \tilde{x}_{n+1}^{n+1}) \tilde{\ll} \eta + \delta.$$

For each $n \in \mathbb{N}$, since $\phi : intP \tilde{\cup} \{\tilde{\theta}\} \to [0, 1)$ is a stronger Meir-Keeler type function, for these η and δ we have that for $\tilde{x}_{\kappa_0+n}, \tilde{x}_{\kappa_0+n+1} \in X$ with

$$\eta \tilde{\preceq} \tilde{\mathcal{B}}(\tilde{x}_{\kappa_0+n}, \tilde{x}_{\kappa_0+n+1}, \tilde{x}_{\kappa_0+n+1}) \tilde{\ll} \delta + \eta,$$

there exists $\gamma_{\tilde{\eta}} \in [0, 1)$ such that

$$\phi(\tilde{\mathcal{B}}(\tilde{x}_{\kappa_0+n}, \tilde{x}_{\kappa_0+n+1}, \tilde{x}_{\kappa_0+n+1})) < \gamma_{\tilde{\eta}}.$$

Thus, by (7), we can deduce

$$\begin{aligned}\tilde{\mathcal{B}}(\tilde{x}_{\kappa_0+n}^{\kappa_0+n}, \tilde{x}_{\kappa_0+n+1}^{\kappa_0+n+1}, \tilde{x}_{\kappa_0+n+1}^{\kappa_0+n+1}) &= \phi(\tilde{\mathcal{B}}(\tilde{x}_{\kappa_0+n-1}^{\kappa_0+n-1}, \tilde{x}_{\kappa_0+n}^{\kappa_0+n}, \tilde{x}_{\kappa_0+n}^{\kappa_0+n})) \cdot \tilde{\mathcal{B}}(\tilde{x}_{\kappa_0+n-1}^{\kappa_0+n-1}, \tilde{x}_{\kappa_0+n}^{\kappa_0+n}, \tilde{x}_{\kappa_0+n}^{\kappa_0+n}) \\ &\tilde{\ll} \gamma_{\tilde{\eta}} \cdot \tilde{\mathcal{B}}(\tilde{x}_{\kappa_0+n-1}^{\kappa_0+n-1}, \tilde{x}_{\kappa_0+n-1}^{\kappa_0+n-1}, \tilde{x}_{\kappa_0+n}^{\kappa_0+n}),\end{aligned}$$

and it follows that for each $n \in \mathbb{N}$,

$$\begin{aligned}\tilde{\mathcal{B}}(\tilde{x}_{\kappa_0+n}^{\kappa_0+n}, \tilde{x}_{\kappa_0+n+1}^{\kappa_0+n+1}, \tilde{x}_{\kappa_0+n+1}^{\kappa_0+n+1}) &\tilde{\ll} \gamma_{\tilde{\eta}} \cdot \tilde{\mathcal{B}}(\tilde{x}_{\kappa_0+n-1}^{\kappa_0+n-1}, \tilde{x}_{\kappa_0+n}^{\kappa_0+n}, \tilde{x}_{\kappa_0+n}^{\kappa_0+n}) \\ &\tilde{\ll} \cdots \\ &\tilde{\ll} \gamma_{\tilde{\eta}}^n \cdot \tilde{\mathcal{B}}(\tilde{x}_{\kappa_0}^{\kappa_0}, \tilde{x}_{\kappa_0+1}^{\kappa_0+1}, \tilde{x}_{\kappa_0+1}^{\kappa_0+1}).\end{aligned}$$

So

$$\lim_{n\to\infty} \tilde{\mathcal{B}}(\tilde{x}_{\kappa_0+n}^{\kappa_0+n}, \tilde{x}_{\kappa_0+n+1}^{\kappa_0+n+1}, \tilde{x}_{\kappa_0+n+1}^{\kappa_0+n+1}) = \tilde{\theta}, \text{ since } \gamma_{\tilde{\eta}} < 1.$$

We next claim that $\lim_{m,n\to\infty} \tilde{\mathcal{B}}(\tilde{x}_{\kappa_0+n+1}^{\kappa_0+n+1}, \tilde{x}_{\kappa_0+m}^{\kappa_0+m}, \tilde{x}_{\kappa_0+m}^{\kappa_0+m}) = \tilde{\theta}$. For $m, n \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned}\tilde{\mathcal{B}}(\tilde{x}_{\kappa_0+n}^{\kappa_0+n}, \tilde{x}_{\kappa_0+m}^{\kappa_0+m}, \tilde{x}_{\kappa_0+m}^{\kappa_0+m}) &\tilde{\preceq} \sum_{i=n}^{m-1} \tilde{\mathcal{B}}((\tilde{x}_{\kappa_0+i}^{\kappa_0+i}, (\tilde{x}_{\kappa_0+i+1}^{\kappa_0+i+1}, (\tilde{x}_{\kappa_0+i+1}^{\kappa_0+i+1}) \\ &\tilde{\ll} \frac{\gamma_{\tilde{\eta}}^{m-1}}{1-\gamma_{\tilde{\eta}}} \tilde{\mathcal{B}}(\tilde{x}_{\kappa_0+1}^{\kappa_0+1}, \tilde{x}_{\kappa_0+2}^{\kappa_0+2}, \tilde{x}_{\kappa_0+2}^{\kappa_0+2}),\end{aligned}$$

and hence $\tilde{\mathcal{B}}(\tilde{x}_{\kappa_0+n}^{\kappa_0+n}, \tilde{x}_{\kappa_0+m}^{\kappa_0+m}, \tilde{x}_{\kappa_0+m}^{\kappa_0+m}) \to \tilde{\theta}$ as $m, n \to \infty$, since $0 < \gamma_{\tilde{\eta}} < 1$. By the properties of the cone ball-metric, we obtain

$$\tilde{\mathcal{B}}(\tilde{x}_{\kappa_0+n}^{\kappa_0+n}, \tilde{x}_{\kappa_0+m}^{\kappa_0+m}, \tilde{x}_{\kappa_0+1}^{\kappa_0+1}) \tilde{\preceq} \tilde{\mathcal{B}}(\tilde{x}_{\kappa_0+n}^{\kappa_0+n}, \tilde{x}_{\kappa_0+m}^{\kappa_0+m}, \tilde{x}_{\kappa_0+m}^{\kappa_0+m}) + \tilde{\mathcal{B}}(\tilde{x}_{\kappa_0+m}^{\kappa_0+m}, \tilde{x}_{\kappa_0+m}^{\kappa_0+m}, \tilde{x}_{\kappa_0+1}^{\kappa_0+1})$$

taking limit as $m,n,l\to\infty$, we get $\tilde{\mathcal{B}}(\tilde{x}^{\kappa_0+n}_{\kappa_0+n},\tilde{x}^{\kappa_0+m}_{\kappa_0+m},\tilde{x}^{\kappa_0+1}_{\kappa_0+1})\to\tilde{\theta}$. So $\{\tilde{x}^n_n\}$ is a Cauchy sequence.

Since $(X,\tilde{\mathcal{B}})$ is a complete cone ball-metric space, there exists $\tilde{\mu}\in X$ such that $\lim_{n\to\infty}\tilde{x}^n_n=\tilde{\mu}$, that is, $\tilde{\mathcal{B}}(\tilde{x}^n_n,\tilde{x}^n_n,\tilde{\mu})\to\tilde{\theta}$.

For $n\in\mathbb{N}$, we have

$$
\begin{aligned}
\tilde{\mathcal{B}}(\tilde{\mu},\tilde{\mu},f\tilde{\mu}) &\,\tilde{\preceq}\, \tilde{\mathcal{B}}(\tilde{\mu},\tilde{\mu},x_n)+\tilde{\mathcal{B}}(x_n,x_n,f\tilde{\mu})\\
&\,\tilde{\preceq}\, \tilde{\mathcal{B}}(\tilde{\mu},\tilde{\mu},x_n)+\tilde{\mathcal{B}}(f\tilde{x}^{n-1}_{n-1},f\tilde{x}^{n-1}_{n-1},f\tilde{\mu})\\
&\,\tilde{\preceq}\, \tilde{\mathcal{B}}(\tilde{\mu},\tilde{\mu},x_n)+\phi(\mathcal{L}(\tilde{x}^{n-1}_{n-1},\tilde{x}^{n-1}_{n-1},\tilde{\mu}))\cdot\mathcal{L}(\tilde{x}^{n-1}_{n-1},\tilde{x}^{n-1}_{n-1},\tilde{\mu})\\
&\,\tilde{\preceq}\, \tilde{\mathcal{B}}(\tilde{\mu},\tilde{\mu},x_n)+\gamma_{\tilde{\eta}}\cdot\mathcal{L}(\tilde{x}^{n-1}_{n-1},\tilde{x}^{n-1}_{n-1},\tilde{\mu}),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{L}(\tilde{x}^{n-1}_{n-1},\tilde{x}^{n-1}_{n-1},\tilde{\mu}) &= \max\{\tilde{\mathcal{B}}(\tilde{x}^{n-1}_{n-1},\tilde{x}^{n-1}_{n-1},\tilde{\mu}),\tilde{\mathcal{B}}(\tilde{x}^{n-1}_{n-1},f\tilde{x}^{n-1}_{n-1},f\tilde{x}^{n-1}_{n-1}),\\
&\qquad \tilde{\mathcal{B}}(\tilde{x}^{n-1}_{n-1},T\tilde{x}^{n-1}_{n-1},f\tilde{x}^{n-1}_{n-1}),\tilde{\mathcal{B}}(f\tilde{x}^{n-1}_{n-1},\tilde{x}^{n-1}_{n-1},\tilde{\mu})\}\\
&= \max\{\tilde{\mathcal{B}}(\tilde{x}^{n-1}_{n-1},\tilde{x}^{n-1}_{n-1},\tilde{\mu}),\tilde{\mathcal{B}}(\tilde{x}^{n-1}_{n-1},\tilde{x}^n_n,\tilde{x}^n_n),\\
&\qquad \tilde{\mathcal{B}}(\tilde{x}^{n-1}_{n-1},\tilde{x}^n_n,\tilde{x}^n_n),\tilde{\mathcal{B}}(\tilde{x}^n_n,\tilde{x}^{n-1}_{n-1},\tilde{\mu})\}.
\end{aligned}
$$

(I) If

$$\mathcal{L}(\tilde{x}^{n-1}_{n-1},\tilde{x}^{n-1}_{n-1},\tilde{\mu})=\tilde{\mathcal{B}}(\tilde{x}^{n-1}_{n-1},\tilde{x}^{n-1}_{n-1},\tilde{\mu}),$$

then

$$\tilde{\mathcal{B}}(\tilde{\mu},\tilde{\mu},f\tilde{\mu}) \,\tilde{\preceq}\, \tilde{\mathcal{B}}(\tilde{\mu},\tilde{\mu},x_n)+\gamma_{\tilde{\eta}}\cdot\tilde{\mathcal{B}}(\tilde{x}^{n-1}_{n-1},\tilde{x}^{n-1}_{n-1},\tilde{\mu}).$$

Letting $n\to\infty$, we conclude that $\tilde{\mathcal{B}}(\tilde{\mu},\tilde{\mu},f\tilde{\mu})=\tilde{\theta}$, and so $\tilde{\mu}=f\tilde{\mu}$. (II) If

$$\mathcal{L}(\tilde{x}^{n-1}_{n-1},\tilde{x}^{n-1}_{n-1},\tilde{\mu})=\tilde{\mathcal{B}}(\tilde{x}^{n-1}_{n-1},\tilde{x}^n_n,\tilde{x}^n_n),$$

then

$$\tilde{\mathcal{B}}(\tilde{\mu},\tilde{\mu},f\tilde{\mu}) \,\tilde{\preceq}\, \tilde{\mathcal{B}}(\tilde{\mu},\tilde{\mu},\tilde{x}^n_n)+\gamma_{\tilde{\eta}}\cdot\tilde{\mathcal{B}}(\tilde{x}^{n-1}_{n-1},\tilde{x}^n_n,\tilde{x}^n_n).$$

Letting $n\to\infty$, we conclude that $\tilde{\mathcal{B}}(\tilde{\mu},\tilde{\mu},T\tilde{\mu})=\tilde{\theta}$, and so $\tilde{\mu}=f\tilde{\mu}$. (III) If

$$\mathcal{L}(\tilde{x}^{n-1}_{n-1},\tilde{x}^{n-1}_{n-1},\tilde{\mu})=\tilde{\mathcal{B}}(\tilde{x}^n_n,\tilde{x}^{n-1}_{n-1},\tilde{\mu})\tilde{\preceq}\tilde{\mathcal{B}}(\tilde{x}^n_n,\tilde{x}^{n-1}_{n-1},\tilde{x}^{n-1}_{n-1})+\tilde{\mathcal{B}}(\tilde{x}^{n-1}_{n-1},\tilde{x}^{n-1}_{n-1},\tilde{\mu}),$$

then

$$\tilde{\mathcal{B}}(\tilde{\mu},\tilde{\mu},f\tilde{\mu}) \le \tilde{\mathcal{B}}(\tilde{\mu},\tilde{\mu},x_n)+\gamma_{\tilde{\eta}}\cdot[\tilde{\mathcal{B}}(\tilde{x}^n_n,\tilde{x}^{n-1}_{n-1},\tilde{x}^{n-1}_{n-1})+\tilde{\mathcal{B}}(\tilde{x}^{n-1}_{n-1},\tilde{x}^{n-1}_{n-1},\tilde{\mu})].$$

Letting $n\to\infty$, we conclude that $\tilde{\mathcal{B}}(\tilde{\mu},\tilde{\mu},f\tilde{\mu})=\tilde{\theta}$, and so $\tilde{\mu}=f\tilde{\mu}$. Follow (I), (II) and (III), we have that $\tilde{\mu}$ is a fixed point of f. Let $\tilde{\nu}$ be another fixed point of f with $\tilde{\mu}\neq\tilde{\nu}$. Then

$$\tilde{\mathcal{B}}(\tilde{\mu},\tilde{\nu},\tilde{\nu}) = \tilde{\mathcal{B}}(f\tilde{\mu},f\tilde{\nu},f\tilde{\nu})$$

$$\begin{aligned} &\tilde{\preceq} \ \psi(\mathcal{L}(\tilde{\mu},\tilde{\nu},\tilde{\nu}))\cdot\mathcal{L}(\tilde{\mu},\tilde{\nu},\tilde{\nu}) \\ &\tilde{\ll} \ \gamma_{\tilde{\eta}}\cdot\mathcal{L}(\tilde{\mu},\tilde{\nu},\tilde{\nu}), \end{aligned}$$

where

$$\begin{aligned} \mathcal{L}(\tilde{\mu},\tilde{\nu},\tilde{\nu}) &= \max\{\tilde{\mathcal{B}}(\tilde{\mu},\tilde{\nu},\tilde{\nu}),\tilde{\mathcal{B}}(\tilde{\mu},f\tilde{\mu},f\tilde{\mu}),\tilde{\mathcal{B}}(\tilde{\nu},f\tilde{\nu},f\tilde{\nu}),\tilde{\mathcal{B}}(T\tilde{\mu},\tilde{\nu},\tilde{\nu})\} \\ &= \max\{\tilde{\mathcal{B}}(\tilde{\mu},\tilde{\nu},\tilde{\nu}),\tilde{\mathcal{B}}(\tilde{\mu},\tilde{\mu},\tilde{\mu}),\tilde{\mathcal{B}}(\tilde{\nu},\tilde{\nu},\tilde{\nu}),\tilde{\mathcal{B}}(\tilde{\mu},\tilde{\nu},\tilde{\nu})\} \\ &= \max\{\tilde{\mathcal{B}}(\tilde{\mu},\tilde{\nu},\tilde{\nu}),\tilde{\theta}\}. \end{aligned}$$

Thus if $\tilde{\mathcal{B}}(\tilde{\mu},\tilde{\nu},\tilde{\nu})\tilde{\ll}\gamma_{\tilde{\eta}}\cdot\tilde{\mathcal{B}}(\tilde{\mu},\tilde{\nu},\tilde{\nu})$, then we get a contradiction. So $\tilde{\mu}=\tilde{\nu}$, and we show that $\tilde{\mu}$ is a unique fixed point of T. To show that f is continuous at $\tilde{\mu}$. Let $\{\tilde{y}_n^n\}$ be any sequence in X such taht $\{\tilde{y}_n^n\}$ convergent to $\tilde{\mu}$. Then

$$\begin{aligned} \tilde{\mathcal{B}}(\tilde{\mu},\tilde{\mu},f\tilde{y}_n^n) &= \tilde{\mathcal{B}}(f\tilde{\mu},f\tilde{\mu},f\tilde{y}_n^n) \\ &\tilde{\preceq} \ \psi(\mathcal{L}(\tilde{\mu},\tilde{\mu},\tilde{y}_n^n))\cdot\mathcal{L}(\tilde{\mu},\tilde{\mu},\tilde{y}_n^n) \\ &\tilde{\ll} \ \gamma_{\tilde{\eta}}\cdot\mathcal{L}(\tilde{\mu},\tilde{\mu},\tilde{y}_n^n), \end{aligned}$$

where

$$\mathcal{L}(\tilde{\mu},\tilde{\mu},\tilde{y}_n^n) = \max\{\tilde{\mathcal{B}}(\tilde{\mu},\tilde{\mu},\tilde{y}_n^n),\tilde{\mathcal{B}}(\tilde{\mu},T\tilde{\mu},T\tilde{\mu}),\tilde{\mathcal{B}}(\tilde{\mu},f\tilde{\mu},f\tilde{\mu}),\tilde{\mathcal{B}}(f\tilde{\mu},\tilde{\mu},\tilde{y}_n^n)\}.$$

Thus

$$\tilde{\mathcal{B}}(\tilde{\mu},\tilde{\mu},T\tilde{y}_n^n)\tilde{\ll}\gamma_{\tilde{\eta}}\cdot\tilde{\mathcal{B}}(\tilde{\mu},\tilde{\mu},\tilde{y}_n^n).$$

Letting $n\to\infty$. Then we deduce that $\{f\tilde{y}_n^n\}$ is convergent to $f\tilde{\mu}=\tilde{\mu}$. Hence f is continuous at $\tilde{\mu}$. □

Acknowledgement

The author is very grateful to the reviewers for their reading the manuscript and valuable comments.

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